1.

$$f'(x) = (1-0)(x^3 - 11x^2 + x + 225) + (x-1)(3x^2 - 22x + 1 + 0)$$
$$= 4x^3 - 36x^2 + 24x + 224$$
$$= 4(x^3 - 9x^2 + 6x + 56) = 4(x-4)(x^2 - 5x - 14)$$
$$= 4(x+2)(x-7)(x-4)$$
$$f''(x) = 4(3x^2 - 18x + 6 + 0) = 12(x^2 - 6x + 2)$$

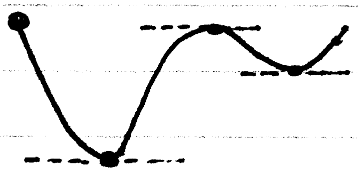
(a) $f'(-2) = 0$, $f''(-2) = 12 \times 18 > 0 \Rightarrow -2$ local minimizer

$f'(4) = 0$, $f''(4) = 12 \times (-6) < 0 \Rightarrow 4$ " maximizer

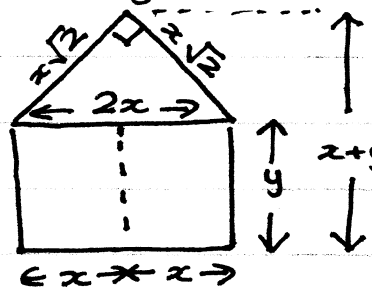
$f'(7) = 0$, $f''(7) = 12 \times 9 > 0 \Rightarrow 7$ " minimizer

(b) $f\left(-\frac{13}{3}\right) = \frac{29056}{81}$, $f(-2) = -513$, $f(4) = 351$, $f(7) = 216$, $f\left(\frac{25}{3}\right) = \frac{28600}{81}$. Because $\frac{29056}{81} > \frac{28600}{81} > 351 > 216 > -513$, the global extrema are $\max\left(f, -\frac{13}{3}, \frac{25}{3}\right) = \frac{29056}{81}$ and $\min\left(f, -\frac{13}{3}, \frac{25}{3}\right) = -513$.

The graph looks like this:



2 (a)



Let the width be $2x$. Then, from the diagram, the height is $x+y$ where

$$x\sqrt{2} + y + 2x + y + x\sqrt{2} = p$$
$$\Rightarrow y = \frac{1}{2}p - (1+\sqrt{2})x$$

PERIMETER

and we have to maximize the area

$$A = A(x) = 2xy + x^2 = px - (1 + 2\sqrt{2})x^2$$

We have $A'(x) = p - 2(1+2\sqrt{2})x$

and $A''(x) = 0 - 2(1+2\sqrt{2})$, which is negative.

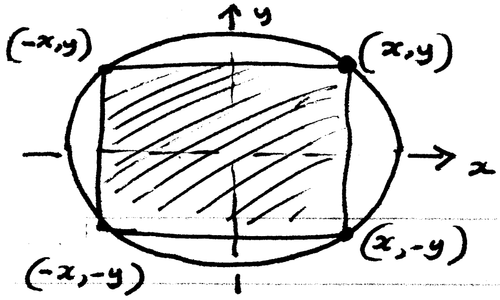
$\therefore$ Graph is concave down, implying unique global maximum where $A'(x) = 0 \Rightarrow 2x = \dfrac{p}{1+2\sqrt{2}}$. That's the width.

The height is $x + y = x + \frac{1}{2}p - (1+\sqrt{2})x = \frac{1}{2}p - \sqrt{2}x = \dfrac{p(1+\sqrt{2})}{2(1+2\sqrt{2})}$

So the ratio of height to width is

$$\frac{x+y}{2x} = \frac{1}{2}(1+\sqrt{2}) \approx 1.207$$

(b)



By symmetry, area is

$$A = A(x) = 4xy = 4x\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$
$$= 4bx\sqrt{1-\frac{x^2}{a^2}}, \quad 0 \le x \le a.$$

So $A^2 = (4b)^2 x^2\left(1-\frac{x^2}{a^2}\right) \Rightarrow$

$$2A\frac{dA}{dx} = (4b)^2\left\{2x\left(1-\frac{x^2}{a^2}\right) + x^2\left(0-\frac{2x}{a^2}\right)\right\}$$
$$= (4b)^2\left\{2x - \frac{4x^3}{a^2}\right\} = (4b)^2 2x\left(1-\frac{2x^2}{a^2}\right)$$
$$\Rightarrow 2\left\{\left(\frac{dA}{dx}\right)^2 + A\frac{d^2A}{dx^2}\right\} = (4b)^2\left\{2 - \frac{12x^2}{a^2}\right\} \quad (*)$$

So $\frac{dA}{dx} = 0 \Rightarrow x = 0$ or $x = \frac{a}{\sqrt{2}}$. But the endpoints $x=0$ and $x=a$ clearly yield a minimum. So the maximizer is $\frac{a}{\sqrt{2}}$, and the maximum is

$A\left(\frac{a}{\sqrt{2}}\right) = 2ab$. Note that $A''\left(\frac{a}{\sqrt{2}}\right) < 0$ because

$(*)$ with $x = \frac{a}{\sqrt{2}} \Rightarrow 2\{0 + AA''\} = (4b)^2(2-6) \Rightarrow$

$$A''\left(\frac{a}{\sqrt{2}}\right) = -\frac{64b^2}{2A} = -\frac{64b^2}{4ab} = \frac{-16b}{a}$$

3 (a) $f(x) = \frac{g(x)}{h(x)}$ where $g(x) = x^3$, $h(x) = x - \sin(x)$

So we have $g(0) = 0$, $h(0) = 0 - \sin(0) = 0$

Also $g'(x) = 3x^2$, $h'(x) = 1 - \cos(x)$

$\Rightarrow g'(0) = 0$, $h'(0) = 1 - 1 = 0$

Also $g''(x) = 6x$, $h''(x) = 0 - (-\sin(x)) = \sin(x)$

$\Rightarrow g''(0) = 0$, $h''(0) = \sin(0) = 0$

Also $g'''(x) = 6$, $h'''(x) = \cos(x)$

$$\Rightarrow \quad g'''(0) = 6, \quad h'''(0) = 1 \quad (\neq 0, \text{ hooray!})$$

$$\text{So} \quad \lim_{x \to 0} f(x) = \frac{g'''(0)}{h'''(0)} = 6$$

(b) $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x) = x - \ln(1+x)$, $h(x) = x\ln(1+x) \Rightarrow$

$g(0) = 0 - \ln(1) = 0$, $h(0) = 0\ln(1) = 0$ and

$g'(x) = 1 - \dfrac{1}{1+x}$, $h'(x) = 1.\ln(1+x) + x.\dfrac{1}{1+x} \Rightarrow$

$g'(0) = 1 - \dfrac{1}{1} = 0$, $h'(0) = \ln(1) + 0.1 = 0$ and

$g''(x) = 0 - \left(\dfrac{-1}{(1+x)^2}.1\right) = \dfrac{1}{(1+x)^2}$, $h''(x) = \dfrac{1}{1+x} + 1.\dfrac{1}{1+x}$

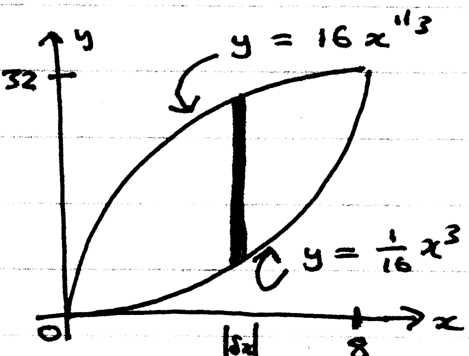
$+ x.\left\{\dfrac{-1}{(1+x)^2}\right\} = \dfrac{2}{(1+x)} - \dfrac{x}{(1+x)^2} \Rightarrow g''(0) = \dfrac{1}{(1+0)^2} = 1$

and $h''(0) = \dfrac{2}{(1+0)} - 0 = 2$. Both $\neq 0$, hooray!

$$\text{So} \quad \lim_{x \to 0} f(x) = \frac{g''(0)}{h''(0)} = \frac{1}{2}$$

DON'T FORGET THAT
$y = Ax^r$, $x \geq 0$, $r > 0$
is upward bending for $r > 1$, a straight line for $r = 1$ and downward bending for $r < 1$, always going through $(0,0)$

4 (a) Boundaries are $y = 16x^{1/3}$ (upper) $\xleftarrow{r<1}$ and
$x^3 = 16y$ or $y = \frac{1}{16}x^3$ (lower) $\leftarrow \boxed{r>1}$
They meet where $16x^{1/3} = x^3/16 \Rightarrow 256 = x^{8/3}$ or $x = 0$
$\Rightarrow x = 0$ or $x = (256)^{3/8}$, i.e., $x = 0$ or $x = 8$.
Correspondingly, $y = 0$ or $y = 16\,8^{1/3} = 32$. So



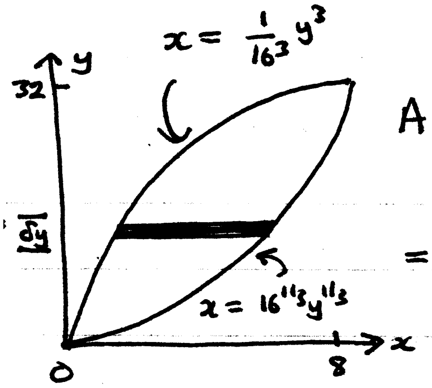
$$\delta A = \left(16x^{1/3} - \frac{1}{16}x^3\right)\delta x + o(\delta x)$$

$$\Rightarrow A = \int_0^8 \left(16x^{1/3} - \frac{1}{16}x^3\right)dx$$

$$= \left(\frac{16x^{4/3}}{4/3} - \frac{1}{16}\frac{x^4}{4}\right)\Bigg|_0^8 = 12\,8^{4/3} - \frac{8^4}{64} = 128$$

(b) Now boundaries are $x = (y/16)^3 = \frac{1}{16^3}y^3$ (left) and
$x = (16y)^{1/3} = 16^{1/3}y^{1/3}$ (right). So

$$\delta A = \left\{16^{1/3}y^{1/3} - \frac{1}{16^3}y^3\right\}\delta y + o(\delta y) \Rightarrow$$



$$A = \int_0^{32} \left\{ 16^{1/3} y^{1/3} - \frac{1}{16^3} y^3 \right\} dy$$

$$= \left. 16^{1/3} \frac{y^{4/3}}{4/3} - \frac{y^4}{4 \cdot 16^3} \right|_0^{32}$$

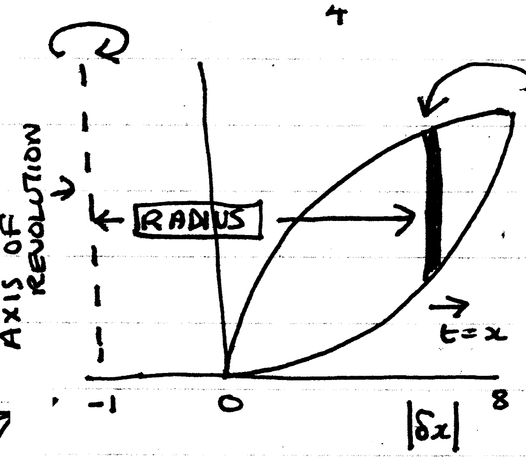
$$= 16^{1/3}\, 32^{4/3} \cdot 3/4 - \frac{32^4}{4 \cdot 16^3} - 0 = 2^{4/3} (2^5)^{4/3} \cdot \frac{3}{4} - \frac{2^4 \cdot 16^4}{4 \cdot 16^3} = \frac{3}{4} 2^{4/3} 2^{20/3} - 2^2 \cdot 16$$

$$= \frac{3}{4} 2^{24/3} - 2^2 \cdot 16 = \frac{3}{4} 2^8 - 2^6 = 3 \cdot 2^6 - 2^6 = 2 \cdot 2^6$$

$$= 2^7 = 128 \text{ as before}$$

DON'T FORGET: $x = Ay^m$, $y \geq 0$, $m > 0$ is upward bending for $m < 1$, straight for $m = 1$ and downward bending for $m > 1$ (i.e., in a sense, "the opposite"). Or if you do forget, test with an m value you know.

5 (a)



x-axis not to scale, obviously (different scale on negative axis)

SEMI VERTICAL CROSS SECTION THROUGH CYLINDER YIELDS STRIP

THICKNESS = $\delta t = \delta x$ (BECAUSE $t = x$)

RADIUS = $r = x - (-1) = x + 1$

HEIGHT = $h = 16x^{1/3} - \frac{1}{16}x^3$ as (before)

So $\delta V = 2\pi r h\, \delta t + o(\delta t)$

$$= 2\pi (1+x)\left(16x^{1/3} - \frac{1}{16}x^3\right)\delta x + o(\delta x)$$

$$\Rightarrow V = 2\pi \int_0^8 (1+x)\left(16x^{1/3} - \frac{1}{16}x^3\right) dx$$

$$= 2\pi \int_0^8 \left\{ 16x^{1/3} + 16x^{4/3} - \frac{1}{16}x^3 - \frac{1}{16}x^4 \right\} dx$$

$$= 2\pi \left. \left\{ \frac{3}{4} 16 x^{4/3} + \frac{3}{7} 16 x^{7/3} - \frac{1}{64}x^4 - \frac{1}{80}x^5 \right\} \right|_0^8$$

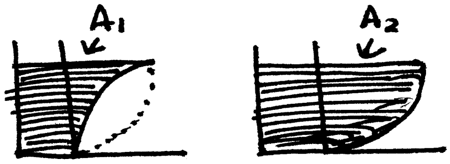
$$= 2\pi \left\{ \frac{3}{4} 16\, 8^{4/3} + \frac{3}{7} 16\, 8^{7/3} - \frac{8^4}{64} - \frac{8^5}{80} - 0 \right\}$$

$$= 2\pi \left\{ 12 \cdot 2^4 + \frac{3}{7} 16 \cdot 2^7 - 8^2 - \frac{8^4}{10} \right\}$$

$$= 2\pi \cdot 2^6 \left\{ 3 + \frac{96}{7} - 1 - \frac{32}{5} \right\} = \pi \cdot 2^7 \left\{ 2 + 32\left(\frac{3}{7} - \frac{1}{5}\right) \right\}$$

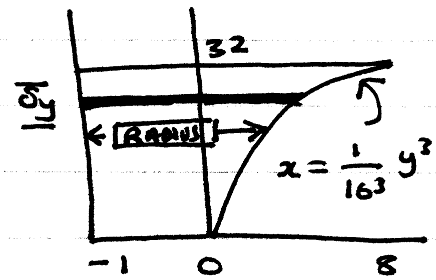
$$= 128\pi \left\{ 2 + 32 \cdot \frac{8}{35} \right\} = 256\pi \left\{ 1 + \frac{128}{35} \right\} = 256\pi \cdot \frac{163}{35} = \frac{41728\pi}{35}$$

(b) Region is difference between these two:



Let V_i be volume from rotating A_i, $i = 1, 2$.

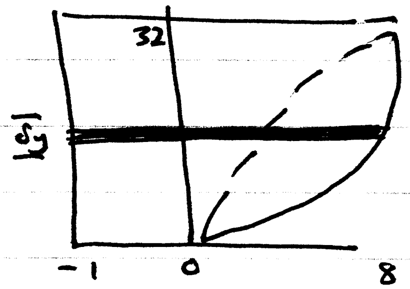
Then clearly $V = V_2 - V_1$. For first region we have



$$\delta V = \pi \, \text{RADIUS}^2 \, \delta y + o(\delta y)$$
$$= \pi \left(1 + \frac{1}{16^3} y^3\right)^2 \delta y + o(\delta y)$$

$$\Rightarrow \quad V_1 = \pi \int_0^{32} \left(1 + \frac{1}{16} y^3\right)^2 dy$$

For second region we have



$$\delta V = \pi \, \text{RADIUS}^2 \, \delta y + o(\delta y)$$
$$= \pi \left\{1 + 16^{1/3} y^{1/3}\right\}^2 \delta y + o(\delta y)$$

$$\Rightarrow \quad V_2 = \pi \int_0^{32} \left(1 + 16^{1/3} y^{1/3}\right)^2 dy .$$

$$\text{So} \quad V = V_2 - V_1 = \pi \int_0^{32} \left(1 + 16^{1/3} y^{1/3}\right)^2 dy - \pi \int_0 \left(1 + \frac{1}{16} y^3\right)^2 dy$$

$$= \pi \left\{ \int_0^{32} \left\{ \left(1 + 16^{1/3} y^{1/3}\right)^2 - \left(1 + \frac{1}{16^3} y^3\right)^2 \right\} dy \right\}$$

$$= \pi \int_0^{32} \left\{ 1 + 2 \cdot 16^{1/3} y^{1/3} + 16^{2/3} y^{2/3} - \left(1 + \frac{2}{16^3} y^3 + \frac{1}{16^6} y^6\right) \right\} dy$$

$$= \pi \int_0^{32} \left\{ 2 \cdot 16^{1/3} y^{1/3} + 16^{2/3} y^{2/3} - \frac{1}{8 \cdot 16^2} y^3 - \frac{1}{16^6} y^6 \right\} dy$$

$$= \pi \left\{ 2 \cdot 16^{1/3} \cdot \frac{3}{4} y^{4/3} + 16^{2/3} \cdot \frac{3}{5} y^{5/3} - \frac{y^4}{16^2 \cdot 32} - \frac{y^7}{7 \cdot 16^6} \right\} \Bigg|_0^{32}$$

$$= \pi \left\{ 2 \cdot 16^{1/3} \cdot \frac{3}{4} \cdot 2^{4/3} \cdot 16^{4/3} + 16^{2/3} \cdot \frac{3}{5} 2^{5/3} 16^{5/3} - \frac{32^3}{16^2} - \frac{2^7 \cdot 16}{7} - 0 \right\}$$

$$= \pi \left\{ 2 \cdot 16^{1/3} \cdot \frac{3}{4} \cdot 16^{1/3} \cdot 16^{4/3} + 2^{8/3} \cdot \frac{3}{5} 2^{5/3} \cdot 2^{20/3} - 128 - \frac{16}{7} 2^7 \right\}$$

$$= \pi \left\{ \frac{3}{2} 16^2 + \frac{3}{5} 2^{11} - 128 - \frac{16}{7} 128 \right\} = 128\pi \left\{ 3 + \frac{48}{5} - 1 - \frac{16}{7} \right\} = \frac{41728\pi}{35}$$