1(a) $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$

$= \lim \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$

$= \lim \frac{x^2+2x+4}{x+2}$

$= \frac{2^2+2\cdot 2+4}{2+2} = 3$

Note: in general, $x^3-a^3 = (x-a)(x^2+ax+a^2)$

(b) $\lim_{x\to 0} \frac{4\sin(3x)+5\sin(x)}{2\sin(x)+5\sin(2x)}$

$= \lim \frac{12\,\frac{\sin(3x)}{3x} + 5\,\frac{\sin(x)}{x}}{2\,\frac{\sin(x)}{x} + 10\,\frac{\sin(2x)}{2x}}$

$= \frac{12\cdot 1+5\cdot 1}{2\cdot 1+10\cdot 1} = \frac{17}{12}$

Note: You know that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Put $\theta = ax$, where $a \neq 0$ is a constant. Then the above becomes $\lim_{ax\to 0} \frac{\sin ax}{ax} = 1$

$\Rightarrow \lim_{x\to 0} \frac{\sin ax}{ax} = 1$

because $ax \to 0 \Leftrightarrow x \to 0$.

(c) See Lecture 4, near end

2(a) Let x increase to $x+\delta x$ in such a way that y increases to $y+\delta y$. Then we have $y+\delta y = \frac{2(x+\delta x)+7}{x+\delta x+3} \Rightarrow \delta y = (y+\delta y) - y = \frac{2(x+\delta x)+7}{(x+\delta x+3)} - \frac{2x+7}{x+3}$

$= \frac{(2x+7+2\delta x)(x+3) - (2x+7)(x+3+\delta x)}{(x+3+\delta x)(x+3)} = \frac{-\delta x}{(x+3+\delta x)(x+3)}$ after simplification in head.

So $\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{-1}{(x+3+\delta x)(x+3)} = \frac{-1}{(x+3+0)(x+3)} = -\frac{1}{(x+3)^2}$.

OR $(x+3)y = 2x+7$ and $(x+\delta x+3)(y+\delta y) = 2(x+\delta x)+7$ yields, after subtraction, $(x+3)\delta y + \delta x\, y + \delta x\,\delta y = 2\delta x \Rightarrow (x+3)\frac{\delta y}{\delta x} + y + \delta y = 2$

Now let $\delta x \to 0$. Get $(x+3)\frac{dy}{dx} + y + 0 = 2$ (because $\delta y \to 0$ as $\delta x \to 0$) $\Rightarrow$

$\frac{dy}{dx} = \frac{2-y}{x+3} = \frac{2 - \frac{2x+7}{x+3}}{x+3} = \frac{\frac{2x+6}{x+3} - \frac{2x+7}{x+3}}{x+3} = \frac{(2x+6)-(2x+7)}{(x+3)^2} = \frac{-1}{(x+3)^2}$

2(b) Again perturb (x,y) to $(x+\delta x, y+\delta y)$ in such a way that you stay on the curve. Then you have $y^2 = 3x^4+5x^2$ and $(y+\delta y)^2 = 3(x+\delta x)^4 + 5(x+\delta x)^2 \Rightarrow$

$\cancel{y^2} + 2y\,\delta y + \delta y^2 = 3\{x^4 + 4x^3\delta x + o(\delta x)\} + 5\{x^2 + 2x\,\delta x + \delta x^2\} = \cancel{3x^4} + \cancel{5x^2} +$

$(12x^3+10x)\delta x + o(\delta x)$. Subtracting, then dividing by δx:

$$2y\frac{\delta y}{\delta x} + \delta y\frac{\delta y}{\delta x} = 12x^3+10x + \frac{o(\delta x)}{\delta x}.$$ Now let $\delta x \to 0$.

Since $\delta x \to 0 \Rightarrow \delta y \to 0$, you get $2y\frac{dy}{dx} + 0\cdot\frac{dy}{dx} = 12x^3+10x+0 \Rightarrow$

$2y\frac{dy}{dx} = 12x^3+10x \Rightarrow y\frac{dy}{dx} = 6x^3+5x \Rightarrow \frac{dy}{dx} = \frac{6x^3+5x}{y} = \frac{6x^3+5x}{\sqrt{3x^4+5x^2}}$

3(a) By the product rule, $f'(x) = \frac{d}{dx}(x^5-3x+2)\sin\left[\frac{1}{4}\pi x\right] + (x^5-3x+2)\frac{d}{dx}\left[\sin\left(\frac{1}{4}\pi x\right)\right]$

$$= (5x^4-3+0)\sin\left(\tfrac{1}{4}\pi x\right) + (x^5-3x+2)\frac{dy}{dx} \quad (*)$$

where $y = \sin(u)$ with $u = \frac{1}{4}\pi x$ so that $\frac{dy}{du} = \cos(u)$ and $\frac{du}{dx} = \frac{1}{4}\pi$

implying $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \cos(u)\frac{1}{4}\pi = \frac{1}{4}\pi\cos\left(\frac{1}{4}\pi x\right)$ by the chain rule.

Substituting into $(*)$ yields $f'(x) = (5x^4-3)\sin\left(\frac{1}{4}\pi x\right) + (x^5-3x+2)\frac{1}{4}\pi\cos\left(\frac{1}{4}\pi x\right)$

So $f'(1) = (5-3)\sin\left(\frac{\pi}{4}\right) + (1-3+2)\cdot\frac{1}{4}\pi\cos\left(\frac{\pi}{4}\right) = 2\cdot\frac{1}{\sqrt{2}} + 0\cdot\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} = \sqrt{2}$

3(b) By the chain rule, $\dfrac{d}{dx}\left[g(u)\right] = g'(u)\dfrac{du}{dx}$ for any g. So with $u = \frac{1}{4}\pi x \Rightarrow$

$\dfrac{du}{dx} = \dfrac{\pi}{4}$ we have $\dfrac{d}{dx}\left[\sin\left(\frac{1}{4}\pi x\right)\right] = \dfrac{d}{dx}\left[\sin(u)\right] = \cos(u)\dfrac{du}{dx} = \cos(u)\cdot\dfrac{\pi}{4} = \frac{1}{4}\pi\cos\left(\frac{1}{4}\pi x\right)$

and $\dfrac{d}{dx}\left[\left(\cos\frac{1}{4}\pi x\right)\right] = \dfrac{d}{dx}\left[\cos(u)\right] = -\sin(u)\dfrac{du}{dx} = -\frac{1}{4}\pi\sin\left(\frac{1}{4}\pi x\right)$.

Now we can use the quotient rule to differentiate $u = \dfrac{3x + \sin\left(\frac{1}{4}\pi x\right)}{3x - \cos\left(\frac{1}{4}\pi x\right)}$

We get
$$\frac{du}{dx} = \frac{\frac{d}{dx}\left\{3x + \sin\left(\frac{1}{4}\pi x\right)\right\}\left\{3x - \cos\left(\frac{1}{4}\pi x\right)\right\} - \left\{3x + \sin\left(\frac{1}{4}\pi x\right)\right\}\frac{d}{dx}\left\{3x - \cos\left(\frac{1}{4}\pi x\right)\right\}}{\left\{3x - \cos\left(\frac{1}{4}\pi x\right)\right\}^2}$$

$$(*) \qquad = \frac{\left\{3 + \frac{1}{4}\pi\cos\left(\frac{1}{4}\pi x\right)\right\}\left\{3x - \cos\left(\frac{1}{4}\pi x\right)\right\} - \left\{3x + \sin\left(\frac{\pi x}{4}\right)\right\}\left\{3 + \frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)\right\}}{\left\{3x - \cos\left(\frac{1}{4}\pi x\right)\right\}^2}$$

Now if we set $f(x) = y$ then we have $y = u^3 \Rightarrow \dfrac{dy}{dx} = \dfrac{d}{du}(u^3)\dfrac{du}{dx}$

$= 3u^2\dfrac{du}{dx}$. So $f'(1) = 3u^2\dfrac{du}{dx}\Big|_{x=1} = 3\left\{\dfrac{3 + \sin(\pi/4)}{3 - \cos(\pi/4)}\right\}^2\dfrac{du}{dx}\Big|_{x=1}$

$$= 3\left(\frac{3\sqrt{2}+1}{3\sqrt{2}-1}\right)^2\frac{du}{dx}\Big|_{x=1}$$

But
$$\frac{du}{dx}\Big|_{x=1} = \frac{\left(3 + \frac{\pi}{4\sqrt{2}}\right)\left(3 - \frac{1}{\sqrt{2}}\right) - \left(3 + \frac{1}{\sqrt{2}}\right)\left(3 + \frac{\pi}{4\sqrt{2}}\right)}{\left(3 - \frac{1}{\sqrt{2}}\right)^2} \quad (\text{by } (*))$$

$$= -\left(3 + \frac{\pi}{4\sqrt{2}}\right)\frac{2\sqrt{2}}{(3\sqrt{2}-1)^2}, \text{ after simplification.}$$

$$\text{So } f'(1) = -6\sqrt{2}\left(3 + \frac{\pi}{4\sqrt{2}}\right)\frac{(3\sqrt{2}+1)^2}{(3\sqrt{2}-1)^4}$$

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4. $f'(x) = \begin{cases} 0 + 2bx & \text{if } x \in [0,2) \\ \frac{d}{dx}\left[\frac{1-x}{1+x}\right] & \text{if } x \in (2,\infty) \end{cases}$ But by the quotient rule, $\dfrac{d}{dx}\left[\dfrac{1-x}{1+x}\right] = \dfrac{(0-1)(1+x) - (1-x)(0+1)}{(1+x)^2} = \dfrac{-2}{(1+x)^2}$

So $f'(x) = \begin{cases} 2bx & \text{if } x \in [0,2) \\ \frac{-2}{(1+x)^2} & \text{if } x \in (2,\infty) \end{cases}$. Now $f(2-) = f(2+) \Rightarrow a + b2^2 = \dfrac{1-2}{1+2}$

and $f'(2-) = f'(2+) \Rightarrow 2b\cdot 2 = \dfrac{-2}{(1+2)^2}$

or $\left.\begin{aligned} a + 4b &= -1/3 \\ 4b &= -2/9 \end{aligned}\right\} \Rightarrow b = -\dfrac{1}{18}, \; a = -1/9$

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5. $y = w^{1/2}$ where $w = \dfrac{7x^2+8}{1+2x^2} \Rightarrow \dfrac{dw}{dx} = \dfrac{\frac{d}{dx}(7x^2+8)(1+2x^2) - (7x^2+8)\frac{d}{dx}(1+2x^2)}{(1+2x^2)^2}$

$= \dfrac{(14x+0)(1+2x^2) - (7x^2+8)(0+4x)}{(1+2x^2)^2}$ by the quotient rule. Also $\dfrac{dy}{dw} = \frac{1}{2}w^{-1/2} \Rightarrow$

$\dfrac{dy}{dx}\Big|_{x=2} = \dfrac{dy}{dw}\dfrac{dw}{dx}\Big|_{x=2} = \dfrac{1}{2}\left(\dfrac{7x^2+8}{1+2x^2}\right)^{-1/2}\dfrac{14x(1+2x^2) - 4x(7x^2+8)}{(1+2x^2)^2}\Bigg|_{x=2} = \dfrac{1}{2}\left(\dfrac{36}{9}\right)^{-1/2}\dfrac{28\cdot 9 - 8\cdot 36}{81}$

$= -\dfrac{1}{9}$. So the tangent line has equation $y - 2 = -\dfrac{1}{9}(x-2) \Rightarrow y = -\dfrac{1}{9}x + \dfrac{20}{9}$

or $x + 9y = 20$.