1 (a) Increasing $x$ by $\delta x$ increases $y$ by $\delta y$ where $y + \delta y = (x+\delta x)^5$.

That is, $y + \delta y = x^5 + 5x^4\delta x + 10x^3\delta x^2 + \ldots + \delta x^5$

$= x^5 + 5x^4 \delta x + o(\delta x)$ [BY THE BINOMIAL THEOREM]

Subtracting $y = x^5$ yields $\delta y = 5x^4\delta x + o(\delta x)$. Hence the differential coefficient is $\frac{dy}{dx} = 5x^4$.

(b) Similarly, $y = \frac{8}{x} \Rightarrow$ both $xy = 8$ and $(x+\delta x)(y+\delta y)$

$= 8 \Rightarrow xy + \delta x y + x\delta y + \delta x \delta y = 8$. Subtracting, we have $\delta x y + x\delta y + \delta x \delta y = 0$. Dividing by $\delta x$, we have $y + x\frac{\delta y}{\delta x} + \delta y = 0$. Now let $\delta x \to 0$ ($\Rightarrow \delta y \to 0$) to obtain $y + x \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x} + \lim_{\delta x \to 0} \delta y = 0 \Rightarrow$

$y + x\frac{dy}{dx} + 0 = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} = -\frac{8}{x^2}$.

2 (a) By the chain rule, $\frac{d}{dx}[\sin(u)] = \frac{d}{du}[\sin(u)]\frac{du}{dx} = \cos(u)\frac{du}{dx}$

$\Rightarrow \frac{d}{dx}\left\{\sin\left(\frac{1}{4}\pi x\right)\right\} = \cos\left(\frac{1}{4}\pi x\right) \cdot \frac{d}{dx}\left(\frac{1}{4}\pi x\right) = \frac{1}{4}\pi\cos\left(\frac{1}{4}\pi x\right)$

Similarly, $\frac{d}{dx}\left\{\cos\left(\frac{1}{4}\pi x\right)\right\} = -\frac{1}{4}\pi\sin\left(\frac{1}{4}\pi x\right)$.

So $f'(x) = \frac{d}{dx}(2x^3)\sin\left(\frac{1}{4}\pi x\right) + 2x^3\frac{d}{dx}\left(\sin\left(\frac{1}{4}\pi x\right)\right) + \frac{d}{dx}\left(\cos\left(\frac{1}{4}\pi x\right)\right)$

$= 2 \cdot 3x^2 \sin\left(\frac{1}{4}\pi x\right) + 2x^3 \cdot \frac{\pi}{4}\cos\left(\frac{1}{4}\pi x\right) - \frac{1}{4}\pi\sin\left(\frac{1}{4}\pi x\right)$

So $f'(1) = 2 \cdot 3 \cdot 1^2 \cdot \sin(\pi/4) + 2 \cdot 1^3 \cdot \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) - \frac{1}{4}\pi\sin\left(\frac{\pi}{4}\right)$

$= 6 \cdot \frac{1}{\sqrt{2}} + \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} = \frac{24+\pi}{4\sqrt{2}}$.

(b) Let $u = \frac{1+x}{1-x-x^2} \Rightarrow \frac{du}{dx} = \frac{(0+1)(1-x-x^2) - (1+x)(0-1-2x)}{(1-x-x^2)^2}$

$= \frac{2+2x+x^2}{(1-x-x^2)^2}$ by the quotient rule. Now $y = u^5$

$\Rightarrow \frac{dy}{dx} = \frac{d}{du}[u^5]\frac{du}{dx} = 5u^4\frac{du}{dx}$ by the chain rule, where $y = f(x)$. But $x = 1 \Rightarrow u = \frac{1+1}{1-1-1^2} = -2$ and $\frac{du}{dx} = \frac{2+2+1^2}{(1-1-1^2)^2} = 5$. So

$$f'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.5u^4\frac{du}{dx}\right|_{x=1} = 5(-2)^4 \cdot 5 = 400$$

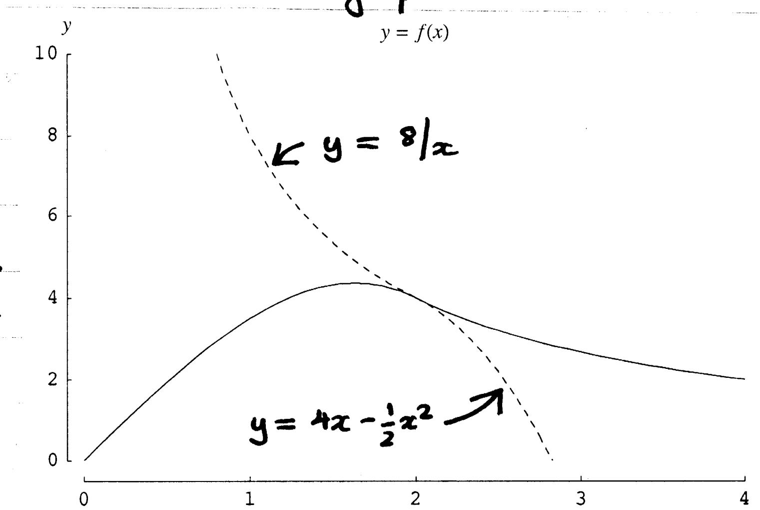
Note that $x > \frac{\sqrt{5}-1}{2} \Rightarrow 1-x-x^2 < 0$ (hence, $\neq 0$).

3. $$f'(x) = \begin{cases} a + 3bx^2 & \text{if } 0 < x < 2 \\ -\frac{8}{x^2} & \text{if } 2 < x < \infty \end{cases}$$

$f$ continuous $\Rightarrow f(2-) = f(2+) \Rightarrow a \cdot 2 + b \cdot 2^3 = \frac{8}{2}$

$f'$ " $\Rightarrow f'(2-) = f'(2+) \Rightarrow a + 3b \cdot 2^2 = \frac{-8}{2^2}$.

So we have $2a + 8b = 4 \Rightarrow a + 4b = 2$ and $a + 12b = -2$. Subtracting, we obtain $-8b = 4$ or $b = -\frac{1}{2}$. Hence $a = 2 - 4b = 4$. Here's the graph:




4. (a) Let $u = \sqrt{x+2} \Rightarrow$
$u^2 = x+2 \Rightarrow \frac{d}{dx}(u^2) = 1+0$
$\Rightarrow 2u\frac{du}{dx} = 1 \Rightarrow \frac{du}{dx} = \frac{1}{2u}$
$= \frac{1}{2\sqrt{x+2}}$, on using the chain rule.

Now $y = (x^2+4)u$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2+4)u + (x^2+4)\frac{du}{dx} = (2x+0)u + (x^2+4)\cdot\frac{1}{2u}$$

When $x = 2$ we have $u = \sqrt{2+2} = \sqrt{4} = 2$. Hence the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=2} = (2\cdot 2+0)2 + (2^2+4)\cdot\frac{1}{2\cdot 2}$$
$$= 10.$$

So the tangent line has equation $y - 16 = 10(x-2)$ or $y = 2(5x-2)$

(b) The line $10x - y = 4$ passes through $(1,6)$ if $10 \cdot 1 - 6 = 4$, which it does.