4. Limits of ordinary functions

We discovered in Lecture 3 that the limit of a convergent sequence is the number to which its "tail" eventually settles down. If L is the limit and $\{s_n\}$ the sequence, then we write

$$\lim_{n\to\infty} s_n = L \tag{1}$$

to indicate that s_n can be made as close as we please to L by making n sufficiently large. For example, we know from Lecture 3 that

$$\lim_{n\to\infty} \frac{1}{n} = 0, \tag{2}$$

which means that I can make $\frac{1}{n}$ as close to zero as you please by taking n sufficiently large. How close to zero do you want to be? If you say within a distance of 0.1, then I say n must be bigger than 10, for that makes $\frac{1}{n}$ less than 0.1; if you say within a distance of 10^{-6}, then I say n must exceed a million, for that makes $\frac{1}{n}$ less than 10^{-6}; if you persist, and say within a distance of 10^{-12}, then I say n must exceed a trillion, for that makes $\frac{1}{n}$ less than 10^{-12}; and so on. However small you decide you want to make the distance between $\frac{1}{n}$ and 0, I can satisfy your wish by making n sufficiently large. This is a game that you can't win. A few moments' thought now reveals, however, that nothing in the

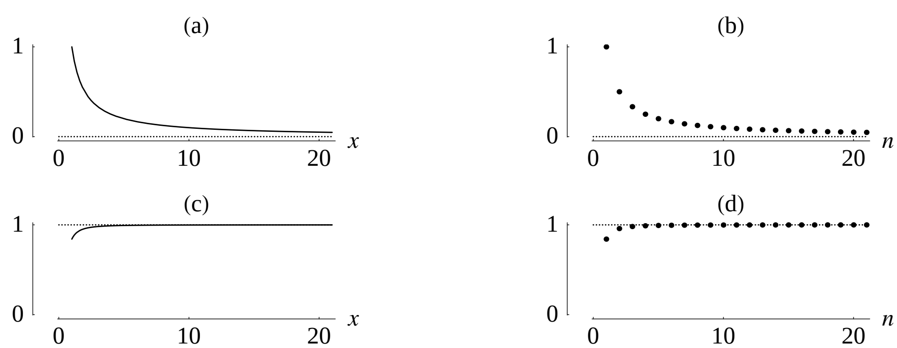


Figure 1: Convergence of a sequence as $n \to \infty$ versus that of an ordinary function as $x \to \infty$.

above argument requires n to be an integer: it could be any positive number. So Lecture 3's concept of limit generalizes at once from sequences to ordinary functions, with curves levelling off to a limit instead of dots settling down to a limit. For example, Figures 1a-b represent

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{versus} \quad \lim_{n\to\infty} \frac{1}{n} = 0, \tag{3}$$

whereas Figures 1c-d represent

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1 \quad \text{versus} \quad \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = 1. \tag{4}$$

But with ordinary functions we can generalize goes even further, because there are many more points that can be approached without actually being reached from within

θ (radians)	$\sin(\theta)$	$\frac{\sin(\theta)}{\theta}$
1	0.84147098	0.84147098
0.5	0.47942554	0.95885108
0.1	$0.99833417 \times 10^{-1}$	0.99833417
0.01	$0.99998333 \times 10^{-2}$	0.99998333
0.001	$0.99999983 \times 10^{-3}$	0.99999983
0.0001	10^{-4}	1.0000

Table 1: How $\frac{\sin(\theta)}{\theta} \to 1$ as $\theta \to 0^+$.

θ (radians)	$\sin(\theta)$	$\frac{\sin(\theta)}{\theta}$
-1	-0.84147098	0.84147098
-0.5	-0.47942554	0.95885108
-0.1	$-0.99833417 \times 10^{-1}$	0.99833417
-0.01	$-0.99998333 \times 10^{-2}$	0.99998333
-0.001	$-0.99999983 \times 10^{-3}$	0.99999983
-0.0001	-10^{-4}	1.0000

Table 2: How $\frac{\sin(\theta)}{\theta} \to 1$ as $\theta \to 0^-$.

Note that θ must be measured in radians. For t measured in degrees, $\frac{\sin(t)}{t} \to \frac{\pi}{180} \approx 0.17453293 \times 10^{-1}$ instead; see Figure 2.

a domain of real numbers than from within a domain of integers (for which the point at infinity is essentially the only option). To see what this means, let us make the substitution

$$\theta \;=\; \frac{1}{x} \quad \text{or} \quad x \;=\; \frac{1}{\theta} \tag{5}$$

in (4) to obtain

$$\lim_{x\to\infty} \frac{1}{\theta}\sin(\theta) \;=\; 1. \tag{6}$$

But $\theta \to 0$ as $x \to \infty$, by (3). So in place of (6) we can write

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} \;=\; 1, \tag{7a}$$

where the superscript $^+$ indicates that θ remains positive as it approaches zero or, as we prefer to say, that θ approaches zero from above, or *from the right* (because the θ-axis is horizontal). Now we have a new kind of limit: instead of a limit as infinity is approached from the left, we have a limit as zero is approached from the right. What (7a) means is that no matter how small you want the distance between $\frac{\sin(\theta)}{\theta}$ and 1 to be, I can satisfy your wish by making θ sufficiently close to zero (though still positive), as illustrated by Table 1. Again, this is a game that you can't win.

But it is not necessary for θ to approach zero from the right: it can also approach zero through negative values, or *from the left*. As you can quickly tell from inspection of Table 2, however, approaching from the other side has no effect on the limit—it is still 1. Using a superscript $^-$ to indicate that θ remains negative as it approaches zero, we write

$$\lim_{\theta\to 0^-} \frac{\sin(\theta)}{\theta} \;=\; 1. \tag{7b}$$

Because it does not matter whether we approach from the right or the left, i.e., because we have no need to distinguish between the *right-handed* limit (7a) and the *left-handed* limit (7b), we can refer to either one as simply the limit. Dropping the superscripts, we write

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \;=\; 1 \tag{8}$$

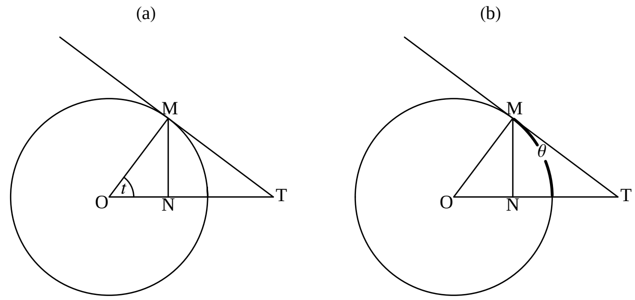


Figure 2: Definitions of $\sin$, $\cos$ and $\tan$. The circle has radius OM = 1. (a) Angle measured in degrees: ON = $\cos(t)$, NM = $\sin(t)$, MT = $\tan(t)$. (b) Angle measured in radians: ON = $\cos(\theta)$, NM = $\sin(\theta)$, MT = $\tan(\theta)$. As $\theta \to 0$, the length of the arc becomes indistinguishable from that of MN. Note that, because t degrees and θ radians represent the same angle, we must have both $\sin(t) = \sin(\theta)$ and $\frac{\theta}{2\pi} = \frac{t}{360}$ or $t = 180\theta/\pi$, so that $\frac{\sin(t)}{t} = \frac{\sin(\theta)}{180\theta/\pi} = \frac{\pi}{180}\frac{\sin(\theta)}{\theta}$. Hence

$$\lim_{t\to 0}\frac{\sin(t)}{t} \;=\; \frac{\pi}{180}\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} \;=\; \frac{\pi}{180}\cdot 1 = \frac{\pi}{180}.$$

to indicate that no matter how small you want the distance between $\frac{\sin(\theta)}{\theta}$ and 1 to be, I can satisfy your wish by making θ sufficiently close to zero, and the direction of approach doesn't matter in the least. But all of that is a bit long-winded. So mathematicians prefer a more concise expression of precisely the same information, namely, that $f(\theta) \to 1$ as $\theta \to 0$ where f is defined on $[-\pi, 0) \cup (0, \pi]$ by*

$$f(\theta) \;=\; \frac{\sin(\theta)}{\theta}. \tag{9a}$$

Note that $f(\theta)$ is undefined at $\theta = 0$ itself because $\frac{\sin(0)}{0} = \frac{0}{0}$ has no meaning as a quotient of numbers: it is a so-called *indeterminate form*. To understand why, consider the meaning of, say, $r = \frac{7}{3}$. It is the number r with the property that 7 results,if you multiply it by 3. So $3 \cdot r = 7$ is in essence the definition of $r = \frac{7}{3}$. If we try this approach with $\frac{0}{0}$, i.e., if we define $\frac{0}{0}$ be the number s with the property that $0 \cdot s = 0$, then s could be 1, because $0 \cdot 1 = 0$; but it could also be 2, because $0 \cdot 2 = 0$; and in fact, because $0 \cdot s$ is always zero, s could be any number whatsoever, which is why we call $\frac{0}{0}$ indeterminate. What this means for f is that the value 0 that θ approaches as $f(\theta)$ approaches the limit 1 lies outside the domain of f. Furthermore, the value 1 that $f(\theta)$ approaches lies outside the range of f. In other words, there is a hole in the graph of f at the point with coordinates $(0, 1)$, as indicated by Figure 3a. The graph is in more than one piece: it is *discontinuous*.

In view of (9a), however, it is natural to fill in the hole at $(0, 1)$ by extending the domain of f to $[-\pi, \pi]$ as follows:

$$f(\theta) \;=\; \begin{cases} \frac{\sin(\theta)}{\theta} & \text{if} \quad \theta \in [-\pi, 0) \cup (0, \pi] \\ 1 & \text{if} \quad \theta = 0. \end{cases} \tag{9b}$$

*In fact, (9a) defines f on $(-\infty, 0) \cup (0, \infty)$, but it is convenient to restrict f to subdomain $(-\pi, 0) \cup (0, \pi)$.

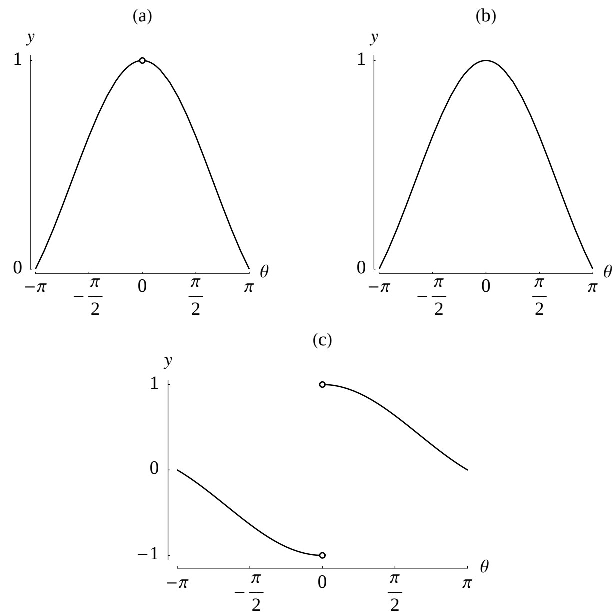


Figure 3: The graph of f as (a) originally defined by (9a) and (b) extended to $[-\pi, \pi]$ by (9b), and (c) the graph of h defined by (10).

The graph of f is now all in one piece, and therefore *continuous*; see Figure 3b. Thus the concept of a limit can make discontinuous functions continuous by suitably extending their domains of definition.

Nevertheless, although the limits as $\theta \to 0^+$ and as $\theta \to 0^-$ are one and the same for f defined by (9), in other cases left- and right-hand limits may be different. Consider, for example, the function h defined by

$$h(\theta) \;=\; \frac{|\sin(\theta)|}{\theta}, \quad \theta \in [-\pi, 0) \cup (0, \pi] \tag{10}$$

whose graph is plotted in Figure 3c. Either from this graph or by suitably modifying Tables 1 and 2 (Exercise 1), we readily discover that

$$\lim_{\theta \to 0^-} h(\theta) \;=\; -1, \quad \lim_{\theta \to 0^+} h(\theta) \;=\; 1. \tag{11}$$

Thus the left-hand limit exists, being -1; and the right-hand limit exists, being +1; but because these numbers are different, the limit itself does not exist.

These ideas are readily generalized as follows (using x in place of θ for the independent variable): Whenever I can make $f(x)$ as close as you please to L by making x a number bigger than a that is as close as necessary to a—but not actually a itself—we say that L is the right-handed limit of $f(x)$ as $x \to a$ and write

$$\lim_{x \to a^+} f(x) \;=\; L; \tag{12}$$

whenever I can make $f(x)$ as close as you please to L by making x a number smaller than a that is as close as necessary to a—but not actually a itself—we say that L is the left-handed limit of $f(x)$ as $x \to a$ and write

$$\lim_{x \to a^-} f(x) = L; \tag{13}$$

and whenever both are true we say that L is simply the limit of $f(x)$ as $x \to a$ and write

$$\lim_{x \to a} f(x) = L. \tag{14}$$

Furthermore, whenever $f(a)$ is actually defined (which, as we have seen, is not necessary—for example, $f(0)$ is undefined by (9a)) *and*

$$\lim_{x \to a^+} f(x) = f(a) \tag{15}$$

we say that f is *continuous from the right* at a; whenever $f(a)$ is defined and

$$\lim_{x \to a^-} f(x) = f(a) \tag{16}$$

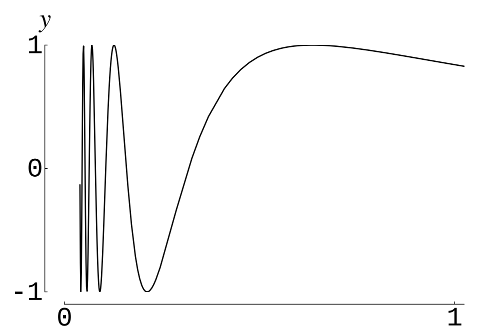
we say that f is *continuous from the left* at a; and whenever (15)-(16) are both true, i.e., whenever $f(a)$ is defined and

$$\lim_{x \to a} f(x) = f(a) \tag{17}$$

we say simply that f is *continuous* at a.

A general result about limits—which we have tacitly used already in Lecture 3—is that the limit of a multiple of a function is that same multiple of the limit of the function—*provided the answer exists*. The importance of this qualification cannot be overemphasized. Consider, for example, the problem of calculating

$$\lim_{t \to 0^+} 5 \sin(1/t). \tag{18}$$



If

$$\lim_{t \to 0^+} \sin(1/t) \tag{19}$$

exists, then (18) is just 5 times (19). But (19) does not exist, because the closer that t gets to zero, the more wildly $\sin(1/t)$ oscillates—it never settles down, as illustrated on the right. So our general result does not apply, and (18) does not exist.

An even more general and important result is that the limit of a sum (or difference) or product or quotient or composition of two functions is the sum (or difference) or product or quotient or composition of their limits—again, *provided the answer exists*. Because sums (or differences), products, quotients and compositions are all combinations, a more concise statement is that the limit of a combination of two functions is the combination of their limits, provided the answer exists. We will refer to this result as the *limit combination rule*.†

†Note that it subsumes our earlier general result for multiples, because a multiple of a function can be regarded as the product of that function with a constant function.

For example, consider p defined on $[-\pi, \pi]$ by

$$p(\theta) \;=\; \frac{2\theta + 3\sin(\theta)}{5\theta + 4\sin(\theta)}. \tag{20}$$

Here

$$\lim_{\theta\to 0} \theta \;=\; 0 \quad \text{and} \quad \lim_{\theta\to 0} \sin(\theta) \;=\; 0 \tag{21}$$

both clearly exist; but because $\frac{0}{0}$ is indeterminate, we *cannot* argue that

$$\lim_{\theta\to 0} p(\theta) \;=\; \lim_{\theta\to 0} \frac{2\theta + 3\sin(\theta)}{5\theta + 4\sin(\theta)} \;=\; \frac{2\lim\limits_{\theta\to 0}\theta + 3\lim\limits_{\theta\to 0}\sin(\theta)}{5\lim\limits_{\theta\to 0}\theta + 4\lim\limits_{\theta\to 0}\sin(\theta)} \;=\; \frac{2\cdot 0 + 3\cdot 0}{5\cdot 0 + 4\cdot 0}. \tag{22}$$

What we can do, however, is divide both numerator and denominator by θ to obtain

$$p(\theta) \;=\; \frac{2\theta + 3\sin(\theta)}{5\theta + 4\sin(\theta)} \;=\; \frac{2 + 3\cdot\frac{\sin(\theta)}{\theta}}{5 + 4\cdot\frac{\sin(\theta)}{\theta}} \;=\; q(\theta), \quad \text{say} \tag{23}$$

and then deduce that because $p(\theta) = q(\theta)$ for all $\theta \neq 0$,

$$\lim_{\theta\to 0} p(\theta) \;=\; \lim_{\theta\to 0} q(\theta) \;=\; \lim_{\theta\to 0} \frac{2 + 3\cdot\frac{\sin(\theta)}{\theta}}{5 + 4\cdot\frac{\sin(\theta)}{\theta}} \;=\; \frac{\lim\limits_{\theta\to 0} 2 + 3\lim\limits_{\theta\to 0}\frac{\sin(\theta)}{\theta}}{\lim\limits_{\theta\to 0} 5 + 4\lim\limits_{\theta\to 0}\frac{\sin(\theta)}{\theta}} \;=\; \frac{2 + 3\cdot 1}{4 + 5\cdot 1} \;=\; \tfrac{5}{9}. \tag{24}$$

The method we have just used is quite general. That is, whenever $p(a)$ is indeterminate, we can often use algebraic manipulations to show that $p(x) = q(x)$ for all $x \neq a$ for some $q(x)$ such that $q(a)$ is not indeterminate, and hence in effect use $q(a)$ to calculate the limit of $p(x)$ as $x \to a$. Often, these algebraic manipulations require a clever multiplication by 1—which, of course, cannot change the value of $p(x)$—and use of the identity

$$A^2 - B^2 = (A - B)(A + B) \tag{25}$$

with a judicious choice of A and B. Moreover, it is usually not necessary to define p and q explicitly when the method is applied.

For example, by this method we obtain

$$\begin{aligned}
\lim_{x\to 0} \tfrac{3-\sqrt{(1-x)(9+x)}}{x} \;&=\; \lim_{x\to 0} \tfrac{3-\sqrt{(1-x)(9+x)}}{x}\cdot 1 \;=\; \lim_{x\to 0} \tfrac{3-\sqrt{(1-x)(9+x)}}{x}\cdot\tfrac{3+\sqrt{(1-x)(9+x)}}{3+\sqrt{(1-x)(9+x)}} \\
&=\; \lim_{x\to 0} \tfrac{3^2-(1-x)(9+x)}{x\{3+\sqrt{(1-x)(9+x)}\}} \;=\; \lim_{x\to 0} \tfrac{8x+x^2}{x\{3+\sqrt{(1-x)(9+x)}\}} \\
&=\; \lim_{h\to 0} \tfrac{8+x}{3+\sqrt{(1-x)(9+x)}} \;=\; \tfrac{8+0}{3+\sqrt{(1-0)(9+0)}} \;=\; \tfrac{4}{3}.
\end{aligned}$$

Here, in effect, $a = 0$, $p(x) = \{3-\sqrt{(1-x)(9+x)}\}/x$ and $p(a) = \{3-\sqrt{(1-0)(9+0)}\}/0 = \{3-\sqrt{9}\}/0 = \frac{3-3}{0} = \frac{0}{0}$ is indeterminate; there is a clever multiplication by 1 in the second line; (25) is judiciously used with $A = 3$ and $B = \sqrt{(1-x)(9+x)}$ to obtain the third line; and the limit is calculated as $q(0) = \frac{4}{3}$, where $q(x) = (8+x)/\{3 + \sqrt{(1-x)(9+x)}\}$. Yet nowhere was it necessary to define p or q explicitly.

The method also applies to indeterminate forms of type $\frac{\infty}{\infty}$. For example, we cannot apply the limit combination rule directly to $\frac{x+1}{x-1}$ in the form

$$\lim_{x\to\infty}\frac{x+1}{x-1} \;=\; \frac{\lim\limits_{x\to\infty} x+1}{\lim\limits_{x\to\infty} x+1} \;=\; \frac{\infty}{\infty}, \tag{26}$$

because $\frac{\infty}{\infty}$ has no meaning as a quotient of numbers. Nevertheless, because $\frac{x+1}{x-1} = \frac{1+1/x}{1-1/x}$ for all finite x and $1/x \to 0$ as $x \to \infty$, we obtain

$$\lim_{x\to\infty}\frac{x+1}{x-1} \;=\; \lim_{x\to\infty}\frac{1+\frac{1}{x}}{1-\frac{1}{x}} \;=\; \frac{\lim\limits_{x\to\infty} 1+\lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty} 1-\lim\limits_{x\to\infty}\frac{1}{x}} \;=\; \frac{1+0}{1-0}=1. \tag{27}$$

On the other hand, although (27) is correct, this is not in practice how one proceeds; rather, one notes that if x is extremely large, then both $x+1$ and $x-1$ are indistinguishable from x, so that $\frac{x+1}{x-1}$ is indistinguishable from $\frac{x}{x}=1$, which must therefore be the limit that $\frac{x+1}{x-1}$ approaches as $x\to\infty$. Similarly, because $3x^2-2x+4$ is dominated by $3x^2$ (the largest term) for extremely large x (what's the difference in practice between \$2999998000004 and three trillion dollars?—I could happily retire on either sum) and because $2x^2+7x-3$ is likewise dominated by $2x^2$, the limit of $\frac{3x^2-2x+4}{2x^2+7x-3}$ as $x\to\infty$ is the same as that of $\frac{3x^2}{2x^2}$, which is $\frac{3}{2}$. But although it is easy enough to breeze through such "dominance" arguments in one's head, it would be just as tedious to write them down as it is to write down (27); and so, in practice, the most efficient way to deal with such examples is often merely to write, e.g., "Clearly, $\lim\limits_{x\to\infty}\frac{x+1}{x-1}=1$"—and that's perfectly acceptable, as long as you're right!

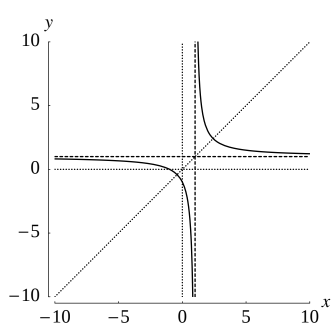


Figure 4: $y=\frac{x+1}{x-1}$ has horizontal asymptote $y=1$ and vertical asymptote $x=1$. Both asymptotes are shown dashed. The dotted lines are the coordinate axes and $y=x$. If the scales are the same on each axis, which they are in this figure, the effect of interchanging axes while holding the origin fixed is to reflect the graph in the line $y=x$. We therefore see that the function whose graph is sketched above is its own inverse: it is *self-inverse*. See Exercise 2.

We conclude by introducing some terminology to help interpret limits geometrically. The geometric significance of (27) is that, as x becomes increasingly large, the curve $y=\frac{x+1}{x-1}$ approaches the horizontal line $y=1$ (from above, because $x>1\Longrightarrow y>1$); and as x becomes increasingly negative, i.e., as $-x\to\infty$ or, as we prefer to write, $x\to-\infty$, the curve $y=\frac{x+1}{x-1}$ again approaches $y=1$ (this time from below, because $x<1\Longrightarrow y<1$). We say that the line is a *horizontal asymptote* to the curve; see Figure 4, where the asymptote is shown dashed. More generally, whenever

$$\lim_{x\to\pm\infty} f(x) \;=\; L \tag{28}$$

for any finite L, where "$\lim_{x\to\pm\infty} f(x) = L$" is a shorthand for "*either* $\lim_{x\to\infty} f(x) = L$ *or* $\lim_{x\to-\infty} f(x) = L$," we say that $y = L$ is a horizontal asymptote to the graph $y = f(x)$.[‡]

You can also see from Figure 4 that f defined by $f(x) = \frac{x+1}{x-1}$ is a decreasing function on $(-\infty, 1) \cup (1, \infty)$, and is therefore invertible. Moreover, from Lecture 1, you know that the graph of f^{-1} is obtained by holding the origin fixed while interchanging the axes; this maneuver converts any horizontal asymptote into a vertical line, but cannot affect the relationship between the asymptote and its curve: therefore, what was once a horizontal asymptote must now be a vertical one instead. If all functions were invertible, then this observation would suffice to frame a formal definition of vertical asymptote; but every function is not invertible, and so first we must formalize infinite limits, as follows.

Whenever I can make $f(x)$ as large as you please by making x a number bigger than a that is as close as necessary to a—but not actually a itself—we write

$$\lim_{x\to a^+} f(x) \;=\; \infty; \tag{29}$$

whenever I can make $f(x)$ as large as you please by making x a number smaller than a that is as close as necessary to a—but not actually a itself—we write

$$\lim_{x\to a^-} f(x) \;=\; \infty; \tag{30}$$

and whenever both are true we write

$$\lim_{x\to a} f(x) \;=\; \infty. \tag{31}$$

Furthermore, we write $f(x) \to -\infty$ whenever $-f(x) \to \infty$, and we use "$f(x) \to \pm\infty$ as $x \to a$" a shorthand for "*either* $f(x) \to \infty$ *or* $f(x) \to -\infty$ not only as $x \to a^+$, but also as $x \to a^-$." Now $x = a$ is a *vertical asymptote* to the graph $y = f(x)$ whenever

$$\lim_{x\to a} f(x) \;=\; \pm\infty. \tag{32}$$

Exercises

1. Verify (11).

2. Show that $f = f^{-1}$ for f defined by $f(x) = \frac{x+1}{x-1}$ (the function whose graph is sketched in Figure 4), i.e., that f is self-inverse.

Suitable problems from standard calculus texts

Stewart (2003): pp. 112-113, ## 10-30, 39-44, 58-60; p. 147, ## 13-30, 37-40; p. 217, ## 35-43.

[‡]So, in particular, $y = 1$ is a horizontal asymptote to the graph of the function K in Lecture 3.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

1. The appropriate modification of Tables 1 and 2 is as follows:

θ (radians)	$\lvert\sin(\theta)\rvert$	$h(\theta)$	θ (radians)	$\lvert\sin(\theta)\rvert$	$h(\theta)$
1	0.84147098	0.84147098	-1	0.84147098	-0.84147098
0.5	0.47942554	0.95885108	-0.5	0.47942554	-0.95885108
0.1	0.99833417×10^{-1}	0.99833417	-0.1	0.99833417×10^{-1}	-0.99833417
0.01	0.99998333×10^{-2}	0.99998333	-0.01	0.99998333×10^{-2}	-0.99998333
0.001	0.99999983×10^{-3}	0.99999983	-0.001	0.99999983×10^{-3}	-0.99999983
0.0001	10^{-4}	1.0000	-0.0001	10^{-4}	-1.0000

2. Use standard algebraic manipulation to show that $y = \frac{x+1}{x-1}$ implies $x = \frac{y+1}{y-1}$.