1 (a) Apply product rule to both sides: $\dfrac{dy}{dx}\cdot\sin(x^2) + y\cdot\dfrac{d}{dx}\left[\sin(x^2)\right] =$

$1\cdot\sin(y^2) + x\,\dfrac{d}{dx}\left[\sin(y^2)\right]$. $(*)$ With $u = x^2$ and $z = \sin u$ we have

$$\frac{d}{dx}\left[\sin(x^2)\right] = \frac{dz}{dx} = \frac{dz}{du}\cdot\frac{du}{dx} = \cos(u)\cdot 2x = 2x\cos(x^2)$$

So also $\dfrac{d}{dx}\left[\sin(y^2)\right] = \dfrac{d}{dy}\left[\sin(y^2)\right]\cdot\dfrac{dy}{dx} = 2y\cos(y^2)\,\dfrac{dy}{dx}$ (on using the chain rule again). So now $(*) \Rightarrow$

$$\frac{dy}{dx}\cdot\sin(x^2) + y\cdot 2x\cos(x^2) = \sin(y^2) + 2xy\cos(y^2)\,\frac{dy}{dx}$$

Put $x = y = \dfrac{\sqrt{\pi}}{2}$. Get $m\sin\left(\frac{\pi}{4}\right) + \frac{\sqrt{\pi}}{2}\cdot 2\frac{\sqrt{\pi}}{2}\cos\left(\frac{\pi}{4}\right) = \sin\left(\pi/4\right)$

$+\ 2\,\frac{\sqrt{\pi}}{2}\,\frac{\sqrt{\pi}}{2}\cos\left(\frac{\pi}{4}\right) m \Rightarrow \dfrac{m}{\sqrt{2}} + \dfrac{\pi}{2}\cdot\dfrac{1}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} + \dfrac{\pi}{2}\,\dfrac{1}{\sqrt{2}}\,m$

$\Rightarrow m + \frac{1}{2}\pi = 1 + \frac{1}{2}\pi m \Rightarrow m\left(\frac{1}{2}\pi - 1\right) = \frac{1}{2}\pi - 1 \Rightarrow m = 1$

(because $\frac{1}{2}\pi \neq 1$).

(b) $y = \left(\dfrac{x^2-1}{x^2+1}\right)^{1/4} \Rightarrow y^4 = \dfrac{x^2-1}{x^2+1} \Rightarrow 4\ln(y) = \ln(x^2-1)$

ASSUME $x > 1$: OK BECAUSE $a > 1$!

$-\ \ln(x^2+1)$ by properties of logarithms. So

$$\frac{d}{dx}\left[4\ln(y)\right] = \frac{d}{dx}\left[\ln(x^2-1)\right] - \frac{d}{dx}\left[\ln(x^2+1)\right] \quad (☆)$$

But $u > 0 \Rightarrow \dfrac{d}{du}\left[\ln(u)\right] = \dfrac{1}{u}$

$$\Rightarrow \frac{d}{dx}\left[\ln(u)\right] = \frac{d}{du}\left[\ln(u)\right]\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

by the chain rule. Using this result three times in succession (with $u = y$, $u = x^2-1$ and $u = x^2+1$) in (☆) we have

$$\frac{4}{y}\frac{dy}{dx} = \frac{1}{x^2-1}(2x-0) - \frac{1}{x^2+1}(2x+0)$$

Now put $x = \sqrt{\frac{17}{15}}$, $y = \frac{1}{2}$ to get

$$\frac{4}{\frac{1}{2}}\,m = \frac{1}{\left(\frac{17}{15}-\frac{15}{15}\right)}\,2\sqrt{\frac{17}{15}} - \frac{1}{\left(\frac{17}{15}+\frac{15}{15}\right)}\,2\sqrt{\frac{17}{15}}$$

$$\Rightarrow m = \frac{15}{128}\sqrt{255} \quad (\approx 1.87)$$

2. $y = \ln(\ln(x)) \Rightarrow e^y = \ln(x) \Rightarrow \frac{d}{dx}(e^y) = \frac{1}{x}$

CHAIN RULE

$\Rightarrow \frac{d}{dy}(e^y)\frac{dy}{dx} = \frac{1}{x} \Rightarrow e^y \frac{dy}{dx} = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{xe^y}$

$= \frac{1}{x\ln(x)}$. So $m = \left.\frac{dy}{dx}\right|_{x=e}$ is $\frac{1}{e\ln(e)} = \frac{1}{e\cdot 1} = \frac{1}{e}$

$\Rightarrow$ tangent line has equation $y - 0 = \frac{1}{e}(x-e)$ or

$$y = \frac{1}{e}x - 1$$

QUOTIENT RULE

3 (a) $v = \frac{dx}{dt} = \frac{1\cdot(1+t^2) - t(0+2t)}{(1+t^2)^2} = \frac{1-t^2}{(1+t^2)^2}$

(b) $a = \frac{dv}{dt} = \frac{d}{dt}\left\{\frac{1-t^2}{(1+t^2)^2}\right\} = \frac{(0-2t)(1+t^2)^2 - (1-t^2)\frac{d}{dt}((1+t^2)^2)}{((1+t^2)^2)^2}$

by the quotient rule. But with $u = 1+t^2$ we have

$\frac{d}{dt}\left[(1+t^2)^2\right] = \frac{d}{dt}\left[u^2\right] = 2u\frac{du}{dt} = 2(1+t^2)(0+2t)$. So

$$a = \frac{-2t(1+t^2)^2 - (1-t^2)\,2(1+t^2)\cdot 2t}{(1+t^2)^4}$$

$$= \frac{2t}{(1+t^2)^3}\left\{-(1+t^2) - 2(1-t^2)\right\} = \frac{2t(t^2-3)}{(1+t^2)^3}$$

(c) The particle starts at $x = 0$, when it is already moving with velocity 1. It then begins to decelerate (because $0 < t < \sqrt{3} \Rightarrow a < 0$) and comes to rest at $t = 1$. Because a is still negative, it continues toward the origin ($t > 1 \Rightarrow v < 0$). At $t = \sqrt{3}$, a becomes positive again, but now $|a|$ is too small to prevent the particle from continuing toward the origin, which it approaches in the limit as $t \to \infty$.

4 (a) $x^{1/3} + y^{1/3} = 5 \Rightarrow \frac{d}{dx}(x^{1/3}) + \frac{d}{dx}(y^{1/3}) = \frac{d}{dx}(5) \Rightarrow$

$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} = -\left(\frac{y}{x}\right)^{2/3}$

Now put $x = 8$, $y = 27$ to get $m = -\frac{27^{2/3}}{8^{2/3}} = -\frac{3^2}{2^2} = -\frac{9}{4}$

(b) $$x^{2/3}\frac{dy}{dx} + y^{2/3} = 0 \Rightarrow \frac{d}{dx}\left(x^{2/3}\frac{dy}{dx}\right) + \frac{d}{dx}\left(y^{2/3}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(x^{2/3}\right)\frac{dy}{dx} + x^{2/3}\frac{d^2y}{dx^2} + \frac{d}{dy}\left(y^{2/3}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{2}{3}x^{-1/3}\frac{dy}{dx} + x^{2/3}\frac{d^2y}{dx^2} + \frac{2}{3}y^{-1/3}\frac{dy}{dx}^{2} = 0$$

Now put $x = 8,\ y = 27$ to get

$$\frac{2}{3}8^{-1/3}m + 8^{2/3}\alpha + \frac{2}{3}27^{-1/3}m = 0$$

$$\Rightarrow \frac{2}{3}\cdot\frac{1}{2}m + 2^2\alpha + \frac{2}{3}\frac{1}{3}m = 0$$

$$\Rightarrow \frac{2}{3}\cdot\frac{1}{2}\cdot\left(-\frac{9}{4}\right) + 4\alpha + \frac{2}{3}\cdot\frac{1}{3}\cdot\left(-\frac{9}{4}\right) = 0$$

$$\Rightarrow \alpha = 5/16$$

(Alternatively, first reduce to $\frac{d^2y}{dx^2} = \frac{10y^{1/3}}{3x^{5/3}}$, but that is easier.)

5 (a) First note that $\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{1}{\sqrt{x^2+1}} \Rightarrow$

$$\frac{d}{dx}\left[\sinh^{-1}(ax)\right] = \frac{d}{du}\left[\sinh^{-1}(u)\right]\frac{du}{dx} \quad \text{with } u = ax$$

$$= \frac{1}{\sqrt{1+u^2}}\cdot a = \frac{a}{\sqrt{1+a^2x^2}} \quad (*)$$

So here
$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(9-x^2\right)^{1/2}\right]\sinh^{-1}(2x) + \left(9-x^2\right)^{1/2}\frac{d}{dx}\left[\sinh^{-1}(2x)\right]$$

$$= \frac{1}{2}\left(9-x^2\right)^{-1/2}(0-2x) \quad '' \quad + \quad '' \quad \frac{2}{\sqrt{1+4x^2}}$$

$$= \frac{-x}{\sqrt{9-x^2}}\sinh^{-1}(2x) + \frac{2\sqrt{9-x^2}}{\sqrt{1+4x^2}}$$

(b) $$y = \ln\left(x^6\right) + \ln\left(\{\sin(x)\}^4\right)$$

$$= 6\ln(x) + 4\ln(\sin(x)) \Rightarrow$$

$$\boxed{\text{CHAIN RULE} \quad \frac{d}{dx}\left[\ln(u)\right] = \frac{1}{u}\frac{du}{dx}}$$

$$\frac{dy}{dx} = 6\cdot\frac{1}{x} + 4\cdot\frac{1}{\sin(x)}\frac{d}{dx}(\sin(x))$$

$$= \frac{6}{x} + 4\cot(x).$$

(c) Put $u = \ln(1+e^{x^2})$ $\Rightarrow$ $\dfrac{du}{dx} = \dfrac{d}{dx}\ln(w)$

$= \ln(w)$, say

$$= \frac{1}{w}\frac{dw}{dx} = \frac{1}{w}\left\{0 + 2xe^{x^2}\right\}$$

So now $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(u^4\right) = 4u^3\dfrac{du}{dx} = \dfrac{8xe^{x^2}u^3}{w}$

$$= \frac{8xe^{x^2}\left[\ln\left(1+e^{x^2}\right)\right]^3}{1+e^{x^2}}$$

$$\boxed{\frac{d}{dx}\left[e^{\phi}\right] = e^{\phi}\frac{d\phi}{dx}}$$

CHAIN RULE

(d) By the product rule and (*),

$$\frac{dy}{dx} = 1\cdot\tanh^{-1}(x) + x\frac{d}{dx}\left[\tanh^{-1}(x)\right] + \frac{d}{dx}\ln\left[\left(3-x^4\right)^{1/2}\right]$$

$$= \tanh^{-1}(x) + \frac{x}{1-x^2} + \frac{d}{dx}\left[\frac{1}{2}\ln\left(3-x^4\right)\right]$$

$$= \quad " \quad + \quad " \quad + \frac{1}{2}\frac{d}{dx}\left[\ln\left(3-x^4\right)\right]$$

$$= \quad " \quad + \quad " \quad + \frac{1}{2}\frac{1}{3-x^4}\left(0-4x^3\right)$$

$$= \tanh^{-1}(x) + \frac{x}{1-x^2} - \frac{2x^3}{3-x^4}.$$

(e) By the product rule and (*),

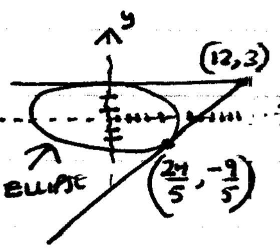
$$\frac{dy}{dx} = 4x^3\cdot\sinh^{-1}(4x) + x^4\cdot\frac{4}{\sqrt{1+16x^2}}$$

$$= 4x^3\left\{\sinh^{-1}(4x) + \frac{x}{\sqrt{1+16x^2}}\right\}$$

6. $x^2 + 4y^2 = 36 \Rightarrow 2x + 4\cdot 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{4y}$.

Let the tangent touch at (a,b). Then its equation must be $y - b = m(x-a)$ where $m = \left.\dfrac{dy}{dx}\right|_{\substack{x=a\\y=b}} = -\dfrac{a}{4b}$

or $y = -\dfrac{ax}{4b} + \dfrac{a^2}{4b} + b = \dfrac{36-ax}{4b}$



because $a^2 + 4b^2 = 36$. This line passes through $(12,3)$ if $3 = \dfrac{36-12a}{4b}$

or $a + b = 3$, $\Rightarrow a^2 + 4(3-a)^2 = 36 \Rightarrow a(5a-24) = 0$.

So either $a = 0$ and $b = 3$ or $a = \frac{24}{5}$ and $b = -\frac{9}{5}$. The tangent lines are $y = 3$ and $y = \frac{2}{3}x - 5$.