(a) $\dfrac{x^2+x-6}{x^2-4} = \dfrac{(x+3)(x-2)}{(x+2)(x-2)} = \dfrac{x+3}{x+2}$ $\forall x \neq 2$. So $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^2-4} =$

$\lim\limits_{x\to 2} \dfrac{x+3}{x+2} = \dfrac{2+3}{2+2} = \dfrac{5}{4}$ (b) As $x \to -2^-$, $\dfrac{x+3}{x+2} \sim \dfrac{1}{\text{small negative}}$

$\Rightarrow \lim\limits_{x\to -2^-} \dfrac{x^2+x-6}{x^2-4} = -\infty$ (c) As $x \to -2^+$, $\dfrac{x+3}{x+2} \sim \dfrac{1}{\text{small positive}} \Rightarrow$

$\lim\limits_{x\to -2^+} \dfrac{x^2+x-6}{x^2-4} = +\infty$ (d) $\nexists$ because $-\infty \neq +\infty$ (e) $\dfrac{3^2+3-6}{3^2-4} = \dfrac{6}{5}$

(f) $\lim\limits_{x\to -\infty} \dfrac{x^2+x-6}{x^2-4} = \lim\limits_{x\to -\infty} \dfrac{x+3}{x+2} = \lim\limits_{x\to -\infty} \dfrac{1+3/x}{1+2/x} = \dfrac{1+0}{1+0} = 1$

(g) Still 1 (just replace $-\infty$ by ∞ here) (h) Set $f(x) = \sqrt{x^2+1} - \sqrt{x^2-3x+1}$.

Then for all finite x, $f(x) = \dfrac{(\sqrt{x^2+1} - \sqrt{x^2-3x+1})(\sqrt{x^2+1} + \sqrt{x^2-3x+1})}{\sqrt{x^2+1} + \sqrt{x^2-3x+1}}$

$= \dfrac{(\sqrt{x^2+1})^2 - (\sqrt{x^2-3x+1})^2}{\text{same denominator}} = \dfrac{x^2+1-(x^2-3x+1)}{\text{same thing}} = \dfrac{3x}{|x|\sqrt{1+\frac{1}{x^2}} + |x|\sqrt{1-\frac{3}{x}+\frac{1}{x^2}}}$

$= \dfrac{3}{\sqrt{1+1/x^2} + \sqrt{1-\frac{3}{x}+\frac{1}{x^2}}}$ because $|x| = x$ for $x > 0$ (and you can't reach $+\infty$ unless first of all you go positive).

So $\lim\limits_{x\to\infty} f(x) = \dfrac{3}{\sqrt{1+0} + \sqrt{1-0+0}} = \dfrac{3}{2}$.

2 (a) $f'(a) = \lim\limits_{x\to a} \dfrac{\frac{1}{\sqrt{2x-3}} - \frac{1}{\sqrt{2a-3}}}{x-a}$

$= \lim\limits_{x\to a} \dfrac{1}{x-a} \dfrac{\sqrt{2a-3} - \sqrt{2x-3}}{\sqrt{2x-3}\sqrt{2a-3}}$

$=$ " " $\dfrac{(2a-3)-(2x-3)}{\sqrt{2x-3}\sqrt{2a-3}\{\sqrt{2a-3}+\sqrt{2x-3}\}}$

$=$ " " $\dfrac{2(a-x)}{\sqrt{2x-3}\sqrt{2a-3}\{\sqrt{2a-3}+\sqrt{2x-3}\}}$

$= \lim\limits_{x\to a} \dfrac{-2}{\sqrt{2x-3}\sqrt{2a-3}\{\sqrt{2a-3}+\sqrt{2x-3}\}}$

$= \dfrac{-2}{\sqrt{2a-3}\sqrt{2a-3}\{\sqrt{2a-3}+\sqrt{2a-3}\}}$

$= \dfrac{-2}{(2a-3)\{2\sqrt{2a-3}\}}$

$= \dfrac{-1}{(2a-3)^{3/2}}$

$\Rightarrow f'(x) = \dfrac{-1}{(2x-3)^{3/2}}$

$\boxed{\text{Both } f \text{ and } f' \text{ have domain } (\tfrac{3}{2}, \infty)}$

(b) $f'(a) = \lim\limits_{x\to a} \dfrac{\frac{x+2}{x+1} - \frac{a+2}{a+1}}{x-a}$

$= \lim\limits_{x\to a} \dfrac{1}{x-a} \left\{\dfrac{(x+2)(a+1)-(a+2)(x+1)}{(x+1)(a+1)}\right\}$

$=$ " " $\dfrac{a-x}{\text{same denominator}}$

$= \lim\limits_{x\to a} \dfrac{-1}{(x+1)(a+1)} = \dfrac{-1}{(a+1)^2}$

$\Rightarrow f'(x) = \dfrac{-1}{(x+1)^2}$

Both f and f' have domain $(-\infty,-1) \cup (-1,\infty)$ (or $\{x \mid x \neq -1\}$)

3. Define $f(x) = x^4 - 3x + 4$

$\Rightarrow f'(x) = 4x^3 - 3 + 0$

and $g(x) = e^x + x^3 + 2$

$\Rightarrow g'(x) = e^x + 3x^2 + 0$

(using linearity and special results)

So $f(1) = 1^4 - 3 + 4 = 2$, $f'(1) = 4 \cdot 1^3 - 3 = 1$, $g(1) = e^1 + 1^3 + 2 = e + 3$ and $g'(1) = e^1 + 3 + 0 = e + 3 \Rightarrow$ (a) $h'(1) = f'(1)g(1) + f(1)g'(1) = 1 \cdot (e+3) + 2(e+3) = \underline{\underline{3(e+3)}}$ and (b) $h'(1) = \dfrac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \dfrac{(e+3)\{f'(1) - f(1)\}}{(e+3)^2}$ $= \dfrac{(e+3)(1-2)}{(e+3)^2} = \underline{\underline{\dfrac{-1}{e+3}}}$

4 $f'(x) = \begin{cases} b & \text{if } x < 2 \\ \dfrac{-1}{(x+1)^2} & \text{if } x > 2 \end{cases}$ So we have $f(2-) = f(2+) \Rightarrow a + 2b = \dfrac{2+2}{2+1}$ and $f'(2-) = f'(2+) \Rightarrow b = \dfrac{-1}{(2+1)^2}$

or $a + 2b = \dfrac{4}{3}$ and $b = \underline{\underline{\dfrac{-1}{9}}} \Rightarrow a = \dfrac{4}{3} + \dfrac{2}{9} = \underline{\underline{\dfrac{14}{9}}}$.

5 (a) Using the quotient rule, $\dfrac{dy}{dx} = \dfrac{(1+2x^2)\frac{d}{dx}(9x^2) - 9x^2 \frac{d}{dx}\{1 + 2x^2\}}{(1+2x^2)^2}$

$= \dfrac{(1+2x^2)\{9 \cdot 2x\} - 9x^2\{0 + 2 \cdot 2x\}}{(1+2x^2)^2} = \dfrac{18x(1 + 2x^2 - 2x^2)}{(1+2x^2)^2}$

$= \dfrac{18x}{(1+2x^2)^2} \Rightarrow \left.\dfrac{dy}{dx}\right|_{x=2} = \dfrac{18 \cdot 2}{(1+2 \cdot 2^2)^2} = \dfrac{36}{9^2} = \dfrac{4}{9}$

So the equation of the tangent line is $y - 4 = \dfrac{4}{9}(x-2)$ or

$\underline{\underline{y = \dfrac{4}{9}(x+7)}}$. (b) Line meets curve at $(2, 4)$ and another point, both of which satisfy $\dfrac{4}{9}(x+7) = \dfrac{9x^2}{1+2x^2}$ or

$81x^2 = 4(x+7)(1+2x^2) = 4\{2x^3 + 14x^2 + x + 7\}$ or

$8x^3 - 25x^2 + 4x + 28 = 0$. You know that $x-2$ must be a factor of the left-hand side (why?). So factor it out, obtaining

$$(x-2)(8x^2 - 9x - 14) = 0$$

or $(x-2)(8x+7)(x-2) = 0$

or $(8x+7)(x-2)^2 = 0$.

But $x \neq 2$ at the other point. So $x = -\dfrac{7}{8}$ and $y = \dfrac{4}{9}\left(-\dfrac{7}{8} + 7\right)$

$= \dfrac{49}{18}$; i.e., the other point is $\left(-\dfrac{7}{8}, \dfrac{49}{18}\right)$.

[Note that the double root at $x = 2$ was inevitable because of tangency, and if you knew that (which I did, of course) it made factorization easier.]