7. Derivatives of combinations

So far we have defined five types of function combination, namely, sum (or difference), product, quotient, composition and join. Moreover, we already have a general result for the derivative of one of these types, namely, equation (17) of Lecture 6:

$$\frac{d}{dt}\{f(t) + g(t)\} \;=\; f'(t) + g'(t). \tag{1}$$

In this lecture we obtain analogous results for the other four types.

Before beginning this task, we briefly digress to note that we also already have a general result for the derivative of a multiple, namely, equation (15) of Lecture 6:

$$\frac{d}{dt}\{\alpha f(t)\} \;=\; \alpha f'(t). \tag{2}$$

Because (2) implies $\frac{d}{dt}\{\beta f(t)\} = \beta g'(t)$ as well, (1)-(2) are easily combined to yield a single result for the derivative of an arbitrary *linear combination* of two functions, namely,

$$\frac{d}{dt}\{\alpha f(t) + \beta g(t)\} \;=\; \alpha f'(t) + \beta g'(t). \tag{3}$$

This equation yields the derivative of both a sum (with $\alpha = \beta = 1$) and a difference (with $\alpha = 1, \beta = -1$). Now back to the task at hand.

We begin by obtaining a general result for the derivative of a product. Suppose that $u = F(t)$, $v = G(t)$ and

$$y \;=\; uv \;=\; F(t)G(t). \tag{4}$$

Then, as t changes infinitesimally to $t + \delta t$, u changes infinitesimally to $u + \delta u$, v changes infinitesimally to $v + \delta v$ and y changes infinitesimally to $y + \delta y$ in such a way that

$$y + \delta y \;=\; (u + \delta u)(v + \delta v) \;=\; uv + u\,\delta v + \delta u\,v + \delta u\,\delta v. \tag{5}$$

Subtracting (4) from (5) yields

$$\delta y \;=\; \delta u\,v + u\,\delta v + \delta u\,\delta v. \tag{6}$$

Dividing by δt yields

$$\frac{\delta y}{\delta t} \;=\; \frac{\delta u}{\delta t}\,v + u\,\frac{\delta v}{\delta t} + \delta u\,\frac{\delta v}{\delta t}. \tag{7}$$

Applying the combination rule yields

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta t} \;=\; \lim_{\delta t\to 0}\frac{\delta u}{\delta t}\lim_{\delta t\to 0} v + \lim_{\delta t\to 0} u \lim_{\delta t\to 0}\frac{\delta v}{\delta t} + \lim_{\delta t\to 0}\delta u \lim_{\delta t\to 0}\frac{\delta v}{\delta t}. \tag{8}$$

But $\delta u \to 0$ as $\delta t \to 0$, while u and v do not change. Hence (8) reduces to

$$\frac{dy}{dt} \;=\; \frac{du}{dt}\,v + u\,\frac{dv}{dt} + 0\cdot\frac{dv}{dt} \;=\; \frac{du}{dt}\,v + u\,\frac{dv}{dt}. \tag{9a}$$

In other words, on using (4), our general result is

$$\frac{d}{dt}\{F(t)G(t)\} \;=\; F'(t)\,G(t)+F(t)\,G'(t), \tag{9b}$$

though we usually apply this *product rule* without explicitly defining F or G. For example:

$$\frac{d}{dt}\{t^2\sin(t)\} = \frac{d}{dt}\{t^2\}\sin(t)+t^2\,\frac{d}{dt}\{\sin(t)\} = 2t\sin(t)+t^2\cos(t) = t\{2\sin(t)+t\cdot\cos(t)\}$$

on using our special results. Note that (9a) is readily extended to deal with a product of any number of functions; for example, with three functions, we have

$$\begin{aligned}\frac{d}{dt}\{uvw\} \;&=\; \frac{d}{dt}\{uv\cdot w\} \;=\; \frac{d}{dt}\{uv\}\,w+uv\,\frac{dw}{dt} \;=\; \left\{\frac{du}{dt}\,v+u\,\frac{dv}{dt}\right\}w+uv\,\frac{dw}{dt}\\ &=\; \frac{du}{dt}\,v\,w+u\,\frac{dv}{dt}\,w+uv\,\frac{dw}{dt}\end{aligned} \tag{10a}$$

or, equivalently,

$$\frac{d}{dt}\{F(t)G(t)H(t)\} \;=\; F'(t)\,G(t)\,H(t)+F(t)\,G'(t)\,H(t)+F(t)\,G(t)\,H'(t). \tag{10b}$$

Next we obtain a general result for the derivative of a quotient. Again suppose that $u = F(t)$ and $v = G(t)$, but now with

$$y \;=\; \frac{u}{v} \;=\; \frac{F(t)}{G(t)} \tag{11}$$

(and, needless to say, $v \neq 0$, so that any t for which $G(t) = 0$ lies outside the domain of the quotient). Then

$$vy \;=\; u. \tag{12}$$

Applying the product rule:

$$\frac{dv}{dt}\,y+v\,\frac{dy}{dt} \;=\; \frac{du}{dt}. \tag{13}$$

Rearranging and using (12):

$$v\,\frac{dy}{dt} \;=\; \frac{du}{dt}-\frac{dv}{dt}\,y \;=\; \frac{du}{dt}-\frac{dv}{dt}\,\frac{u}{v}. \tag{14}$$

Hence, dividing by v,

$$\frac{dy}{dt} \;=\; \frac{1}{v}\frac{du}{dt}-\frac{dv}{dt}\,\frac{u}{v^2} \;=\; \frac{\dfrac{du}{dt}v-u\dfrac{dv}{dt}}{v^2}. \tag{15a}$$

In other words, on using (11), our general result is

$$\frac{d}{dt}\left\{\frac{F(t)}{G(t)}\right\} \;=\; \frac{F'(t)\,G(t)-F(t)\,G'(t)}{\{G(t)\}^2}. \tag{15b}$$

Again, we usually apply this *quotient rule* without explicitly defining F or G. For example:

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\sin(t)}{\cos(t)}\right\} &= \frac{\frac{d}{dt}\{\sin(t)\}\ \cos(t) - \sin(t)\ \frac{d}{dt}\{\cos(t)\}}{\{\cos(t)\}^2} \\ &= \frac{\cos(t)\cdot\cos(t) - \sin(t)\,\{-\sin(t)\}}{\cos^2(t)} \\ &= \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)} = \left(\frac{1}{\cos(t)}\right)^2\end{aligned}$$

on using our special results. A neater way to rewrite this result is

$$\frac{d}{dt}\{\tan(t)\} = \sec^2(t). \tag{16}$$

We can use the quotient rule to extend the result that

$$\frac{d}{dx}\{x^r\} = r\,x^{r-1} \tag{17}$$

if r is a positive integer to negative-integer exponents. For $s > 0$, we have

$$\frac{d}{dx}\{x^{-s}\} = \frac{d}{dx}\left\{\frac{1}{x^s}\right\} = \frac{\frac{d}{dx}(1)\cdot x^s - 1\cdot\frac{d}{dx}(x^s)}{(x^s)^2} = \frac{0\cdot x^s - 1\cdot sx^{s-1}}{x^{2s}} = -sx^{-s-1}.$$

This result agrees with (17) for $r = -s$. Now we know that (17) holds for any integer, regardless of whether it is positive or negative.

A general result for the derivative of a composition is even simpler to derive than the product or the quotient rule.* Let x be the independent variable, let y depend upon x, and let z in turn depend on y. Then three derivatives are involved, because y is changing with x and z is changing with y, which in turn makes z change with x. The three derivatives are

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}, \quad \frac{dz}{dy} = \lim_{\delta y\to 0}\frac{\delta z}{\delta y} \quad \text{and} \quad \frac{dz}{dx} = \lim_{\delta x\to 0}\frac{\delta z}{\delta x},$$

respectively, and we assume that they all exist, which requires in particular that $\delta y \to 0$ as $\delta x \to 0$, and vice versa. Thus applying the combination rule to

$$\frac{\delta z}{\delta x} = \frac{\delta z}{\delta y}\frac{\delta y}{\delta x} \tag{18}$$

we obtain

$$\lim_{\delta x\to 0}\frac{\delta z}{\delta x} = \lim_{\delta x\to 0}\frac{\delta z}{\delta y}\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta y\to 0}\frac{\delta z}{\delta y}\lim_{\delta x\to 0}\frac{\delta y}{\delta x} \tag{19}$$

or

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{20a}$$

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*If we assume, as we are going to, that if the function called U in (20b) is a join, then it doesn't have subdomains on which it is constant. Then, on the one hand, we don't need the result for that particular subdomain to begin with; and on the other hand, the result is still true for the entire domain—but the derivation is trickier, and we prefer to avoid unnecessary complications.

Note that if $y = U(x)$, $z = P(y)$ and the composition is called R, as in Lecture 2, so that $z = R(x)$, then (20a) becomes $R'(x) = P'(y)\,U'(x)$; that is,

$$R(x) \;=\; P(U(x)) \Longrightarrow R'(x) \;=\; P'(U(x))\,U'(x). \tag{20b}$$

This general result for the derivative of a composition should perhaps be called the composition rule—but it isn't, it's called the *chain rule*.

For example, suppose you wish to calculate

$$\frac{d}{dx}\left\{\sin\left(\frac{1}{x}\right)\right\}. \tag{21}$$

Then set

$$y \;=\; \frac{1}{x} \quad\Longrightarrow\quad \frac{dy}{dx} \;=\; -\frac{1}{x^2} \tag{22}$$

(from Lecture 5) and set

$$z \;=\; \sin(y) \quad\Longrightarrow\quad \frac{dz}{dy} \;=\; \cos(y) \tag{23}$$

(again from Lecture 5) so that you can calculate (21) in terms of (20a) as

$$\begin{aligned}\frac{d}{dx}\left\{\sin\left(\frac{1}{x}\right)\right\} \;&=\; \frac{d}{dx}\left\{\sin\left(y\right)\right\} \;=\; \frac{dz}{dx} \;=\; \frac{dz}{dy}\,\frac{dy}{dx} \;=\; \cos(y)\left\{-\frac{1}{x^2}\right\}\\ &=\; -\frac{1}{x^2}\cos(y) \;=\; -\frac{1}{x^2}\cos\left(\frac{1}{x}\right).\end{aligned} \tag{24}$$

In practice, the most useful expression of the chain rule is often neither (20a) nor (20b), but rather a hybrid of the two: we set $z = P(y)$ in (20a) to obtain

$$\frac{d}{dx}\{P(y)\} \;=\; \frac{d}{dy}\{P(y)\}\,\frac{dy}{dx}. \tag{25}$$

Thus, for example, $\frac{d}{dx}\{\sin(y)\} = \frac{d}{dy}\{\sin(y)\}\,\frac{dy}{dx} = \cos(y)\,\frac{dy}{dx}$ so that

$$\frac{d}{dx}\{\sin(y)\} \;=\; \cos(y)\,\frac{dy}{dx} \tag{26}$$

holds for an arbitrary relationship between x and y (regardless of whether we know it). In the case where $y = 1/x$, (26) reduces to (24); in the case where $y = x^2$, (26) yields

$$\frac{d}{dx}\{\sin(x^2)\} \;=\; \cos(x^2)\,\frac{d}{dx}\{x^2\} \;=\; 2x\cos(x^2); \tag{27}$$

in the case where $y = x^3$, (26) yields

$$\frac{d}{dx}\{\sin(x^3)\} \;=\; \cos(x^3)\,\frac{d}{dx}\{x^3\} \;=\; 3x^2\cos(x^3); \tag{28}$$

and so on.

We can use the chain rule to extend our result for the derivative of a square root from Lecture 3 to a result for the derivative of an arbitrary n-th root. Let

$$y = \sqrt[n]{x} = x^{\frac{1}{n}} \tag{29}$$

so that

$$\begin{aligned} y^n = x \quad &\Longrightarrow \quad \frac{d}{dx}\{y^n\} = \frac{d}{dx}\{x\} \\ &\Longrightarrow \quad \frac{d}{dy}\{y^n\}\frac{dy}{dx} = 1 \\ &\Longrightarrow \quad ny^{n-1}\frac{dy}{dx} = 1 \end{aligned} \tag{30}$$

by the chain rule. Hence

$$\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{n}y^{1-n} = \frac{1}{n}\left(x^{\frac{1}{n}}\right)^{1-n} = \frac{1}{n}x^{\frac{1}{n}-1}, \tag{31}$$

which agrees with (17) for $r = \frac{1}{n}$. But any rational number r can be written as $r = m/n$, where m is an integer and n is a positive integer. So we can apply the chain rule with $y = x^{1/n}$ and $z = y^m$ to obtain

$$\begin{aligned} \frac{d}{dx}\{x^r\} &= \frac{d}{dx}\left\{x^{m/n}\right\} = \frac{d}{dx}\left\{\left(x^{1/n}\right)^m\right\} = \frac{d}{dx}\{y^m\} = \frac{dz}{dx} \\ &= \frac{dz}{dy}\frac{dy}{dx} = m\,y^{m-1}\frac{1}{n}x^{-1+1/n} = m\left(x^{1/n}\right)^{m-1}\frac{1}{n}x^{-1+1/n} \\ &= mx^{\{m-1\}/n}\frac{1}{n}x^{-1+1/n} = \frac{m}{n}x^{m/n-1/n-1+1/n} = r\,x^{r-1} \end{aligned} \tag{32}$$

for any rational number.

Finally, the simplest case in many ways is that of a join: all you do is differentiate separately on each contiguous subdomain. That is, if W is defined on $[a, b]$ by *either*

$$W(t) = \begin{cases} F(t) & \text{if} \quad a \le t < c \\ G(t) & \text{if} \quad c \le t \le b \end{cases} \tag{33a}$$

or

$$W(t) = \begin{cases} F(t) & \text{if} \quad a \le t \le c \\ G(t) & \text{if} \quad c < t \le b \end{cases} \tag{33b}$$

then its derivative W' is defined on at least $(a, c) \cup (c, b)$ by†

$$W'(t) = \begin{cases} F'(t) & \text{if} \quad a < t < c \\ G'(t) & \text{if} \quad c < t < b. \end{cases} \tag{34}$$

†It is also defined on $[a, c) \cup (c, b]$ if we have a right- and left-hand derivative at $t = a$ and $t = b$, respectively—see the remark at the end of Lecture 6, which also implies that $F'(c)$ and $G'(c)$ in (37) must be interpreted as left- and right-hand derivatives, respectively.

On the other hand, there is a question that doesn't arise in the other three cases, namely, whether the resulting derivative (34) must have a hole in its domain at $t = c$ (as in Figure 4 of Lecture 4), or whether it is *removable* (as described in Figure 3 of Lecture 4).

To deal with this question, we find it convenient to have a more compact notation for left- and right-handed limits. Accordingly, we define

$$w(a^+) \;=\; \lim_{x \to a^+} w(x), \qquad w(a^-) \;=\; \lim_{x \to a^-} w(x) \tag{35}$$

for *any* function called w—including any derivative, so (35) automatically implies

$$W'(a^+) \;=\; \lim_{x \to a^+} W'(x), \qquad W'(a^-) \;=\; \lim_{x \to a^-} W'(x). \tag{36}$$

It follows immediately from (34) that the left- and right-handed limits of W' at $t = c$ are

$$W'(c^-) = \lim_{t \to c^-} W'(t) = F'(c^-) \quad \text{and} \quad W'(c^+) = \lim_{x \to c^+} W'(t) = G'(c^+), \tag{37}$$

respectively. If these two limits are equal, that is, if

$$F'(c^-) \;=\; G'(c^+), \tag{38}$$

then—exactly as in Lecture 4—we can remove the hole by defining $W'(c)$ to be their common value. If $F'(c^-) \neq G'(c^+)$, on the other hand, then $W'(c)$ is undefined.

We illustrate these ideas with two familiar joins from Lecture 2, namely, photosynthesis rate

$$L(u) \;=\; \begin{cases} \frac{2-\sqrt{2}}{2+\sqrt{2}}\left(2+\sqrt{2}-\frac{1}{2}u\right)u & \text{if} \quad 0 \le u \le 2+\sqrt{2} \\ 1 & \text{if} \quad 2+\sqrt{2} < u < \infty \end{cases} \tag{39}$$

and diastolic inflow

$$v(t) \;=\; \begin{cases} \frac{81600}{1127}(30t-23)(5t-2)(4t-3) & \text{if} \quad 0.4 \le t < 0.75 \\ \frac{14000}{33}(12t-11)(4t-3)(10t-9) & \text{if} \quad 0.75 \le t \le 0.9. \end{cases} \tag{40}$$

From (33)-(34), (3) and (9) with the obvious modifications, we obtain

$$L'(u) \;=\; \begin{cases} \dfrac{d}{du}\left\{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}(2+\sqrt{2}-\tfrac{1}{2}u)u\right\} & \text{if} \quad 0 \le u < 2+\sqrt{2} \\ \dfrac{d}{du}\{1\} & \text{if} \quad 2+\sqrt{2} < u < \infty \end{cases} \tag{41a}$$

$$=\; \begin{cases} \dfrac{2-\sqrt{2}}{2+\sqrt{2}}\dfrac{d}{du}\left\{\left(2+\sqrt{2}-\tfrac{1}{2}u\right)u\right\} & \text{if} \quad 0 \le u < 2+\sqrt{2} \\ 0 & \text{if} \quad 2+\sqrt{2} < u < \infty \end{cases} \tag{41b}$$

$$=\; \begin{cases} \frac{2-\sqrt{2}}{2+\sqrt{2}}\left(2+\sqrt{2}-u\right) & \text{if} \quad 0 \le u < 2+\sqrt{2} \\ 0 & \text{if} \quad 2+\sqrt{2} < u < \infty \end{cases} \tag{41c}$$

in the first instance, with $L'(2+\sqrt{2})$ undefined; see Exercise 1. From (41c), however, with $c = 2+\sqrt{2}$ we have $L'(c^-) = \frac{2-\sqrt{2}}{2+\sqrt{2}}\left(2+\sqrt{2}-c\right) = 0$, which equals $L'(c^+)$. Hence (38) is satisfied with $W = L$, and so we can re-define L' as a continuous function without a hole:

$$L'(u) \;=\; \begin{cases} \frac{2-\sqrt{2}}{2+\sqrt{2}}\left(2+\sqrt{2}-u\right) & \text{if} \quad 0 \le u \le 2+\sqrt{2} \\ 0 & \text{if} \quad 2+\sqrt{2} < u < \infty. \end{cases} \tag{42}$$

On the other hand, when we differentiate v (Exercise 2), we obtain

$$v'(t) \;=\; \begin{cases} \frac{81600}{1127}\{1800t^2 - 2300t + 709\} & \text{if} \quad 0.4 \le t < 0.75 \\ \frac{28000}{33}\{720t^2 - 1232t + 525\} & \text{if} \quad 0.75 < t \le 0.9. \end{cases} \tag{43}$$

after simplification, so that $v'(0.75^-) = \frac{81600}{1127}\{1800(0.75)^2 - 2300(0.75) + 709\} \approx -253$ while $v'(0.75^+) = \frac{28000}{33}\{720(0.75)^2 - 1232(0.75) + 525\} \approx 5091$. Thus $v'(0.75^-) \neq v'(0.75^+)$, and the graph of v' has a hole at $t = 0.75$; see Figure 1.

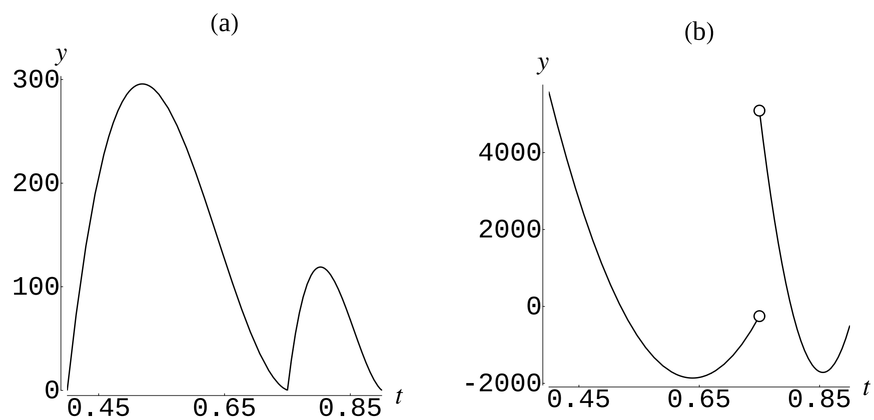


Figure 1: (a) $y = v(t)$ and (b) $y = v'(t)$ defined by (40) and (43), respectively.

We conclude by noting that a function whose derivative is a continuous function—if necessary, after a removable hole has been filled—is called a *smooth* function. Thus L defined by (39) is smooth because L' defined by (42) is continuous; whereas v defined by (40) is not smooth, because v' defined by (43) is discontinuous where $t = 0.75$. Alternatively, given the geometrical interpretation of the differential coefficient (Lecture 5), a function is smooth if its graph has no corners, and otherwise is not smooth (as illustrated by Figure 1). In practice, however, even non-smooth functions are usually *piecewise-smooth*, in the sense that if a graph has n corners then the function's domain can be decomposed into $n+1$ subdomains with the corners always at endpoints, in such a way that the function is smooth on every subdomain, despite not being smooth on its entire domain.

Exercises

1. Verify (42).
Hint: Use (33)-(34), (3) and (9) with the obvious modifications.

2. Verify (43).
Hint: Apply (10) and (34) to (40).

3. Find $\dfrac{dy}{dt}$ for $y = \dfrac{1-t}{1+t}$ where $t \neq -1$.

4. (a) Find $\dfrac{dy}{dt}$ for $y = \dfrac{t}{B+t}$, where B is a constant and $t \neq -B$.

(b) Find the equation of the tangent line to the curve $y = \frac{2x}{1+2x}$ at the point with coordinates $(0, 0)$.

(c) Find the equation of the tangent line to the curve $y = \frac{2x}{1+2x}$ at the point with coordinates $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(d) Where do these two tangent lines meet?

(e) Sketch the graph of $y = \frac{2x}{1+2x}$ on $\left(-\frac{1}{2}, 1\right)$ together with its vertical asymptote and both tangent lines, clearly indicating both their points of tangency and their point of intersection.

5. Find $\dfrac{dy}{dt}$ for $y = \dfrac{t^2}{B-t}$, where B is a constant and $t \neq B$.

6. Find $\dfrac{dy}{dt}$ for $y = \sqrt{t^3 + 2t}$ where $t > 0$.

7. Find $\dfrac{dy}{dt}$ for $y = t^2\sqrt{t^3 + 2t}$, $t > 0$.

8. For f defined by $f(t) = t\sin(\pi t)\sqrt{t^3 + 2t}$, find $f'(1)$.

9. Find $\dfrac{dy}{dt}$ for $y = \tan\left(\sqrt{t^3 + 2t}\right)$ where $0 < t < \frac{1}{2}$.

10. A smooth function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} At + Bt^2 & \text{if} \quad 0 \leq t < 2 \\ \dfrac{1}{t} & \text{if} \quad 2 \leq t < \infty \end{cases}$$

where A and B are constants. What must be their values?

11. A smooth function W is defined on $[0, \infty)$ by

$$W(t) = \begin{cases} \frac{1}{4}t(A-t) & \text{if} \quad 0 \leq t < 1 \\ \dfrac{t}{B+t} & \text{if} \quad 1 \leq t < \infty \end{cases}$$

where A and B are positive constants. What must be their values?

12. A smooth function W is defined on $[0, \infty)$ by

$$W(t) \;=\; \begin{cases} At + Bt^3 & \text{if} \quad 0 \le t < 1 \\ \dfrac{1-t}{1+t} & \text{if} \quad 1 \le t < \infty \end{cases}$$

where A and B are constants. What must be their values?

13. A smooth function W is defined on $[0, 3]$ by

$$W(t) \;=\; \begin{cases} At^3 & \text{if} \quad 0 \le t < 2 \\ \dfrac{t^2}{B-t} & \text{if} \quad 2 \le t \le 3 \end{cases}$$

where A and B are positive constants. What must be their values?

14. A smooth function W is defined on $[0, \infty)$ by

$$W(t) \;=\; \begin{cases} At - Bt^2 & \text{if} \quad 0 \le t < 3 \\ \dfrac{16t}{t+1} & \text{if} \quad 3 \le t < \infty \end{cases}$$

where A and B are positive constants. What must be their values?

15. A smooth function W is defined on $[0, \infty)$ by

$$W(t) \;=\; \begin{cases} At - Bt^2 & \text{if} \quad 0 \le t < 1 \\ \dfrac{9t}{t+2} & \text{if} \quad 1 \le t < \infty \end{cases}$$

where A and B are positive constants. What must be their values?

16. Calculate $\dfrac{d}{dx}\left\{\sqrt[4]{1+\sqrt[3]{2+\sqrt{x^2+3}}}\right\}$.

Suitable problems from standard calculus texts

Stewart (2003): p. 191, ## 15-16 and 19-27; p. 197, ## 7-11, 19-24, 27, 28, 31, 32 and 34-38; p. 216, ## 1-7 and 9-19; and p. 224, ## 1-4, 7-14, 17-20, 25-27, 29, 30, 32-35, 37-41, 43-45, 48 and 51-54.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions to selected exercises

3. From the quotient rule we have

$$\frac{dy}{dt} = \frac{\frac{d}{dt}\{1-t\}\cdot(1+t)-(1-t)\cdot\frac{d}{dt}\{1+t\}}{(1+t)^2}$$
$$= \frac{(0-1)\cdot(1+t)-(1-t)(0+1)}{(1+t)^2} = \frac{-2}{(1+t)^2}$$

because $\frac{d}{dt}\{1\pm t\} = \frac{d}{dt}\{1\} \pm \frac{d}{dt}\{t\} = 0 \pm 1$. Alternatively

$$y = \frac{1-t}{1+t} = \frac{2-1-t}{1+t} = \frac{2-(1+t)}{1+t} = \frac{2}{1+t} - 1 = 2(1+t)^{-1} - 1$$

implies

$$\frac{dy}{dt} = 2\frac{d}{dt}\{(1+t)^{-1}\} - \frac{d}{dt}\{1\} = 2\frac{d}{dt}\{(1+t)^{-1}\} - 0 = 2\frac{d}{dt}\{(1+t)^{-1}\}.$$

But from the chain rule we have

$$\frac{d}{dt}\{x^{-1}\} = \frac{d}{dx}\{x^{-1}\}\cdot\frac{dx}{dt} = -\frac{1}{x^2}\frac{dx}{dt}$$

on using a special result from Lecture 5. Hence with $x = 1+t$ we obtain

$$\frac{dy}{dt} = 2\frac{d}{dt}\{x^{-1}\} = -\frac{2}{x^2}\frac{dx}{dt} = -\frac{2}{(1+t)^2}\frac{d}{dt}\{1+t\} = -\frac{2}{(1+t)^2}\{0+1\} = \frac{-2}{(1+t)^2}$$

as before.

4. (a) From the quotient rule we have

$$\frac{dy}{dt} = \frac{\frac{d}{dt}\{t\}\cdot(B+t) - t\cdot\frac{d}{dt}\{B+t\}}{(B+t)^2}$$
$$= \frac{1\cdot(B+t) - t\cdot(0+1)}{(B+t)^2} = \frac{B}{(B+t)^2}.$$

(b) Note that

$$y = \frac{2x}{1+2x} = \frac{x}{\frac{1}{2}+x}.$$

But from (a) with $B = \frac{1}{2}$ we have

$$\frac{d}{dt}\left\{\frac{t}{\frac{1}{2}+t}\right\} = \frac{\frac{1}{2}}{(\frac{1}{2}+t)^2}.$$

It follows immediately that

$$\frac{d}{dx}\left\{\frac{x}{\frac{1}{2}+x}\right\} = \frac{\frac{1}{2}}{(\frac{1}{2}+x)^2} = \frac{2}{(1+2x)^2}.$$

So the first tangent line has slope

$$m_1 \;=\; \frac{d}{dx}\left\{\frac{x}{\frac{1}{2}+x}\right\}\bigg|_{x=0} \;=\; \frac{2}{(1+2\cdot 0)^2} \;=\; 2$$

and hence equation $y - 0 = m_1(x-0)$ or $y = 2x$.

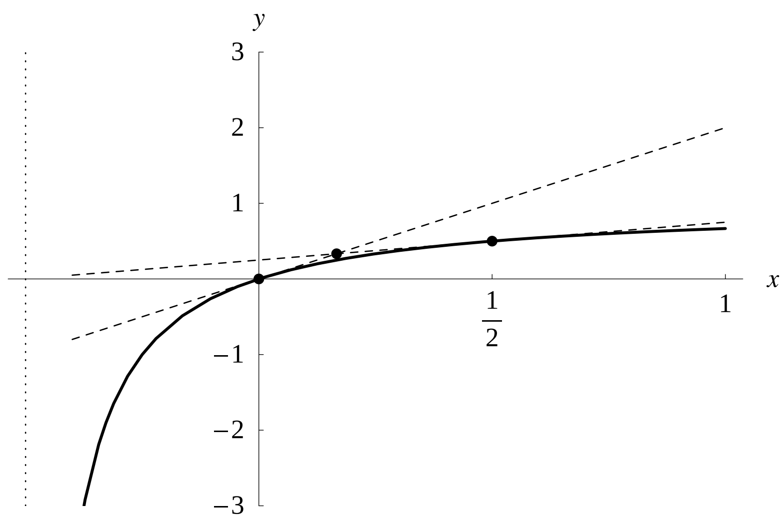
(c) Likewise, the second tangent line has slope

$$m_2 \;=\; \frac{d}{dx}\left\{\frac{x}{\frac{1}{2}+x}\right\}\bigg|_{x=\frac{1}{2}} \;=\; \frac{2}{\left(1+2\cdot\frac{1}{2}\right)^2} \;=\; \frac{1}{2}$$

and hence equation $y - \frac{1}{2} = m_2(x - \frac{1}{2})$ or $y = \frac{1}{2}x + \frac{1}{4}$.

(d) These lines meet where $2x = \frac{1}{2}x + \frac{1}{4}$ or $x = \frac{1}{6}$, and hence $y = \frac{1}{3}$; in other words, at the point with coordinates $\left(\frac{1}{6}, \frac{1}{3}\right)$.

(e)



6. Squaring, we have

$$y^2 \;=\; t^3 + 2t \Longrightarrow \frac{d}{dt}\{y^2\} \;=\; \frac{d}{dt}\{t^3+2t\} \Longrightarrow 2y\frac{dy}{dt} \;=\; 3t^2+2.$$

Hence

$$\frac{dy}{dt} \;=\; \frac{3t^2+2}{2y} \;=\; \frac{3t^2+2}{2\sqrt{t^3+2t}}.$$

Alternatively, set $x = t^3 + 2t$. Then, on using the chain rule, the linear-combination rule and special results, we have

$$\begin{aligned}\frac{dy}{dt} \;&=\; \frac{d}{dt}\{\sqrt{x}\} \;=\; \frac{d}{dx}\{\sqrt{x}\}\frac{dx}{dt} \;=\; \frac{1}{2\sqrt{x}}\frac{dx}{dt} \;=\; \frac{1}{2\sqrt{x}}\frac{d}{dt}\{t^3+2t\}\\ &=\; \frac{1}{2\sqrt{x}}\left(\frac{d}{dt}\{t^3\} + 2\frac{d}{dt}\{t\}\right) \;=\; \frac{1}{2\sqrt{x}}\left(3t^2 + 2\cdot 1\right) \;=\; \frac{3t^2+2}{2\sqrt{t^3+2t}}\end{aligned}$$

as before.

10. Using our general results for the derivative of a join or sum together with a special result from Lecture 5, we find that

$$W'(t) = \begin{cases} A + 2Bt & \text{if} \quad 0 \le t < 2 \\ -\dfrac{1}{t^2} & \text{if} \quad 2 < t < \infty. \end{cases}$$

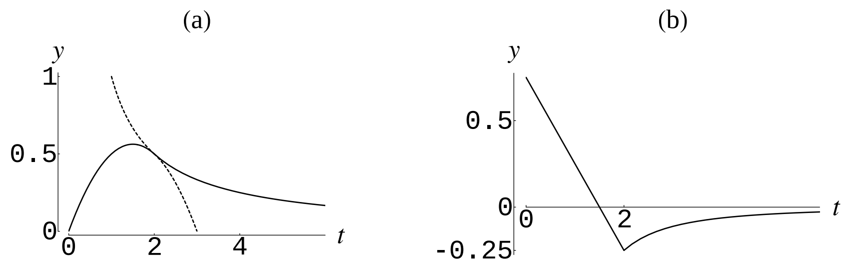
So the left-handed derivative as $t \to 2^-$ is $W'(2^-) = A + 2B \cdot 2 = A + 4B$, and the right-handed derivative as $t \to 2^+$ is $W'(2^+) = -\frac{1}{2^2} = -\frac{1}{4}$. For W to be smooth, its derivative must be continuous everywhere, and hence in particular at $t = 2$; so we require $W'(2^-) = W'(2+)$, or $A + 4B = -\frac{1}{4}$. But W can't have a continuous derivative unless it is continuous itself, so we also require $W(2^-) = W(2^+)$, i.e., $A \cdot 2 + B \cdot 2^2 = \frac{1}{2}$ or $2A + 4B = \frac{1}{2}$. Subtracting $A + 4B = -\frac{1}{4}$ from $2A + 4B = \frac{1}{2}$ yields $A = \frac{3}{4}$, and substituting back into one of these equations yields $B = -\frac{1}{4}$. Now we have ensured that W defined on $[0, \infty)$ by

$$W(t) = \begin{cases} \frac{1}{4}t(3 - t) & \text{if} \quad 0 \le t < 2 \\ \dfrac{1}{t} & \text{if} \quad 2 \le t < \infty \end{cases}$$

is smooth, with derivative W' defined on $[0, \infty)$ by

$$W'(t) = \begin{cases} \frac{1}{4}(3 - 2t) & \text{if} \quad 0 \le t < 2 \\ -\dfrac{1}{t^2} & \text{if} \quad 2 \le t < \infty. \end{cases}$$

The figure below shows the corresponding graphs, (a) $y = W(t)$ and (b) $y = W'(t)$. Note the smoothness of the join between different curves.



11. Similarly, we have

$$W'(t) = \begin{cases} \frac{1}{4}A - \frac{1}{2}t & \text{if} \quad 0 \le t < 1 \\ \dfrac{B}{(B + t)^2} & \text{if} \quad 1 < t < \infty. \end{cases}$$

So the left-handed derivative as $t \to 1^-$ is $W'(1^-) = \frac{1}{4}A - \frac{1}{2}$, and the right-handed derivative as $t \to 1^+$ is $W'(1^+) = \frac{B}{(B+1)^2}$. Also, the left-handed limit of W itself as $t \to 1^-$ is $W(1^-) = \frac{1}{4}(A - 1)$, and the right-handed limit of W itself is $W(1^+) = \frac{1}{B+1}$. For W to be smooth, we require both $W(1^-) = W(1^+)$ and $W'(1^-) = W'(1+)$, hence

$$\begin{aligned} \tfrac{1}{4}A - \tfrac{1}{4} &= \tfrac{1}{B+1} \\ \tfrac{1}{4}A - \tfrac{1}{2} &= \tfrac{B}{(B+1)^2} \end{aligned}$$

Subtraction yields $\frac{1}{4} = \frac{1}{B+1} - \frac{B}{(B+1)^2}$, or $B^2 + 2B - 3 = (B+3)(B-1) = 0$, after simplification. So either $B = -3$ or $B = 1$. But W would be discontinuous at $t = 3$ for $B = -3$; therefore, we must take $B = 1$, with $A = 1 + \frac{4}{B+1} = 3$. Now we have ensured that W defined on $[0, \infty)$ by

$$W(t) \;=\; \begin{cases} \frac{1}{4}t(3-t) & \text{if} \quad 0 \le t < 1 \\ \frac{t}{1+t} & \text{if} \quad 1 \le t < \infty \end{cases}$$

is smooth, with derivative W' defined on $[0, \infty)$ by

$$W'(t) \;=\; \begin{cases} \frac{1}{4}(3-2t) & \text{if} \quad 0 \le t < 1 \\ \frac{1}{(1+t)^2} & \text{if} \quad 1 \le t < \infty. \end{cases}$$

12. $A = \frac{1}{4}, B = -\frac{1}{4}$.

13. $A = \frac{1}{4}, B = 4$.

14. $A = 7, B = 1$.

15. $A = 4, B = 1$.

16. Making multiple use of our special results from Lecture 6, set:

$$\begin{array}{lllllllll} y &=& x^2+3 & \Longrightarrow & \dfrac{dy}{dx} &=& 2x+0=2x & v \;=\; w^{1/3} & \Longrightarrow \quad \dfrac{dv}{dw} \;=\; \frac{1}{3}w^{-2/3} \\ z &=& \sqrt{y} & \Longrightarrow & \dfrac{dz}{dy} &=& \frac{1}{2}y^{-1/2} & u \;=\; 1+v & \Longrightarrow \quad \dfrac{du}{dv} \;=\; 0+1=1 \\ w &=& 2+z & \Longrightarrow & \dfrac{dw}{dz} &=& 0+1=1 & s \;=\; u^{1/4} & \Longrightarrow \quad \dfrac{ds}{du} \;=\; \frac{1}{4}u^{-3/4} \end{array}$$

Now, from repeated application of (20a):

$$\begin{aligned} \frac{d}{dx}\left\{\sqrt[4]{1+\sqrt[3]{2+\sqrt{x^2+3}}}\right\} \;&=\; \frac{d}{dx}\left\{\sqrt[4]{1+\sqrt[3]{2+\sqrt{y}}}\right\} \;=\; \frac{d}{dx}\left\{\sqrt[4]{1+\sqrt[3]{2+z}}\right\} \\ &=\; \frac{d}{dx}\left\{\sqrt[4]{1+w^{1/3}}\right\} \;=\; \frac{d}{dx}\left\{\sqrt[4]{1+v}\right\} \;=\; \frac{d}{dx}\left\{u^{1/4}\right\} \;=\; \frac{ds}{dx} \\ &=\; \frac{ds}{dz}\frac{dz}{dx} \;=\; \left\{\frac{ds}{dv}\frac{dv}{dz}\right\}\frac{dz}{dx} \;=\; \frac{ds}{dv}\frac{dv}{dz}\frac{dz}{dx} \\ &=\; \left\{\frac{ds}{du}\frac{du}{dv}\right\}\left\{\frac{dv}{dw}\frac{dw}{dz}\right\}\left\{\frac{dz}{dy}\frac{dy}{dx}\right\} \\ &=\; \frac{ds}{du}\frac{du}{dv}\frac{dv}{dw}\frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx} \\ &=\; \tfrac{1}{4}u^{-3/4}\cdot 1\cdot \tfrac{1}{3}w^{-2/3}\cdot 1\cdot \tfrac{1}{2}y^{-1/2}\cdot 2x \\ &=\; \frac{x}{12\left\{1+\left(2+\sqrt{x^2+3}\right)^{1/3}\right\}^{3/4}\left(2+\sqrt{x^2+3}\right)^{2/3}\sqrt{x^2+3}} \end{aligned}$$

after simplification—absolutely gruesome, but perfectly straightforward.