1. $\dfrac{dy}{dx} = (3x^2+0)\sin\left(\frac{1}{2}\pi x\right) + (x^3+1)\cos\left(\frac{1}{2}\pi x\right)\cdot\frac{1}{2}\pi \Rightarrow \dfrac{dy}{dx}\Big|_{x=1} = 3\sin\left(\frac{\pi}{2}\right) + 0 = 3$

So tangent line is $y - 2 = 3(x-1)$ or $y = 3x - 1$.

2 (a) $f'(t) = 4\int t^{-3}\,dt = -2t^{-2} + c$; $f'(1) = -1 \Rightarrow c = 1$. Now $f'(t) = -2t^{-2} + 1 \Rightarrow f(t) = -2\int t^{-2}\,dt + \int 1\,dt = 2t^{-1} + t + b$. But $f(1) = 3 \Rightarrow b = 0$. So $f(t) = t + \dfrac{2}{t}$.

(b) $u = \sqrt{5x+1} \Rightarrow u^2 = 5x+1 \Rightarrow x = \frac{1}{5}u^2 - \frac{1}{5} \Rightarrow \dfrac{dx}{du} = \dfrac{2u}{5} - 0 = \dfrac{2u}{5}$.

So $I = \displaystyle\int_{\sqrt{5\cdot 0+1}}^{\sqrt{5\cdot 3+1}} \frac{5x-1}{\sqrt{5x+1}}\,\frac{dx}{du}\,du = \int_1^4 \frac{5\left(\frac{u^2-1}{5}\right)-1}{u}\,\frac{2u}{5}\,du = \frac{2}{5}\int_1^4 (u^2-2)\,du$

$= \dfrac{2}{5}\left(\dfrac{u^3}{3} - 2u\right)\Big|_1^4 = \dfrac{2}{5}\left\{\dfrac{64}{3} - 8 - \dfrac{1}{3} + 2\right\} = \dfrac{2}{5}\cdot 15 = 6$

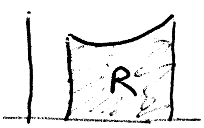
(c) $I = \displaystyle\int_0^{\ln(2)} (2-e^x)\,dx + \int_{\ln(2)}^1 (e^x-2)\,dx = (2x - e^x)\Big|_0^{\ln(2)} + (e^x - 2x)\Big|_{\ln(2)}^1$

$= 2\ln(2) - 2 - (0 - e^0) + e^1 - 2 - (2 - 2\ln(2)) = e + 4\ln(2) - 5$.

3. $1 - \dfrac{dy}{dx} + 3x^2 - 3y^2\dfrac{dy}{dx} = 0 \Rightarrow 0 - \dfrac{d^2y}{dx^2} + 6x - 6y\left(\dfrac{dy}{dx}\right)^2 - 3y^2\dfrac{d^2y}{dx^2} = 0$.

Now put $x = 1$, $y = -1 \Rightarrow 1 - m + 3 - 3m = 0$ or $m = 1$ and

$6 - \alpha + 6 + 6m^2 - 3\alpha = 0$ or $\alpha = 6(1+m^2)/4 = 3$.



4. (a) $\displaystyle\int_1^4 \left(x + \frac{3}{\sqrt{x}}\right)dx = \left(\frac{x^2}{2} + 6\sqrt{x}\right)\Big|_1^4 = 8 + 12 - \frac{1}{2} - 6 = \frac{27}{2}$

(b) $\displaystyle\int_1^4 2\pi x\left(x + \frac{3}{\sqrt{x}}\right)dx = 2\pi\int_1^4 (x^2 + 3x^{1/2})\,dx = 2\pi\left(\frac{x^3}{3} + 2x^{3/2}\right)\Big|_1^4$

$= 2\pi\left\{\dfrac{64}{3} + 16 - \dfrac{1}{3} - 2\right\} = 70\pi$

(c) $\displaystyle\int_1^4 \pi\left(x + \frac{3}{\sqrt{x}}\right)^2 dx = \pi\int_1^4 \left(x^2 + 6\sqrt{x} + \frac{9}{x}\right)dx = \pi\left(\frac{x^3}{3} + 4x^{3/2} + 9\ln(x)\right)\Big|_1^4$

$= \pi\left\{\dfrac{64}{3} + 32 + 9\ln(4) - \dfrac{1}{3} - 4 - 9\ln(1)\right\} = \{49 + 18\ln(2)\}\pi$

5. $g(x) = 1 - \cos(2x) \Rightarrow g'(x) = 0 + 2\sin(2x) \Rightarrow g''(x) = 4\cos(2x)$

$h(x) = \ln(1+3x^2) \Rightarrow h'(x) = \dfrac{6x}{1+3x^2} \Rightarrow h''(x) = \dfrac{6(1+3x^2) - 6x\cdot(0+6x)}{(1+3x^2)^2}$.

So $g(0) = 1-1 = 0$, $g'(0) = 2\sin(0) = 0$, $g''(0) = 4\cos(0) = 4$ and $h(0) = \ln(1) = 0$, $h'(0) = 0$, $h''(0) = 6 \Rightarrow \lim\limits_{x\to 0}\dfrac{g(x)}{h(x)} = \dfrac{g''(0)}{h''(0)} = \dfrac{4}{6} = \dfrac{2}{3}$

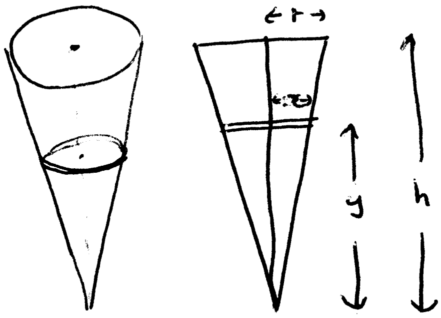
6. $f(x) = \ln(7) - \ln(x) - \dfrac{6}{x} \Rightarrow f'(x) = 0 - \dfrac{1}{x} + \dfrac{6}{x^2} \Rightarrow f''(x) = \dfrac{1}{x^2} - \dfrac{12}{x^3}$

$= \dfrac{x-12}{x^3} < 0$ for $x \in [4,10] \Rightarrow$ unique maximum where $f'(x) = 0$ or $x = 6$.

Also $f(4) - f(10) = \ln\left(\frac{7}{4}\right) - \frac{6}{4} - \ln\left(\frac{7}{10}\right) + \frac{6}{10} = \ln\left(\frac{5}{2}\right) - \frac{9}{10} > 0 \Rightarrow$

$f(4) > f(10)$. So global max is $f(6) = \ln\left(\frac{7}{6}\right) - 1$ and global min is $f(10) = \ln\left(\frac{7}{10}\right) - \frac{3}{5}$.

7.



Similar Δs $\Rightarrow$ $\frac{y}{x} = \frac{h}{r} \Rightarrow x = \frac{ry}{h}$.

$$\delta V = \pi x^2 \delta y + o(\delta y) = \frac{\pi r^2}{h^2} y^2 \delta y + o(\delta y)$$

$$\delta W = \rho \, \delta V g (h - y + o(\delta y))$$

$$= \rho g \frac{\pi r^2}{h^2} y^2 (h-y) \delta y + o(\delta y)$$

So $W = \int_0^h \rho \frac{g\pi r^2}{h^2} y^2 (h-y) \, dy = \rho g \frac{\pi r^2}{h^2} \int_0^h (h y^2 - y^3) \, dy = \rho g \frac{\pi r^2}{h^2} \left(\frac{h y^3}{3} - \frac{y^4}{4} \right) \Big|_0^h$

$= \rho g \frac{\pi r^2}{h^2} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{1}{12} \rho \pi r^2 h^2 g = \frac{1}{4} M g h$ where

$M = \frac{1}{3} \pi r^2 h \rho$ is the mass of the liquid.