1 (a) Because $-16 \notin [-15, 15]$, $16 \notin [-15, 15]$ and f is piecewise-rational, the only possible discontinuities are at $x = -5$ and $x = 12$. But $f(-5^-) = \frac{33(-5)}{16-5} + 657 = 642$ and $f(-5^+) = 17 + 60(-5) + 27(-5)^2 - 2(-5)^3 = 17 - 300 + 675 + 250 = 642$ also $\Rightarrow$ continuous at -5; similarly, $f(12^-) = 17 + 60 \cdot 12 + 27 \cdot 12^2 - 2 \cdot 12^3 = 17 + 720 + 3 \cdot 12^2 = 737 + 432 = 1169 = 48 + 1121 = \frac{16 \cdot 12}{16-12} + 1121 = f(12^+) \Rightarrow$ continuous at 12.

(b) First note that $\frac{d}{dx}\left(\frac{33x}{16+x} + 657\right) = \frac{33 \cdot (16+x) - 33x \cdot 1}{(16+x)^2} + 0 = \frac{528}{(16+x)^2}$ and $\frac{d}{dx}\left(\frac{16x}{16-x} + 1121\right) = \frac{16(16-x) - 16x(-1)}{(16-x)^2} + 0 = \left(\frac{16}{16-x}\right)^2$ by the quotient rule. Hence

$$f'(x) = \begin{cases} 528\,(16+x)^{-2} & \text{if } -15 < x < -5 \\ 60 + 54x - 6x^2 & \text{if } -5 < x < 12 \\ 256\,(16-x)^{-2} & \text{if } 12 < x < 15 \end{cases}$$

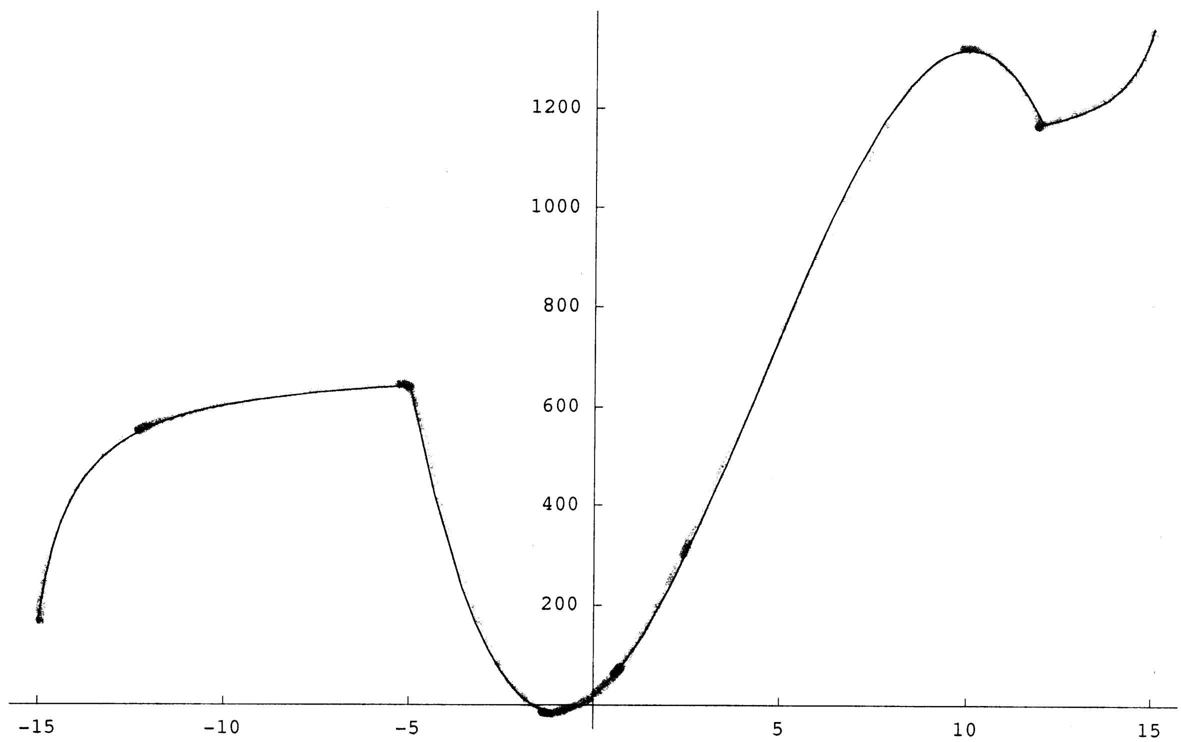
$$\Rightarrow f''(x) = \begin{cases} -1056\,(16+x)^{-3} & \text{if } -15 < x < -5 \\ 54 - 12x & \text{if } -5 < x < 12 \\ -512\,(16-x)^{-3}(-1) & \text{if } 12 < x < 15 \end{cases}$$

(c) $f'(-5^-) = \frac{528}{11^2}$, $f'(-5^+) = 60 + 54(-5) - 6(-5)^2 = -360$ $\Rightarrow f'(-5^-) > 0 > f'(-5^+)$, hence local max where $x = -5$; $f'(12^-) = 60 + 54 \cdot 12 - 6 \cdot 12^2 = -156$, $f'(12^+) = 256\,(16-12)^{-2} = 16 \Rightarrow f'(12^-) < 0 < f'(12^+)$, hence local min where $x = 12$. These are the only possible corner extrema. All other local extrema must be smooth. Clearly, $f'(x) > 0$ for $x \in (-15, -5)$ and for $x \in (2, 15)$. So the only possibilities for $f'(x) = 0$ are on $(-5, 12)$. We have $f'(x) = 0 \Rightarrow 60 + 54x - 6x^2 = 0 \Rightarrow 10 + 9x - x^2 = 0 \Rightarrow (10 - x)(1 + x) = 0 \Rightarrow x = -1$ or $x = 10$. Because $f''(-1) = 54 + 12 > 0$, -1 is a local minimizer.

Because $f''(10) = 54 - 120 < 0$, 10 is a local maximizer

(d) Clearly, $f''(x) < 0$ for $x \in (-15,-5)$ and $f''(x) > 0$ for $x \in (12,15)$. On $(-5,12)$, $f''(x)$ changes sign where $54 - 12x = 0$, or $x = \frac{9}{2}$, and the sign change is from + to −. Hence concave down on $(-15,-5)$, concave up on $(-5, 9/2)$, concave down on $(9/2, 12)$ and concave up on $(12, 15)$. There are three inflection points, namely, $x = -5$, $x = 9/2$ and $x = 12$.

(e) The candidates for global extremizer are the corner local extremizers $x = -5$ and $x = 12$, the smooth local extremizers $x = -1$ and $x = 10$, and the endpoints $x = -15$ and $x = 15$. We have $f(-15) = 162$, $f(-5) = 642$, $f(-1) = -14$, $f(10) = 1317$, $f(12) = 1169$ and $f(15) = 1361$. So the global min and max are $f(-1) = -14$ and $f(15) = 1361$, respectively



2 (a) $g(2^-) = 4(5-2) = 12$; $g(2^+) = 2^2 - 2^3 + 2^4 = 12$. So $g(2^-) = g(2^+) \Rightarrow g$ continuous at $t=2$, $\Rightarrow g$ continuous on $[0,3]$ because g is piecewise-polynomial.

(b) For $0 \le t \le 2$, $G(t) = \int_0^t 4(5-x)\,dx = 4\int_0^t 5-x\ dx$
$= 4\left\{\int_0^t 5\,dx - \int_0^t x\,dx\right\} = 4\left\{5\int_0^t 1\,dx - \int_0^t x\,dx\right\} =$

$$4\left\{5(t-0) - \frac{(t^2-0^2)}{2}\right\} = 4\left\{5t - \tfrac{1}{2}t^2\right\} = 20t - 2t^2.$$

In particular, $G(2) = 40 - 8 = 32$.

For $2 \le t \le 3$, $G(t) = \int_0^t g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^t g(x)\,dx$
$= G(2) + \int_2^t (x^2 - x^3 + x^4)\,dx = 32 + \int_2^t x^2\,dx - \int_2^t x^3\,dx$

$$+ \int_2^t x^4\,dx = 32 + \frac{t^3-2^3}{3} - \left(\frac{t^4-2^4}{4}\right) + \left(\frac{t^5-2^5}{5}\right)$$

$$= 32 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 - \frac{8}{3} + 4 - \frac{32}{5}$$

Hence
$$G(t) = \begin{cases} 2(10-t^2) & \text{if } 0 \le t \le 2 \\ \frac{404}{15} + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \frac{1}{5}t^5 & \text{if } 2 \le t \le 3 \end{cases}$$

3 (a) For $f(x) = \left(x - \frac{2}{\sqrt{x}}\right)^3 = x^3 + 3x^2\left(\frac{-2}{\sqrt{x}}\right) + 3x\left(\frac{-2}{\sqrt{x}}\right)^2 + \left(\frac{-2}{\sqrt{x}}\right)^3$

$$= x^3 - 6x^{3/2} + 12 - 8x^{-3/2}$$

we have
$$\int_1^2 f(x)\,dx = \int_1^2 x^3\,dx - 6\int_1^2 x^{3/2}\,dx + 12\int_1^2 1\,dx$$

$$- 8\int_1^2 x^{-3/2}\,dx = \int_1^2 \frac{d}{dx}\left(\frac{x^4}{4}\right)dx - 6\int_1^2 \frac{d}{dx}\left[\frac{2}{5}x^{5/2}\right]dx$$

$$+ 12\int_1^2 \frac{d}{dx}\{x\}\,dx - 8\int_1^2 \frac{d}{dx}\left\{-2x^{-1/2}\right\}dx = \left.\frac{x^4}{4}\right|_1^2$$

$$- 6\cdot\left.\frac{2}{5}x^{5/2}\right|_1^2 + \left.12x\right|_1^2 - \left.8\left(-2x^{-1/2}\right)\right|_1^2 = \frac{2^4}{4} - \frac{1^4}{4}$$

$$-6\left\{\frac{2}{5}2^{5/2}-\frac{2}{5}1^{5/2}\right\}+12(2-1)-8\left\{-2\cdot 2^{-1/2}-\left(-2\cdot 1^{-1/2}\right)\right\}$$

$$= 4-\frac{1}{4}-\frac{12}{5}4\sqrt{2}+\frac{12}{5}+24-12+\frac{16}{\sqrt{2}}-16$$

$$= \frac{43}{20}-\frac{16}{5\sqrt{2}} \approx -0.1127$$

(b) $$I=\int_{-2}^{1}\left(15-26x+8x^2\right)dx=\int_{-2}^{1}\frac{d}{dx}\left\{15x-13x^2+\frac{8x^3}{3}\right\}dx$$

$$=\left(15x-13x^2+\frac{8x^3}{3}\right)\Big|_{-2}^{1}=\left(15-13+\frac{8}{3}\right)-\left(-30-52-\frac{64}{3}\right)$$

$$=15-13+30+52+\frac{8+64}{3}=108$$

(c) The integrand is $f(x)=\left|e^{2x}-3\right|=\begin{cases}e^{2x}-3 & \text{if } e^{2x}>3\\ 3-e^{2x} & \text{if } e^{2x}<3\end{cases}$

$$=\begin{cases}3-e^{2x} & \text{if } x<\frac{1}{2}\ln(3)\\ e^{2x}-3 & \text{if } x>\frac{1}{2}\ln(3)\end{cases}$$ (because $e^{2x}>3 \iff 2x>\ln(3)$)

So $$I=\int_0^1 f(x)\,dx=\int_0^{\ln(3)/2}\left(3-e^{2x}\right)dx+\int_{\ln(3)/2}^{1}\left(e^{2x}-3\right)dx$$

$$=\int_0^c\frac{d}{dx}\left\{3x-\frac{1}{2}e^{2x}\right\}dx+\int_c^1\frac{d}{dx}\left\{\frac{1}{2}e^{2x}-3x\right\}dx \quad \text{where } c=\frac{\ln(3)}{2}$$

$$=\left(3x-\frac{1}{2}e^{2x}\right)\Big|_0^c+\left(\frac{1}{2}e^{2x}-3x\right)\Big|_c^1$$

$$=3c-\frac{1}{2}e^{2c}-\left(3\cdot 0-\frac{1}{2}e^{2\cdot 0}\right)+\frac{1}{2}e^{2\cdot 1}-3-\left(\frac{1}{2}e^{2c}-3c\right)$$

$$=3c-\frac{1}{2}e^{2c}-0+\frac{1}{2}+\frac{1}{2}e^{2}-3-\frac{1}{2}e^{2c}+3c$$

$$=\frac{1}{2}e^2-e^{2c}+6c-\frac{5}{2}=\frac{1}{2}e^2-e^{\ln(3)}+3\ln(3)-\frac{5}{2}$$

$$=\frac{1}{2}e^2-3-\frac{5}{2}+3\ln(3)=\frac{1}{2}\left(e^2-11\right)+3\ln(3)$$

$$\approx 1.49$$

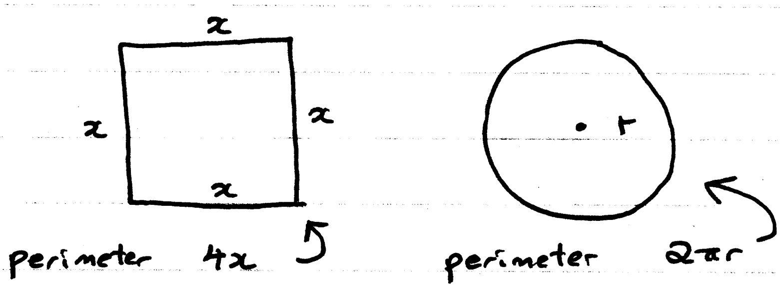
4. $F'(t) = \dfrac{d}{dt}\left\{-\displaystyle\int_{-4}^{t^3}\sqrt{1+x^4}\,dx\right\}$. Put $u = t^3$ and

$y = \displaystyle\int_{-4}^{u}\sqrt{1+x^4}\,dx$. Then $\dfrac{du}{dt} = 3t^2$ and $\dfrac{dy}{du} = \sqrt{1+u^4}$

So $F'(t) = \dfrac{d}{dt}(-y) = -\dfrac{dy}{dt} = -\dfrac{dy}{du}\dfrac{du}{dt} = -\sqrt{1+u^4}\cdot 3t^2$

$= -3t^2\sqrt{1+t^{12}}$, on using the chain rule.

5.



Let the square cookie cutter have side x, let the circular one have radius r. Then $4x + 2\pi r = L \Rightarrow r = \dfrac{L-4x}{2\pi}$

So the total area of cookie is $A = x^2 + \pi r^2$

$= x^2 + \pi\left(\dfrac{L-4x}{2\pi}\right)^2 = x^2 + \dfrac{1}{4\pi}(L-4x)^2$

(a) $\dfrac{dA}{dx} = 2x + \dfrac{1}{4\pi}\cdot 2(L-4x)(-4) = 2\left\{x\left(1+\dfrac{4}{\pi}\right) - \dfrac{L}{\pi}\right\}$

$\dfrac{d^2A}{dx^2} = 2\left\{1.\left(1+\dfrac{4}{\pi}\right) - 0\right\} = 2\left(1+\dfrac{4}{\pi}\right) > 0$

So the area has a unique global min where $\dfrac{dA}{dx} = 0$

or $x = \dfrac{L/\pi}{1+4/\pi} = \dfrac{L}{\pi+4}$

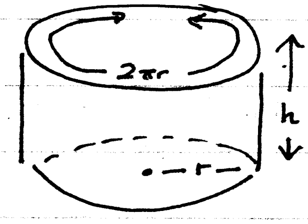
(b) The radius of the circular cookie is then $r = \dfrac{L-4x}{2\pi} = \dfrac{L}{2(\pi+4)}$

(c) $\square$ cookie has area $x^2 = \dfrac{L^2}{(\pi+4)^2}$, $\bigcirc$ cookie has area πr^2

$= \dfrac{\pi L^2}{4(\pi+4)^2}$. So square cookie is larger $(\pi > 4)$ and the ratio is $\dfrac{4}{\pi}$.

6. First note that since the only given parameter is an area S, lengths must be in terms of $\sqrt{S}$ and volume must be in terms of $S^{3/2}$. This will provide a check on your answer.

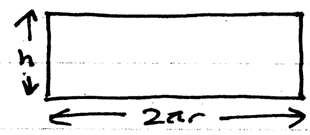
Let r be the radius, h the height. Then volume is $V = \pi r^2 h$ and surface area is



$$2\pi r \cdot h + \pi r^2 = S \Rightarrow$$

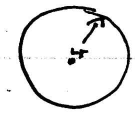
$$h = \frac{S - \pi r^2}{2\pi r} \Rightarrow$$

Side wall when unrolled:



$$V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) = \frac{1}{2} r S - \frac{1}{2}\pi r^3$$

base:



So $$\frac{dV}{dr} = \frac{1}{2} S - \frac{3}{2}\pi r^2 \Rightarrow \frac{d^2V}{dr^2} = 0 - 3\pi r$$

is always negative. Hence there is a unique global maximum where

$$\frac{dV}{dr} = 0 \quad \text{or} \quad S = 3\pi r^2 \Rightarrow r = \sqrt{\frac{S}{3\pi}} = \frac{\sqrt{S}}{\sqrt{3\pi}}$$

Then the height is $$\frac{S - \pi r^2}{2\pi r} = \frac{S - (S/3)}{2\pi\left(\sqrt{S}/\sqrt{3\pi}\right)} = \frac{\sqrt{S}}{\sqrt{3\pi}}.$$

and the volume is $$\pi r^2 h = \frac{S}{3} \cdot \frac{\sqrt{S}}{\sqrt{3\pi}} = \frac{S^{3/2}}{\sqrt{27\pi}}$$

Hence: (i) maximum volume is $\dfrac{S^{3/2}}{\sqrt{27\pi}}$

(ii) both radius and height are $\sqrt{\dfrac{S}{3\pi}}$.

For example, if the surface area is 27π cm^2 then the maximum volume is 27π cm^3 and is achieved by a radius and height of 3cm.

Note that our initial expectations were confirmed by our analysis.