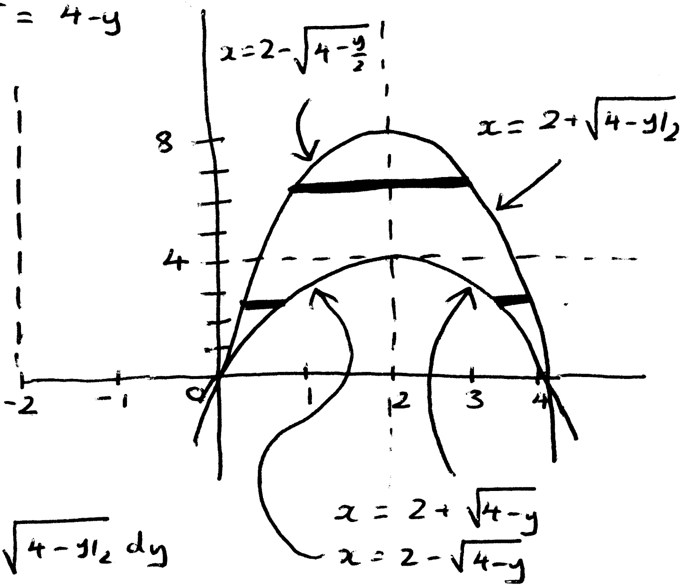
First, the "wrong" way: $y = 4x - x^2 \Rightarrow (x-2)^2 = 4-y$
$\Rightarrow x = 2 \pm \sqrt{4-y}$ and $y = 8x - 2x^2 = 2(4x - x^2) \Rightarrow \frac{y}{2} = 4x - x^2 \Rightarrow x = 2 \pm \sqrt{4-y/2}$
determine the boundaries in the diagram. The axis of revolution is $x = -2$. So for $4 \leq y \leq 8$ the inner radius is $2 + (2 - \sqrt{4-y/2})$ and the outer radius is $2 + (2 + \sqrt{4-y/2}) \Rightarrow$



$$\delta V = \pi\left\{\left(4+\sqrt{4-y/2}\right)^2 - \left(4-\sqrt{4-y/2}\right)^2\right\}\delta y + o(\delta y)$$
$$= \pi\left\{16\sqrt{4-y/2}\right\}\delta y + o(\delta y).$$

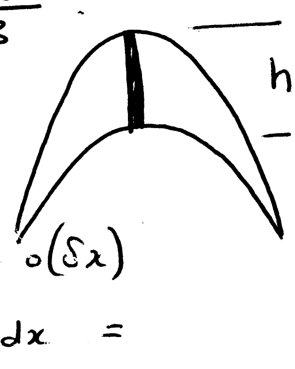
So the volume <u>above</u> $y = 4$ is $V_1 = \int_4^8 \pi\, 16\sqrt{4-y/2}\, dy$
$$= 16\pi\int_4^8 (4-y/2)^{1/2}\,dy = 16\pi\left\{-\tfrac{4}{3}(4-y/2)^{3/2}\right\}\Big|_4^8 = 16\pi\left\{0 - \left(-\tfrac{4}{3}\,2\sqrt{2}\right)\right\} = \frac{128\pi\sqrt{2}}{3}.$$

Below $y = 4$ there are two elementary disks at each height, a smaller one with
$$IR = 2 + \left(2 - \sqrt{4-y/2}\right) \quad\text{and}\quad OR = 2 + \left(2 - \sqrt{4-y}\right)$$
and a larger one with
$$IR = 2 + \left(2 + \sqrt{4-y}\right) \quad\text{and}\quad OR = 2 + \left(2 + \sqrt{4-y/2}\right)$$

Hence
$$\delta V = \pi\left\{\left(4-\sqrt{4-y}\right)^2 - \left(4-\sqrt{4-y/2}\right)^2\right\} + o(\delta y)$$
$$+ \pi\left\{\left(4+\sqrt{4-y/2}\right)^2 - \left(4+\sqrt{4-y}\right)^2\right\} + o(\delta y)$$
$$= \pi\left\{8\sqrt{4-y/2} - 8\sqrt{4-y} - y/2\right\} + \pi\left\{8\sqrt{4-y/2} - 8\sqrt{4-y} + y/2\right\} + o(\delta y)$$
$$= 16\pi\left\{\sqrt{4-y/2} - \sqrt{4-y}\right\} + o(\delta y) \quad \text{after simplification.}$$

Hence the volume <u>below</u> $y = 4$ is
$$V_2 = \int_0^4 16\pi\left\{\sqrt{4-y/2} - \sqrt{4-y}\right\}dy = 16\pi\int_0^4 \left\{(4-y/2)^{1/2} - (4-y)^{1/2}\right\}dy$$
$$= 16\pi\left\{-\tfrac{4}{3}(4-y/2)^{3/2} + \tfrac{2}{3}(4-y)^{3/2}\right\}\Big|_0^4 = 16\pi\left\{-\tfrac{4}{3}\,2\sqrt{2} + 0 - \left(-\tfrac{4}{3}\,4^{3/2} + \tfrac{2}{3}\,4^{3/2}\right)\right\}$$
$$= \frac{-128\pi\sqrt{2}}{3} + \frac{256\pi}{3}.$$

So the total volume is $V = V_1 + V_2 = \dfrac{256\pi}{3}$.



Second, the "right" way. The height of a cylindrical shell is $8x - 2x^2 - (4x - x^2) = 4x - x^2$ and the radius is $2 + x$.
Hence $\delta V = 2\pi r h\,\delta x + o(\delta x) = 2\pi(2+x)(4x-x^2)\,\delta x + o(\delta x)$
$$\Rightarrow V = \int_{x=0}^{x=4} 2\pi(2+x)(4x-x^2)\,dx = 2\pi\int_0^4 (8x + 2x^2 - x^3)\,dx =$$
$$2\pi\left(4x^2 + \tfrac{2}{3}x^3 - \tfrac{x^4}{4}\right)\Big|_0^4 = 2\pi\left\{4^3 + \tfrac{2}{3}4^3 - 4^3 - 0\right\} = 2\pi\cdot 4^3\left\{1 + \tfrac{2}{3} - 1\right\}$$
$$= 2\pi\cdot 4^3\cdot\tfrac{2}{3} = \frac{256\pi}{3}.$$