Suppose that g with domain Q and range R is differentiable, and that f with domain R and range S is differentiable. Then the composition $F = f \circ g$ defined by

$$F(x) \;=\; f(g(x))$$

has domain Q and range S and is differentiable with

$$F'(x) \;=\; f'(g(x))\,g'(x).$$

In other words, whenever x and y are differentially related through an intermediate u such that $y = f(u)$ and $u = g(x)$, implying $y = F(x)$ with $F = f \circ g$, then the three derivatives $\frac{dy}{du}$, $\frac{du}{dx}$ and $\frac{dy}{dx}$ are related by

$$\frac{dy}{dx} \;=\; \frac{dy}{du}\,\frac{du}{dx}.$$

For all practical purposes, the proof of this result—known as the chain rule—requires us only to observe that

$$\frac{dy}{dx} \;=\; \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \;=\; \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x} \;=\; \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \;=\; \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \cdot \frac{du}{dx},$$

that

$$\frac{dy}{du} \;=\; \lim_{\delta u \to 0} \frac{\delta y}{\delta u},$$

and that $\delta x \to 0$ implies $\delta u \to 0$ (because g is continuous, otherwise it couldn't be differentiable). Strictly speaking, however, the existence of F' can be inferred from the existence of f' and g' only if $\delta u \to 0$ implies $\delta x \to 0$—which usually holds, but is not guaranteed to hold, because it is possible to have $\delta u = 0$ when $\delta x \neq 0$. Equivalently, the argument that

$$\begin{aligned}
F'(a) \;&=\; \lim_{x \to a} \frac{F(x) - F(a)}{x - a} \\
&=\; \lim_{x \to a} \frac{F(x) - F(a)}{g(x) - g(a)}\,\frac{g(x) - g(a)}{x - a} \;=\; \lim_{x \to a} \frac{F(x) - F(a)}{g(x) - g(a)}\,\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&=\; \lim_{x \to a} \frac{F(x) - F(a)}{g(x) - g(a)}\,g'(a) \;=\; \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\,g'(a) \\
&=\; \lim_{g(x) \to g(a)} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}\,g'(a) \;=\; f'(g(a))\,g'(a)
\end{aligned}$$

is, strictly speaking, valid only if we can't have $g(x) = g(a)$ when $x \neq a$.

Well, suppose that we did have $g(x) = g(a)$ for $x \neq a$. What would that imply? Remember that we can make x as close as we please to a, as long as x is not actually equal to a. So if there's an x ($\neq a$) for which $g(x) = g(a)$, just move a bit closer to x. And if there's another x ($\neq a$) for which $g(x) = g(a)$, just move a bit closer still. And if there's yet another x ($\neq a$) for which $g(x) = g(a)$, just move even closer again. And so on. Eventually, it must be possible to move close enough to a so that there are no more x ($\neq a$) for which $g(x) = g(a)$—unless g is constant near a. Then, because g is continuous, we must have $g(x) = \text{constant} = g(a)$ for all x near a. Hence $g'(a) = 0$ *and* $F(x) = f(g(x) = f(g(a)) = F(a)$ for all x near a, implying $F = \text{constant}$; hence $F'(a) = 0$. Thus, because

$$0 \;=\; f'(g(a)) \cdot 0$$

(given that f' exists), we must have

$$F'(a) \;=\; f'(g(a))\, g'(a)$$

regardless of whether g is ever constant on any subdomain.

Several successive applications of the chain rule are often necessary to calculate the derivative of a composition, because several functions may be compounded, and some of these functions may be joins. To illustrate: suppose that

$$u \;=\; g(x) \;=\; \sqrt{x(1-x)},$$

which is differentiable on $Q = (0, 1)$: although the domain of g is actually $[0, 1]$, for the purposes of the chain rule we must restrict g to where it is differentiable. The range of g is $R = \big(0, \tfrac{1}{2}\big)$—why? On this domain we can define

$$y \;=\; f(u) \;=\; \begin{cases} \big(u - \tfrac{3}{10}\big)^2 & \text{if} \quad 0 < u \le \tfrac{3}{10} \\ 0 & \text{if} \quad \tfrac{3}{10} < u \le \tfrac{4}{10} \\ \big(u - \tfrac{2}{5}\big)^2 & \text{if} \quad \tfrac{2}{5} < u < \tfrac{1}{2}. \end{cases}$$

The range of f is $S = \big(0, \tfrac{9}{100}\big)$—why? An application of the chain rule* yields

$$\frac{dy}{du} \;=\; f'(u) \;=\; \begin{cases} 2\left(u - \tfrac{3}{10}\right) & \text{if} \quad 0 < u \le \tfrac{3}{10} \\ 0 & \text{if} \quad \tfrac{3}{10} < u \le \tfrac{4}{10} \\ 2\left(u - \tfrac{2}{5}\right) & \text{if} \quad \tfrac{2}{5} < u < \tfrac{1}{2}. \end{cases}$$

Another application of the chain rule† yields

$$\frac{du}{dx} \;=\; g'(x) \;=\; \frac{1-2x}{2\sqrt{x(1-x)}}.$$

Yet another application of the chain rule‡ yields

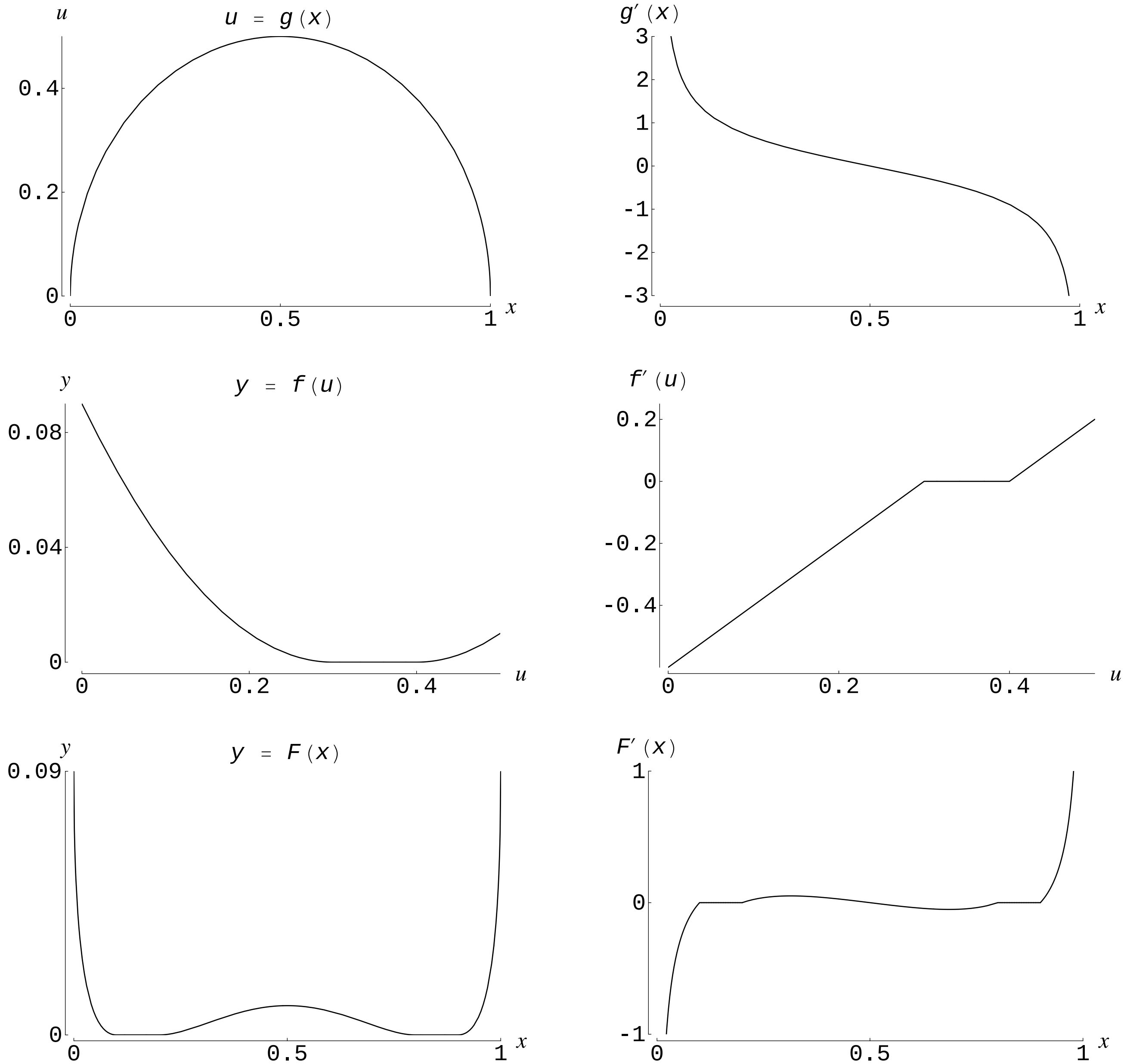
$$\frac{dy}{dx} \;=\; \frac{dy}{du}\frac{du}{dx} \;=\; \begin{cases} (1-2x)\Big(1 - \frac{3}{10\sqrt{x(1-x)}}\Big) & \text{if} \quad 0 < x \le \tfrac{1}{10} \\ 0 & \text{if} \quad \tfrac{1}{10} < x \le \tfrac{1}{5} \\ (1-2x)\Big(1 - \frac{2}{5\sqrt{x(1-x)}}\Big) & \text{if} \quad \tfrac{1}{5} < x \le \tfrac{4}{5} \\ 0 & \text{if} \quad \tfrac{4}{5} < x \le \tfrac{9}{10} \\ (1-2x)\Big(1 - \frac{3}{10\sqrt{x(1-x)}}\Big) & \text{if} \quad \tfrac{9}{10} < x < 1. \end{cases}$$

These results are illustrated overleaf.

*Work out the details for yourself.

†Again work out the details for yourself.

‡Yet again work out the details for yourself.


u = g(x)
g′(x)
y = f(u)
f′(u)
y = F(x)
F′(x)
u
y
x
0.4
0.2
0
0.5
1
3
2
1
-1
-2
-3
0.08
0.04
0.2
-0.2
-0.4
0.09
-1