6. The derivative

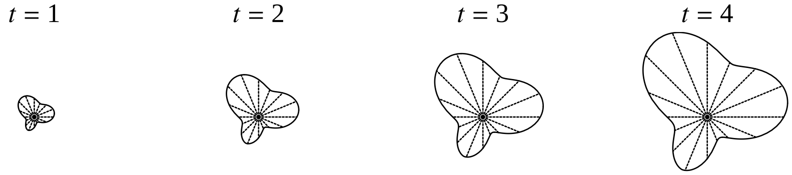

Figure 1: A growing patch of weeds.

An important idea in calculus is that of an instantaneous rate of change, which is exactly what it claims to be, namely, the value at a given instant of some quantity's rate of change. Consider, for example, our patch of weeds from Lecture 2, whose growth is reproduced above. The area of this patch of weeds at age t months is

$$y \;=\; A(t) \;=\; \alpha t^2 \tag{1}$$

square meters, where

$$\alpha \;=\; \tfrac{4\pi}{81} \;\approx\; 0.155. \tag{2}$$

From Figure 1, the area is clearly increasing with time—but how rapidly?

To answer this question, we use infinitesimals. Between time t and the slightly later time $t + \delta t$, the area of the weed patch increases from $y = A(t)$ to $y + \delta y = A(t + \delta t)$, and so the average rate of increase during this short interval is

$$\frac{\delta y}{\delta t} \;=\; \frac{(y + \delta y) - y}{\delta t} \;=\; \frac{A(t + \delta t) - A(t)}{\delta t} \;=\; \frac{\alpha(t + \delta t)^2 - \alpha t^2}{\delta t} \;=\; 2\alpha t + \delta t \tag{3}$$

after simplification. As $\delta t \to 0$, the interval between time t and time $t + \delta t$ collapses onto the instant t itself, and so the average rate of increase between time t and time $t + \delta t$ collapses onto the rate of increase at the instant t itself. Thus the instantaneous rate of increase at time t is just the differential coefficient

$$\frac{dy}{dt} \;=\; \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \;=\; \lim_{\delta t \to 0} 2\alpha t + \delta t \;=\; 2\alpha t + 0 \;=\; 2\alpha t. \tag{4}$$

For example, on using (2), at age 2 months the weed patch is increasing at the rate of $2\alpha \cdot 2 = 4\alpha = \frac{16\pi}{81} \approx 0.62$ square meters per month.

Observe from (4) that the instantaneous rate of change itself is changing with time. It thus defines a brand new function, which needs a brand new name. Because the first function is called A, we might be tempted to call the new function B; but this notation would not be terribly evocative of the new function's special relationship to A. So instead we call the new rate-of-change function A'. That is, on using (4):

$$A'(t) \;=\; 2\alpha t. \tag{5}$$

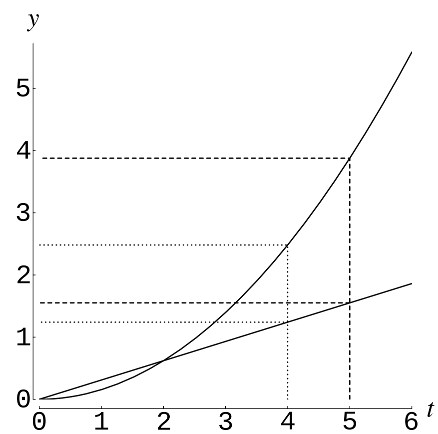

Figure 2: The graphs of A and A' defined by (1)-(2) and (5). These graphs supply both the area and its differential coefficient at any instant. For example, the dotted lines show that $A(4) \approx 2.48$ with $A'(4) \approx 1.24$, while the dashed lines show that $A(5) \approx 3.88$ with $A'(5) \approx 1.55$.

The graphs of both functions are sketched in Figure 2.

Both this result and the new notation generalize in the following way to any function f whose graph is both continuous (i.e., all in one piece) and *smooth*—i.e., without corners: the relationship $y = f(t)$ between the independent variable t and the dependent variable y implies a corresponding relationship between the infinitesimals δt and δy belonging to t and y, respectively, and the differential coefficient for this relationship is the instantaneous rate of change of $f(t)$ at time t, denoted by $f'(t)$. That is,

$$y \;=\; f(t) \Longrightarrow \delta y \;=\; f'(t)\delta t + o(\delta t), \tag{6}$$

and the instantaneous rate of change of $f(t)$ is

$$f'(t) \;=\; \frac{dy}{dt} \;=\; \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \;=\; \lim_{\delta t \to 0} \frac{y + \delta y - y}{\delta t} \;=\; \lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t}. \tag{7}$$

Moreover, the brand new function thus defined, namely, f', is known as the *derivative* of f; and because of the close relationship between derivative and differential coefficient, the process of finding a function's derivative is usually called *differentiating* the function (or simply *differentiation*). Note an important but subtle distinction: the derivative f' is a function, whereas the differential coefficient $f'(t)$ is just a number—the label assigned to t by the derivative. Henceforward, we will drop the adjective instantaneous and assume that "rate of change" implies it (in the absence of any indication to the contrary).

Here several remarks are in order. First, some rates of change are so frequently encountered that they have special names. For example, the rate of change of distance (which is nonnegative, and never decreases) is called *speed* (which is likewise nonnegative); the rate of change of displacement (which may be positive or negative, and may increase or decrease) is called *velocity* (which likewise may be positive or negative); and the rate of change of volume (which is nonnegative, but may decrease) is called *inflow*. So, in particular, the functions V and v defined by the graphs in Figure 3 of Lecture 1 are related according to $v(t) = V'(t)$.

Second, in principle, we can use infinitesimals to calculate the derivative of any function.* Most of the functions we meet in practice, however, are combinations of simple functions, such as linear or power functions; and as we saw in Lecture 2, such combinations can be sums, products, quotients, joins or compositions. It is therefore more efficient in the long run to have general results for the derivatives of combinations and to use them in conjunction with special results for the derivatives of simple functions. And because we believe in long-term efficiency, that is exactly how we shall proceed.

We begin by finding a special result for the derivative of a linear function. Accordingly, suppose that

$$y \;=\; f(t) \;=\; \alpha t + \beta, \tag{8}$$

where α and β are constants (and therefore, in particular, do not depend on t). Then

$$y + \delta y \;=\; f(t + \delta t) \;=\; \alpha(t + \delta t) + \beta \;=\; \alpha t + \alpha\delta t + \beta. \tag{9}$$

Subtracting (8) from (9), we obtain $\delta y = \alpha\delta t$, implying

$$\lim_{\delta t \to 0} \frac{\delta y}{\delta t} \;=\; \lim_{\delta t \to 0} \frac{\alpha\delta t}{\delta t} \;=\; \lim_{\delta t \to 0} \alpha \;=\; \alpha, \tag{10}$$

and so $f'(t) = \alpha$, on using (7). Thus our special result is that

$$y \;=\; f(t) \;=\; \alpha t + \beta \quad\Longrightarrow\quad \frac{dy}{dt} \;=\; f'(t) \;=\; \alpha. \tag{11}$$

It is often convenient to record this information without explicitly invoking the symbols f and y, however, in which case we write

$$\frac{d}{dt}\{\alpha t + \beta\} \;=\; \alpha. \tag{12}$$

Note in particular (from setting $\alpha = 0$) that the derivative of a constant is zero:

$$\frac{d\beta}{dt} \;=\; 0. \tag{13}$$

Well honestly, what did you expect? If something is constant, it cannot change!

Next we obtain a general result for the derivative of a constant multiple. Accordingly, suppose that $u = f(t)$ and $y = \alpha u$, where f is an arbitrary function and α is an arbitrary constant. Then changing t to $t + \delta t$ changes u to $u + \delta u$ and y to $y + \delta y$ in such a way that $y + \delta y = \alpha(u + \delta u)$. So $\delta y = \alpha(u + \delta u) - y = \alpha(u + \delta u) - \alpha u = \alpha\delta u$, implying

$$\frac{dy}{dt} \;=\; \lim_{\delta t \to 0} \frac{\delta y}{\delta t} \;=\; \lim_{\delta t \to 0} \frac{\alpha\delta u}{\delta t} \;=\; \alpha \lim_{\delta t \to 0} \frac{\delta u}{\delta t} \;=\; \alpha\frac{du}{dt} \tag{14}$$

on using the limit combination rule. In other words,

$$\frac{d}{dt}\{\alpha f(t)\} \;=\; \alpha f'(t). \tag{15}$$

*Whose derivative exists, i.e., a smooth one. We will return to the question of existence later, but for now we will finesse the issue by selecting only functions whose derivatives exist.

Next we obtain a general result for the derivative of a sum. Suppose that $u = f(t)$, $v = g(t)$ and $y = u + v$, where where f and g are arbitrary functions. Then changing t to $t+\delta t$ changes u to $u+\delta u$, v to $v+\delta v$ and y to $y+\delta y$ in such a way that $y+\delta y = u+\delta u+v+\delta v$. Hence $\delta y = \delta u + \delta v + u + v - y = \delta u + \delta v$, implying

$$\frac{dy}{dt} = \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \lim_{\delta t \to 0} \frac{\delta u + \delta v}{\delta t} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} + \frac{\delta v}{\delta t} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} + \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{du}{dt} + \frac{dv}{dt} \quad (16)$$

on using the limit combination rule. In other words,

$$\frac{d}{dt}\{f(t) + g(t)\} = f'(t) + g'(t). \quad (17)$$

The result is readily extended to a sum of three functions, say, p, q and r. For setting $f = p + q$ and $g = r$ in (17) yields

$$\frac{d}{dt}\{p(t) + q(t) + r(t)\} = \frac{d}{dt}\{p(t) + q(t)\} + \frac{d}{dt}\{r(t)\} = p'(t) + q'(t) + r'(t) \quad (18)$$

on using (17) again, this time with $f = p$ and $g = q$.

Next we find a special result for the derivative of a power function with integer exponent and coefficient 1. Suppose that

$$y = f(t) = t^n. \quad (19)$$

Then

$$y + \delta y = f(t + \delta t) = (t + \delta t)^n. \quad (20)$$

Now, from the binomial theorem, we have

$$(1 + x)^n = 1 + nx + \tfrac{n(n-1)}{1.2}x^2 + \tfrac{n(n-1)(n-2)}{1.2.3}x^3 + \tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}x^4 + \ldots \quad (21)$$

where the dots mean a lot more terms, in fact $n - 4$ of them; however, by setting

$$x = \frac{\delta t}{t} \quad (22)$$

we can reduce all the useful information that (21) contains[†] from

$$\left(1 + \tfrac{\delta t}{t}\right)^n = 1 + n\,\tfrac{\delta t}{t} + \tfrac{n(n-1)}{1.2}\left(\tfrac{\delta t}{t}\right)^2 + \tfrac{n(n-1)(n-2)}{1.2.3}\left(\tfrac{\delta t}{t}\right)^3 + \tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\tfrac{\delta t}{t}\right)^4 + \ldots \quad (23)$$

to

$$\left(1 + \frac{\delta t}{t}\right)^n = 1 + n\frac{\delta t}{t} + o(\delta t). \quad (24)$$

All but the first two terms are junk. Now, subtracting (19) from (20) and using (24), we obtain

$$\begin{aligned} \delta y &= (t + \delta t)^n - t^n = t^n \left(1 + \frac{\delta t}{t}\right)^n - t^n = t^n \left\{\left(1 + \frac{\delta t}{t}\right)^n - 1\right\} \\ &= t^n \left\{1 + n\frac{\delta t}{t} + o(\delta t) - 1\right\} = t^n \left\{n\frac{\delta t}{t} + o(\delta t)\right\} = t^n \cdot n\frac{\delta t}{t} + t^n o(\delta t) \\ &= nt^{n-1}\,\delta t + o(\delta t) \end{aligned} \quad (25)$$

[†]At least for our purposes.

because anything[‡] times $o(\delta t)$ is still $o(\delta t)$. We immediately deduce that the differential coefficient is nt^{n-1}. In other words, our special result is that

$$\frac{d}{dt}(t^n) \;=\; nt^{n-1}. \tag{26}$$

A differential coefficient defines a derivative even when time is not the independent variable, and this derivative is still a rate-of-change function: the only difference is that the change is with respect to a variable other than time. What this means in practice is that we already know other special results for derivatives from Lecture 5 (including the exercises at the end). For example, we already know from Lecture 5 that

$$\frac{d}{dx}\left\{\frac{1}{x}\right\} \;=\; -\frac{1}{x^2} \tag{27}$$

and

$$\frac{d}{dx}\{\sin(x)\} \;=\; \cos(x). \tag{28}$$

Furthermore, any of our special results can be used in conjunction with any of our general results to extend the list of functions whose derivatives we regard as known. For example,

$$\begin{aligned} \frac{d}{dx}\left\{7x^6 + \frac{5}{x} + 10\sin(x)\right\} &= \frac{d}{dx}\left\{7x^6\right\} + \frac{d}{dx}\left\{\frac{5}{x}\right\} + \frac{d}{dx}\left\{10\sin(x)\right\} \\ &= 7\,\frac{d}{dx}\left\{x^6\right\} + 5\,\frac{d}{dx}\left\{\frac{1}{x}\right\} + 10\,\frac{d}{dx}\left\{\sin(x)\right\} \\ &= 7\cdot 6x^5 + 5\left(-\frac{1}{x^2}\right) + 10\cdot\cos(x) \\ &= 42x^5 - \frac{5}{x^2} + 10\cos(x) \end{aligned} \tag{29}$$

from (18), followed by three applications of (15), followed by use of both (26) with $n = 6$ (and $t = x$) and (27)-(28). With a little practice, the intermediate steps are readily done in one's head.

It cannot be too strongly emphasized that the limit in (7) exists only because δt and δy *both* approach zero: you cannot have $\delta t \to 0$ without simultaneously having $\delta y \to 0$ as well (or there would be no finite limit).[§] We can exploit this observation to find a special result for the derivative of a square root. Suppose that

$$y \;=\; f(t) \;=\; \sqrt{t} \tag{30}$$

so that

$$y^2 \;=\; t \tag{31}$$

[‡]That is, anything independent of δt

[§]So in one sense, there are two kinds of junk—$o(\delta t)$ junk and $o(\delta y)$ junk—and in a more important sense, there is only one kind of junk, because $o(\delta y)$ and $o(\delta t)$ *both* approach zero as $\delta t \to 0$ in such a way that $o(\delta y) = o(\delta t)$, because $\lim\limits_{\delta t\to 0}\frac{o(\delta y)}{\delta t} = \lim\limits_{\delta t\to 0}\frac{o(\delta y)}{\delta y}\cdot\frac{\delta y}{\delta t} = \lim\limits_{\delta t\to 0}\frac{o(\delta y)}{\delta y}\lim\limits_{\delta t\to 0}\frac{\delta y}{\delta t} = \lim\limits_{\delta y\to 0}\frac{o(\delta y)}{\delta y}\lim\limits_{\delta t\to 0}\frac{\delta y}{\delta t} = 0\cdot\frac{dy}{dt} = 0.$

by definition. Then also

$$(y + \delta y)^2 \;=\; t + \delta t \tag{32}$$

so that

$$y^2 + 2y\,\delta y + \delta y^2 \;=\; t + \delta t. \tag{33}$$

Subtracting (31) from(33), we obtain

$$2y\,\delta y + \delta y^2 \;=\; \delta t \tag{34}$$

and hence, dividing by δt,

$$2y\frac{\delta y}{\delta t} + \delta y\frac{\delta y}{\delta t} \;=\; 1. \tag{35}$$

Taking the limit as $\delta t \to 0$, we obtain

$$\lim_{\delta t \to 0}\left\{2y\frac{\delta y}{\delta t} + \delta y\frac{\delta y}{\delta t}\right\} \;=\; \lim_{\delta t \to 0} 1 \;=\; 1, \tag{36}$$

because 1 doesn't change as $\delta t \to 0$. Furthermore, $2y$ doesn't change as $\delta t \to 0$ either: it just stays $2y$. Hence, applying the limit combination rule to (36), we obtain

$$2y \lim_{\delta t \to 0}\frac{\delta y}{\delta t} + \lim_{\delta t \to 0}\delta y \lim_{\delta t \to 0}\frac{\delta y}{\delta t} \;=\; 1. \tag{37}$$

But $\delta y \to 0$ as $\delta t \to 0$, i.e., $\lim_{\delta t \to 0}\delta y = 0$. So (37) implies $2y\frac{dy}{dt} + 0 \cdot \frac{dy}{dt} = 1$ or

$$2y\frac{dy}{dt} \;=\; 1, \tag{38}$$

from which

$$\frac{dy}{dt} \;=\; \frac{1}{2y} \;=\; \frac{1}{2\sqrt{t}} \tag{39}$$

by (30). In other words, our special result is that

$$\frac{d}{dt}\left\{\sqrt{t}\right\} \;=\; \frac{1}{2\sqrt{t}} \;=\; \tfrac{1}{2}\,t^{-1/2}. \tag{40}$$

We conclude by discussing the subtle question of what the domain of f' should be, given that the domain of f itself is $[a, b]$. The question arises because

$$f'(t) \;=\; \lim_{\delta t \to 0}\frac{f(t + \delta t) - f(t)}{\delta t} \tag{41}$$

makes sense in (7) only if $f(t + \delta t)$ is well defined, and hence only if $t + \delta t$ belongs to the domain of f. No difficulty arises at interior points of the domain, i.e., for t satisfying $a < t < b$, because for such values of t we always also have $a < t + \delta t < b$ in the limit as $\delta t \to 0$. But for $t = b$ we have $t + \delta t \in [a, b]$ only if $\delta t < 0$, and for $t = a$ we have $t + \delta t \in [a, b]$ only if $\delta t > 0$. There are two ways to deal with this matter. The first solution is to say that the domain of f' is the open interval (a, b), as opposed to the closed interval $[a, b]$; then

f and f' have different domains, although they differ by only two points. The second solution is to say that the domain of f' is still $[a, b]$, but only a left-handed derivative exists at $t = b$ and only a right-handed derivative exists at $t = a$; then, strictly speaking, the definition of f' is the following:

$$f'(t) = \begin{cases} \lim\limits_{\delta t \to 0^+} \frac{f(t+\delta t)-f(t)}{\delta t} & \text{if} \quad t = a \\ \lim\limits_{\delta t \to 0} \frac{f(t+\delta t)-f(t)}{\delta t} & \text{if} \quad a < t < b \\ \lim\limits_{\delta t \to 0^-} \frac{f(t+\delta t)-f(t)}{\delta t} & \text{if} \quad t = b. \end{cases} \tag{42}$$

We will have occasion to return to this point in Lecture 7.

Exercises

In each case, find the derivative.

1. $y = f(x) = 3x + 4\cos(x) + 5\sin(x).$

2. $y = f(x) = \sqrt{x(x+1)}.$

3. $y = f(x) = \sqrt{\dfrac{x}{x+1}}.$

Suitable problems from standard calculus texts

Stewart (2003): p. 174, ## 21-31; p. 191, ## 3-10; p. 216, ## 1, 3, 4, 6 and 12.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

1. By analogy with (29) and using the special result obtained in Exercise 1 of Lecture 5,

$$\frac{dy}{dx} = f'(x) = 3\tfrac{d}{dx}\{x\} + 4\tfrac{d}{dx}\{\cos(x)\} + 5\tfrac{d}{dx}\{\sin(x)\}$$
$$= 3 \cdot 1 + 4 \cdot (-\sin(x) + 5 \cdot \cos(x) = 3 - 4\sin(x) + 5\cos(x).$$

2. Squaring, we have both

$$y^2 = x(x+1) = x^2 + x$$

and

$$(y + \delta y)^2 = (x + \delta x)^2 + (x + \delta x)$$

or, expanding both sides,

$$y^2 + 2y\delta y + \delta y^2 \;=\; x^2 + 2x\delta x + \delta x^2 + x + \delta x \;=\; x^2 + x + (2x+1)\delta x + \delta x^2.$$

Subtracting, we obtain

$$2y\delta y + \delta y^2 \;=\; (2x+1)\delta x + \delta x^2.$$

Dividing by δx, we obtain

$$2y\frac{\delta y}{\delta x} + \delta y\frac{\delta y}{\delta x} \;=\; 2x + 1 \;+\; \delta x.$$

Now, taking the limit as $\delta x \to 0$ and using the limit combination rule, we obtain

$$2y \lim_{\delta x\to 0}\frac{\delta y}{\delta x} + \lim_{\delta x\to 0}\delta y \cdot \lim_{\delta x\to 0}\frac{\delta y}{\delta x} \;=\; \lim_{\delta x\to 0}(2x+1) \;+\; \lim_{\delta x\to 0}\delta x$$

or

$$2y\frac{dy}{dx} + 0\cdot\frac{dy}{dx} \;=\; 2x + 1 + 0.$$

Hence

$$\frac{dy}{dx} \;=\; \frac{2x+1}{2y} \;=\; \frac{2x+1}{2\sqrt{x(x+1)}}.$$

3. Squaring, we have both

$$y^2 \;=\; \frac{x}{x+1}$$

and

$$(y+\delta y)^2 \;=\; \frac{x+\delta x}{(x+\delta x)+1} \;=\; \frac{x+\delta x}{x+1+\delta x}$$

or, expanding the left-hand side,

$$y^2 + 2y\delta y + \delta y^2 \;=\; \frac{x+\delta x}{x+1+\delta x}.$$

Subtracting, we obtain

$$2y\delta y + \delta y^2 \;=\; \frac{x+\delta x}{x+1+\delta x} \;-\; \frac{x}{x+1} \;=\; \frac{(x+\delta x)(x+1) - (x+1+\delta x)x}{(x+1+\delta x)(x+1)}$$

which simplifies to

$$2y\delta y + \delta y^2 \;=\; \frac{\delta x}{(x+1+\delta x)(x+1)}$$

so that

$$2y\frac{\delta y}{\delta x} + \delta y\frac{\delta y}{\delta x} \;=\; \frac{1}{(x+1+\delta x)(x+1)}.$$

Now, taking the limit as $\delta x \to 0$ and proceeding as above, we obtain

$$2y\frac{dy}{dx} + 0 \;=\; \lim_{\delta x\to 0}\frac{1}{(x+1+\delta x)(x+1)} \;=\; \frac{1}{(x+1+0)(x+1)}$$

or

$$2y\frac{dy}{dx} = \frac{1}{(x+1)^2}.$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2(x+1)^2 y} = \frac{1}{2y(x+1)^2} = \frac{1}{2(x+1)^2}\frac{1}{y} \\ &= \frac{1}{2(x+1)^2}\sqrt{\frac{x+1}{x}} = \frac{1}{2x^{1/2}(x+1)^{3/2}}.\end{aligned}$$