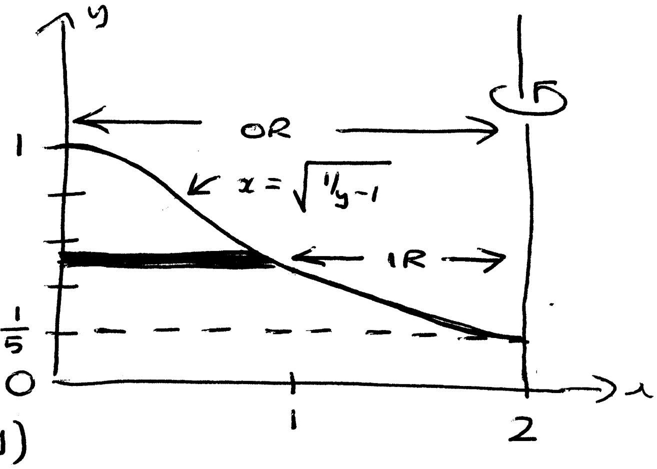
First, the "wrong" way: $y = \dfrac{1}{1+x^2} \Rightarrow$



$1+x^2 = \dfrac{1}{y} \Rightarrow x^2 = \dfrac{1}{y} - 1 \Rightarrow x = \sqrt{\dfrac{1}{y} - 1}$

Hence for $\frac{1}{5} \le y \le 1$ we have $OR = 2$ and $IR = 2 - x = 2 - \sqrt{\frac{1}{y} - 1}$. So for $y > \frac{1}{5}$ we have

$$\delta V = \pi\left\{OR^2 - IR^2\right\}\delta y + o(\delta y)$$

$$= \pi\left\{2^2 - \left(2 - \sqrt{\tfrac{1}{y}-1}\right)^2\right\}\delta y + o(\delta y)$$

while for $y < \frac{1}{5}$ we have $\delta V = \pi\{2^2 - 0^2\}\delta y + o(\delta y) = 4\pi\,\delta y + o(\delta y)$.

Hence the total volume is

$$V = \int_0^{1/5} 4\pi\, dy + \int_{1/5}^1 \pi\left\{4 - \left(2 - \sqrt{\tfrac{1}{y}-1}\right)^2\right\} dy$$

$$= 4\pi\int_0^{1/5} 1\, dy + \pi\int_{1/5}^1 \left\{4 - \left(2^2 - 4\sqrt{\tfrac{1}{y}-1} + \tfrac{1}{y} - 1\right)\right\} dy$$

$$= 4\pi\cdot\frac{1}{5} + \pi\int_{1/5}^1 \left\{4\sqrt{\tfrac{1}{y}-1} + 1 - \tfrac{1}{y}\right\} dy =$$

$$\frac{4\pi}{5} + 4\pi\int_{1/5}^1 \sqrt{\tfrac{1}{y}-1}\, dy + \pi\int_{1/5}^1\left(1 - \tfrac{1}{y}\right) dy = \frac{4\pi}{5} + 4\pi I +$$

$\pi\left\{y - \ln(y)\right\}_{1/5}^{1}$ where $I = \int_{1/5}^1 \sqrt{\frac{1}{y}-1}\, dy$. That is, $V = \frac{4\pi}{5} + 4\pi I + \pi\left\{1 - \ln(1) - \frac{1}{5} + \ln\left(\frac{1}{5}\right)\right\} = \frac{4\pi}{5} + 4\pi I + \frac{4\pi}{5} - \pi\ln(5)$. To evaluate I, we use the substitution $y = \cos^2\theta \Rightarrow \frac{1}{y} - 1 = \sec^2\theta - 1 = \tan^2\theta$ and $\frac{dy}{d\theta} = 2\cos\theta\,\frac{d}{d\theta}[\cos\theta] = 2\cos\theta(-\sin\theta) = -2\sin\theta\cos\theta$. Also, $y = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$ and $y = 1/5 \Rightarrow \cos^2\theta = 1/5 \Rightarrow \cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \frac{2}{\sqrt{5}}$ $\Rightarrow \tan\theta = 2 \Rightarrow \theta = \arctan(2)$. So

$$I = \int_{y=1/5}^{y=1}\sqrt{\tfrac{1}{y}-1}\, dy = \int_{\theta=\arctan(2)}^{\theta=0}\sqrt{\tan^2\theta}\,\frac{dy}{d\theta}\, d\theta = \int_{\arctan(2)}^{0}\tan\theta\left\{-2\sin\theta\cos\theta\right\}d\theta$$

$$= \int_0^{\arctan(2)} 2\sin^2\theta\, d\theta = \int_0^{\arctan(2)}\left[1 - \cos(2\theta)\right]d\theta = \left\{\theta - \tfrac{1}{2}\sin(2\theta)\right\}_0^{\arctan(2)}$$

$$= \arctan(2) - \sin(\arctan(2))\cos(\arctan(2)) = \arctan(2) - \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} =$$

$\arctan(2) - \frac{2}{5} \Rightarrow V = \frac{4\pi}{5} + 4\pi\arctan(2) - \frac{8\pi}{5} + \frac{4\pi}{5} - \pi\ln(5) =$

$\pi\left\{4\arctan(2) - \ln(5)\right\}$. Second, the "right" way: $r = 2 - x$,

$h = \dfrac{1}{1+x^2} \Rightarrow \delta V = 2\pi r h\,\delta t + o(\delta t) = 2\pi\dfrac{(2-x)}{1+x^2}\delta x + o(\delta x) \Rightarrow$

$$V = 2\pi\int_0^2 \frac{2-x}{1+x^2}\, dx = 4\pi\int_0^2 \frac{1}{1+x^2}\, dx - \pi\int_0^2 \frac{2x}{1+x^2}\, dx = 4\pi\arctan(x)\Big|_0^2 - \pi\ln(1+x^2)\Big|_0^2$$

$$= 4\pi\arctan(2) - \pi\ln(5)$$