Integration by Substitution

Roughly speaking, the fundamental theorem of calculus says that differentiation (of a function on an appropriate domain) is the opposite of integration, and vice versa. But there is also an asymmetry: if integrals are like parents and derivatives like children, then a parent function has a unique child function, but a child function has a family of parent functions. So we have to be more precise: differentiation is the opposite of integration in the sense that

$$F'(u) \;=\; f(u) \qquad \Longleftrightarrow \qquad \int f(u)\,du \;=\; F(u) + C$$

for some constant C. Equivalently, differentiation is the opposite of integration in the sense that

$$F'(u) \;=\; f(u) \qquad \Longleftrightarrow \qquad \int_a^b f(u)\,du \;=\; F(b) - F(a)$$

for arbitrary a and b, where $\int_a^b f(u)\,du$ denotes the definite integral of f over the subdomain $[a, b]$, i.e., the *signed* area enclosed by the graph $y = f(u)$, the horizontal coordinate axis $y = 0$ and vertical line segments at $u = a$ and $u = b$ (with area counted positively above the axis and negatively below it).

Note that the fundamental theorem is about a relationship between f and F on an entire domain, and so it does not matter what symbol we use to denote an arbitrary element of that domain. Furthermore, the information content of the fundamental theorem is in no way altered by labelling a parent function and a child function by G and g, respectively, instead of by F and f. Thus an identical statement of the fundamental theorem is that

$$g(x) \;=\; G'(x) \qquad \Longleftrightarrow \qquad G(x) + C = \int g(x)\,dx$$

for some constant C. Equivalently, differentiation is the opposite of integration in the sense that

$$g(x) \;=\; G'(x) \qquad \Longleftrightarrow \qquad G(b) - G(a) \;=\; \int_a^b g(x)\,dx$$

for arbitrary a and b, where $\int_a^b g(x)\,dx$ denotes the definite integral of g over the subdomain $[a, b]$, i.e., the *signed* area enclosed by the graph $y = g(x)$, the horizontal coordinate axis $y = 0$ and vertical line segments at $x = a$ and $x = b$ (again, of course, area is counted positively above the axis and negatively below it).

Now suppose that u and x are related by $u = \phi(x)$, where ϕ is an *invertible* (either increasing or decreasing) function on a subdomain of interest. Then there exists an inverse function, say ζ, such that

$$u \;=\; \phi(x) \qquad \Longleftrightarrow \qquad x \;=\; \zeta(u).$$

Furthermore, define a composition G by

$$G(x) \;=\; F(\phi(x)),$$

which implies, of course, that

$$F(u) \;=\; G(\zeta(u)).$$

So, applying the chain rule to each of these equations in turn, we have BOTH that

$$G'(x) \;=\; F'(\phi(x))\,\phi'(x)$$

AND

$$F'(u) \;=\; G'(\zeta(u))\zeta'(u).$$

But $F'(u) = f(u) \Longrightarrow F'(\phi(x)) = f(\phi(x))$, and $G'(x) = g(x) \Longrightarrow G'(\zeta(u)) = g(\zeta(u))$. So, from above,

$$g(x) \;=\; f(\phi(x))\,\phi'(x)$$

and

$$f(u) \;=\; g(\zeta(u))\zeta'(u).$$

Now we put it all together to get BOTH

$$\begin{aligned}\int f(u)\,du \;&=\; F(u)+C \;=\; F(\phi(x))+C \\ &=\; G(x)+C \;=\; \int g(x)\,dx \\ &=\; \int f(\phi(x))\,\phi'(x)\,dx \;=\; \int \left\{ f(u)\,\frac{du}{dx} \right\}\,dx\end{aligned}$$

AND

$$\begin{aligned}\int g(x)\,dx \;&=\; G(x)+C \;=\; G(\zeta(u))+C \\ &=\; F(u)+C \;=\; \int f(u)\,du \\ &=\; \int g(\zeta(u))\,\zeta'(u)\,du \;=\; \int \left\{ g(x)\,\frac{dx}{du} \right\}\,du.\end{aligned}$$

So the substitution $u = \phi(x)$ can be used to convert an integral with respect to u into an integral with respect to x; and, correspondingly, the inverse substitution $x = \zeta(u)$ can be used to convert an integral with respect to x into an integral with respect to u. This process is known as *integration by substitution*.

The corresponding equations for definite integrals are as follows. First,

$$\int_{u=a}^{u=b} f(u)\,du \;=\; \int_{x=\zeta(a)}^{x=\zeta(b)} f(\phi(x))\,\phi'(x)\,dx$$

uses the substitution $u = \phi(x)$ to convert an integral with respect to u into an integral with respect to x. Second,

$$\int_{x=\alpha}^{x=\beta} g(x)\,dx \;=\; \int_{u=\phi(\alpha)}^{u=\phi(\beta)} g(\zeta(u))\,\zeta'(u)\,du$$

uses the substitution $x = \zeta(u)$ to convert an integral with respect to x into an integral with respect to u. In each case, note the important point that integration by substitution requires both the original substitution and the inverse substitution; specifically, the original substitution is used to

1. Rewrite the old integrand in terms of the new integration variable
2. Differentiate to find the (nonlinear) scaling factor by which the old integrand must be multiplied to become the new integrand

and the inverse substitution is used to

3. Convert the old integration limits into the new integration limits.

Problems

1. Calculate $\int_0^1 x(x^2+1)^4\,dx$ by
   (a) using the substitution $x = \sqrt{u-1}$ and
   (b) some other method.

2. Use the substitution $x = 1 + u^2$ to show that $\int_1^5 x\sqrt{x-1}\,dx = \frac{272}{15}$.

3. Use the substitution $x = \sqrt{1+u^2}$ to show that $\int_1^{\sqrt{5}} x^3\sqrt{x^2-1}\,dx = \frac{136}{15}$.

4. Use the substitution $x = 1 + u^2$ to show that
   (a) $\int_1^2 (x+2)\sqrt{x-1}\,dx = \frac{12}{5}$
   (b) $\int_1^2 (2x+1)\sqrt{x-1}\,dx = \frac{14}{5}$.

5. Use the substitution $x = \frac{u}{16-u}$ to show that $\int\limits_0^b \frac{64\sqrt{x}}{(1+x)^{\frac{7}{2}}}\,dx = \frac{317}{3840}$, where $b = \frac{1}{63}$.