1 (a) $\dfrac{3^2-9}{3^2+6-3} = 0$ (because rational function continuous where denominator $\neq 0$)

(b) $\dfrac{x^2-9}{x^2+2x-3} = \dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \dfrac{x-3}{x-1}$ for all $x \neq -3$. Hence $\lim\limits_{x\to -3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{-3-3}{-3-1} = \dfrac{3}{2}$

(c) $+\infty$ (because near $x = 1$, numerator ≈ -2 and denominator small and < 0)

(d) $-\infty$ (" " " ", " ≈ -2 " " " " > 0)

(e) 1 $\left(= \lim\limits_{x\to\infty} \dfrac{1 - \frac{9}{x^2}}{1 + 2/x - 3/x^2} = \dfrac{1-0}{1+0-0} \right)$

(f) 1 $\left(= \lim\limits_{x\to -\infty} \quad " \quad = \quad " \right)$

(g) $\nexists$ $\left(\text{because } \lim\limits_{x\to 1+} \dfrac{7}{x-1} = +\infty \text{ and } \lim\limits_{x\to 1-} \dfrac{7}{x-1} = -\infty \right)$

(h)
$$\lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x^2-2x} = \lim_{x\to 2} \frac{\sqrt{x+2}-\sqrt{2x}}{x(x-2)} \cdot \frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}}$$

$$= \lim_{x\to 2} \frac{x+2-2x}{x(x-2)\{\sqrt{x+2}+\sqrt{2x}\}} \quad \left(\text{because } (A-B)(A+B) = A^2-B^2 \text{ with } A = \sqrt{x+2},\ B = \sqrt{2x} \right)$$

$$= \quad " \quad \frac{2-x}{x(x-2)\{\sqrt{x+2}+\sqrt{2x}\}} = \lim_{x\to 2} \frac{-1}{x\{\sqrt{x+2}+\sqrt{2x}\}} = \frac{-1}{2\{\sqrt{4}+\sqrt{4}\}} = \frac{-1}{8}$$

2 (a)
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \quad " \quad \frac{\frac{6}{\sqrt{x+2}} - \frac{6}{\sqrt{a+2}}}{x-a}$$

$$= \lim_{x\to a} \frac{6}{x-a}\left\{\frac{1}{\sqrt{x+2}} - \frac{1}{\sqrt{a+2}}\right\}$$

$$= \lim_{x\to a} \frac{6}{x-a}\left\{\frac{\sqrt{a+2}-\sqrt{x+2}}{\sqrt{x+2}\sqrt{a+2}}\right\}$$

$$= \quad " \quad \frac{6}{x-a} \frac{(\sqrt{a+2}-\sqrt{x+2})(\sqrt{a+2}+\sqrt{x+2})}{\sqrt{x+2}\sqrt{a+2}\{\sqrt{a+2}+\sqrt{x+2}\}}$$

$$= \quad " \quad \frac{6}{x-a} \frac{(a+2)-(x+2)}{\sqrt{x+2}\sqrt{a+2}\{\sqrt{a+2}+\sqrt{x+2}\}}$$

$$= \quad " \quad \frac{6(a-x)}{(x-a)\sqrt{x+2}\sqrt{a+2}\{\sqrt{a+2}+\sqrt{x+2}\}}$$

$$= \quad " \quad \frac{-6}{\sqrt{x+2}\sqrt{a+2}\{\sqrt{a+2}+\sqrt{x+2}\}}$$

$$= \frac{-6}{\sqrt{a+2}\sqrt{a+2}\{2\sqrt{a+2}\}} = \frac{-3}{(a+2)^{3/2}}$$

$$\Rightarrow \quad f'(x) = \frac{-3}{(x+2)^{3/2}}$$

Both f and f' have domain $(-2, \infty)$

ESSENTIAL FOR CONSISTENCY

(b)
$$f'(a) = \lim_{x\to a} \frac{\frac{x^2+1}{\sqrt{x-2}} - \frac{a^2+1}{\sqrt{a-2}}}{x-a}$$

$$= \lim_{x\to a} \frac{1}{x-a} \frac{(x^2+1)\sqrt{a-2} - (a^2+1)\sqrt{x-2}}{\sqrt{x-2}\sqrt{a-2}}$$

$$= \lim_{x\to a} \frac{1}{x-a} \frac{(x^2+1)^2(a-2) - (a^2+1)^2(x-2)}{\sqrt{x-2}\sqrt{a-2}\{(x^2+1)\sqrt{a-2}+(a^2+1)\sqrt{x-2}\}}$$

$$= \lim_{x\to a} \frac{1}{x-a}\left\{\frac{(x^2+1)^2 a - (a^2+1)^2 x + 2(a^4-x^4) + 4(a^2-x^2)}{\text{same denominator}}\right\}$$

$$= \lim_{x\to a} \frac{\{ax^3+a^2x^2+a^3x+2ax-1 - 2(a+x)(a^2+x^2) - 4(a+x)\}}{\text{same thing}}$$

$$= \frac{a^4+a^4+a^4+2a^2-1-8a^3-8a}{\sqrt{a-2}\sqrt{a-2}\{(a^2+1)\sqrt{a-2}+(a^2+1)\sqrt{a-2}\}}$$

$$= \frac{3a^4-8a^3+2a^2-8a-1}{2(a^2+1)(a-2)^{3/2}}$$

$$= \frac{3a^2-8a-1}{2(a-2)^{3/2}}$$

$$\Rightarrow \quad f'(x) = \frac{3x^2-8x-1}{2(x-2)^{3/2}}$$

Domain: $(2, \infty)$ for both f & f'

[MUCH TOO HARD FOR A REAL TEST, BY THE WAY]

3 (a) Set $f(x) = x^3 - 3x + 2 \Rightarrow f'(x) = 3x^2 - 3 + 0 \Rightarrow f'(1) = 0$

and $g(x) = e^x + 4x^2 + 1 \Rightarrow g'(x) = e^x + 4\cdot 2x + 0$

$\Rightarrow g'(1) = e + 8$

Then $h'(x) = \frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

$\Rightarrow h'(1) = f'(1)g(1) + f(1)g'(1) = 0 + (1^3 - 3 + 2)(e+8)$

$= 0$

(But the method would be essentially the same even if the answer were not zero.)

(b) $h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

$\Rightarrow h'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2} = \frac{(e+5)\cdot 0 - 0\cdot(e+8)}{(e+5)^2} = 0$

(Same comment.)

4. $f'(x) = \begin{cases} a + 2bx & \text{if } x < 2 \\ \frac{-1}{x^2} & \text{if } x > 2 \end{cases}$ (by linearity and power law)

Continuity of f requires $f(2-) = f(2+)$ or $a\cdot 2 + b\cdot 2^2 = \frac{1}{2}$

" " f' " $f'(2-) = f'(2+)$ or $a + 2b\cdot 2 = -\frac{1}{2^2}$

So $\left.\begin{array}{l} 2a + 4b = \frac{1}{2} \\ a + 4b = -\frac{1}{4} \end{array}\right\} \Rightarrow a = \frac{3}{4},\ b = -\frac{1}{4}$

5. By the quotient rule, $\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x+1)}{(x+1)^2} = \frac{(x+1)\cdot 1 - x(1+0)}{(x+1)^2}$

$= \frac{1}{(x+1)^2}$

So tangent line at $\left(a, \frac{a}{a+1}\right)$ has slope $\frac{1}{(a+1)^2}$

So its equation is $y - \frac{a}{a+1} = \frac{1}{(a+1)^2}\{x - a\}$

It passes through $(1, 2)$ if $2 - \frac{a}{a+1} = \frac{1-a}{(a+1)^2} \Rightarrow$

$2(a+1)^2 - a(a+1) = 1 - a$ or $a^2 + 4a + 1 = 0 \Rightarrow (a+2)^2 = 3$

$\Rightarrow a = -2 \pm \sqrt{3}$. So there are two such tangent lines. They touch the curve at $\left(-2\pm\sqrt{3}, \frac{-2\pm\sqrt{3}}{-2\pm\sqrt{3}+1}\right)$, or $\left(\sqrt{3}-2, \frac{\sqrt{3}-2}{\sqrt{3}-1}\right)$ and $\left(-\sqrt{3}-2, \frac{\sqrt{3}+2}{\sqrt{3}+1}\right)$