MAC 2311, Sections 07 & 09, Fall 2004 (Dr M-G)

Mock Final

Monday, December 06, 2004

You are allowed to use a TI-30Xa (or any four-function calculator). No other calculator is allowed. You have 75 minutes. Present your solutions clearly. Show all necessary steps in your method. Include enough comments or diagrams to convince me that you thoroughly understand. Begin each question (as opposed to part of question) on a fresh sheet of paper, use *one* side of the paper only, and ensure that your solutions are stapled together in the proper order at the end of the test. You may assume that

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int \frac{\ln(x)}{x}\,dx = \tfrac{1}{2}\{\ln(x)\}^2 + C$$

$$\int x\ln(x)\,dx = \tfrac{1}{4}x^2\{\ln(x^2) - 1\} + C$$

$$\int \{\ln(x)\}^2\,dx = x\left(\{\ln(x)\}^2 - \ln(x^2) + 2\right) + C$$

(where C is an arbitrary constant).

DO NOT WRITE ON THIS QUESTION PAPER, WHICH MUST BE TURNED IN AT THE END OF THE TEST (BUT NOT STAPLED TO YOUR SOLUTIONS)

1. **(a)** For $f(x) = 6x\{\pi\sin(x) + 3\cos(x)\}$, find $f'(\frac{\pi}{3})$. [?]

(b) For $f(x) = \left\{\frac{x+3}{x^2-3x+4}\right\}^4$, find $f'(1)$. [?]

2. Find the equation of the tangent line to $y = (x+3)\sqrt{3x^2+1}$ at the point $(4, 49)$. [?]

3. For $y = \sqrt{\dfrac{3x+1}{x^2+3}}$ (with $x > -\frac{1}{3}$), find $m = \left.\dfrac{dy}{dx}\right|_{x=1}$ *exactly* [?]

4. What is the tangent line to the curve $y = (2x+1)e^{x^2}$ at the point $(1, 3e)$? [?]

5. Find both $m = \left.\dfrac{dy}{dx}\right|_{(x,y)=(1,-1)}$ and $\alpha = \left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(1,-1)}$ for $x^2 + y^3 + y^4 = 1$ [?]

6. In each of the following cases, find $m = \left.\dfrac{dy}{dx}\right|_{x=0}$ *exactly*:

(a) $y = \{\ln(1 + \{x+1\}^4)\}^3$ [?]

(b) $y = x\,\tanh^{-1}(3x) + \sinh^{-1}(4x), \quad -\frac{1}{3} < x < \frac{1}{3}$ [?]

7. A particle moves along the x-axis with position $x = \dfrac{2t-1}{e^{2t}}$ at time t.

(a) Find the particle's velocity at time t [?]

(b) Find the particle's acceleration at time t [?]

(c) *Briefly* describe its motion for $t \geq 0$ [?]

8. Given that $f(1) = 4$, $f'(1) = -10$ and

$$f''(t) = \frac{36 + 3t\sqrt{t} + 4t^3}{2t^3}$$

for all $t > 0$, find $f(t)$ *exactly*. [?]

9. Use the substitution $u = \sqrt[3]{21x+1}$ to find the *exact* value of $I = \int_0^3 \frac{35x+1}{\sqrt[3]{21x+1}}\,dx$. [?]

10. In each of the following cases, find the exact value of the definite integral:

(a) $I = \int_1^4 \left\{ x - \frac{3}{\sqrt{x}} \right\}^2 dx$ [?]

(b) $I = \int_{-3}^{3} x(x+1)(2x-1)\,dx$ [?]

(c) $I = \int_0^2 |e^x - 3|\,dx$ [?]

11. A square cookie cutter and a circular cookie cutter are to be made from a thin strip of metal of length L by cutting it into two pieces and bending each piece appropriately. If the sum of the areas of the two resulting cookies must be as small as possible (perhaps to ensure that whoever eats them ingests as few calories as possible) then

(a) How long is a side of the square cookie? [?]

(b) What is the radius of the circular cookie? [?]

(c) What is the ratio of the larger cookie's area to the area of the smaller one? [?]

12. **(a)** A non-Norman window consists of a rectangle surmounted by a right-angled isosceles triangle, with the right angle at the apex of the window. If the window's perimeter is fixed, find the ratio of height to width that admits the greatest possible amount of light. [?]

(b) Find the area of the largest rectangle inscribable in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [?]

13. Use L'Hôpital's rule to calculate $\lim_{x \to 0} f(x)$ where:

(a) $f(x) = \dfrac{x^3}{x - \sin(x)}$ [?]

(b) $f(x) = \dfrac{x - \ln(1+x)}{x \ln(1+x)}$ [?]

(c) $f(x) = \dfrac{1 - \cos(2x)}{\sin(x) - \ln(1+x)}$ [?]

14. **(a)** The function g is defined on $(-\infty, \infty)$ by $g(x) = \frac{\sqrt{x^2+4}}{x^2+2}$. Find $g'(1)$.

(b) Given that $y^3 + \sin(x^2 y) = 1 + \frac{1}{\sqrt{2}}$, find $\frac{dy}{dx}$ when $x = \frac{1}{2}\sqrt{\pi}$ and $y = 1$. [?]

15. Calculate the following exactly (i.e., without using decimal points):

(a) $\int_1^2 \left(x^6 - \frac{1}{7x^6}\right) dx$ **(b)** $\int_2^4 \left(\frac{t-1}{t}\right)^2 dt$ **(c)** $\int_0^{\pi/4} \{\sin\theta - 2\cos\theta\}\, d\theta$

(d) $\int_1^2 (x+1)^3\, dx$ **(e)** $\int_3^4 \left(\frac{1}{y} + e^y + 3\right) dy$ **(f)** $\int_0^{\sqrt{\pi}/2} \theta \cos(\theta^2)\, d\theta$ [?]

16. A function F is defined on $(-\infty, \infty)$ by $F(x) = \int_0^x |e^{6t} - 4|\, dt$. Find an explicit and exact expression for $F(x)$. Hence find the only value of c such that $F(c) = 3$. [?]

17. Developers own adjacent patches of land. Their boundaries in the x-y plane have equations $y = 0, 0 \le x \le 3$ and $y = x(x-1)(x-3), 0 \le x \le 3$. How much more valuable per square unit of land must the smaller patch be to make the two patches equally valuable? [?]

18. A rooster throws an egg vertically upwards from the roof of a henhouse at 40 meters per second on a perfectly calm day. The egg's (vertical) velocity t seconds after launch is $v(t) = 40 - 32t$ m/s.

(a) If the roof of the henhouse is 5 meters above the ground, find an expression for the height of the egg above the ground at any time t.

(b) How long does the egg take to reach its highest point?

(c) What is its maximum height?

(d) What is the average velocity of the egg during its first second of flight?

(e) A 2-meter politician is standing on the ground next to the henhouse, directly below the egg. How much time elapses before the egg strikes the politician?

(f) Show that the egg is travelling more than 2 m/s faster than the rooster threw it initially when it hits the politician. [?]

19. **(a)** For $y = \ln(e^{-x} + xe^{-x})$ on $(-\infty, \infty)$, find the simplest possible expression for $\frac{dy}{dx}$.

(b) For f defined on $(0, 4)$ by $f(x) = \sqrt[3]{2 - \sqrt{\sin\left(\frac{1}{4}\pi x\right)}}$, find $f'(1)$. [?]

20. If $y^3 + \sin(x^2 y) = 1 + \frac{1}{\sqrt{2}}$, find **(a)** $\frac{dy}{dx}$ and **(b)** $\frac{d^2y}{dx^2}$ for $x = \frac{1}{2}\sqrt{\pi}$ and $y = 1$. [?]

21. An open crate has vertical sides, a square bottom and a volume of 4 cubic meters. If the crate has the least possible surface area, find the crate's dimensions. [?]

22. **(a)** For $y = \ln\left(x + \sqrt{x^2 - 1}\right)$ on $(1, \infty)$, find the simplest possible expression for $\frac{dy}{dx}$.

(b) Given that f is defined for $-\frac{1}{3} < x < \sqrt{\frac{2}{3}}$ by $f(x) = \left(\frac{2-3x^2}{2(1+3x)}\right)^{2/3}$, find $f'(0)$. [?]

23. Given that $x^6 + y^6 = 54$, find **(a)** $\frac{dy}{dx}$ and **(b)** $\frac{d^2y}{dx^2}$ for $x = \sqrt{3} = y$. [?]

24. How many tangent lines to the curve

$$y = \frac{x}{x+1}$$

pass through the point $(1,2)$? At which points do these lines touch the curve? [?]

25. A paper cup has the shape of an inverted cone with height 9 cm and radius 3 cm.

(a) If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 6 cm deep? [?]

(b) If, at that precise moment, the pouring stops and the water is all sucked out in a single gulp, how much work is done.(Assume, e.g., that the cup is glued to a table in the upright position.) [?]

26. **(a)** Calculate $\dfrac{d}{dx}\left\{\ln\left(\sqrt{x^2+1}\right)\right\}$. [2] **(b)** Find the *exact* value of $\int_1^{\sqrt{3}} \dfrac{1+4x}{x^2+1}\,dx$. [?]

Hint: Rewrite the integral as a sum of integrals you know how to calculate.

27. The region bounded by the lines $x = 0$, $x = 1$, $y = 0$ and the curve $y = e^{-x}$ or $x = \ln(1/y)$ is rotated (through angle 2π) about the axis of symmetry $y = -1$. Find the *exact* volume of the solid thus generated

(a) by integrating with respect to x and [?]

(b) by integrating with respect to y. [?]

28. A function f is defined on $[2,8]$ by

$$f(x) \;=\; \ln(7x) + \frac{4}{x}.$$

(a) Find the global minimum and maximum of f. Sketch a *very* rough graph. [?]

(b) Find the area of the two-dimensional region R enclosed by $y = 0$, $y = f(x)$, $x = 2$ and $x = 8$. [?]

(c) Find the volume of the three-dimensional region generated by rotating R about the axis of symmetry $x = 0$ (i.e., the y-axis). [?]

(d) Find the volume of the three-dimensional region generated by rotating R about the axis of symmetry $y = 0$ (i.e., the x-axis). [?]

[Perfect score: $?^? + ? \times ? + ? = 120$]