8. Function sequences. The exponential and logarithm

The next most fundamental concept in calculus (after those of function, sequence, limit, infinitesimal, differential coefficient and derivative) is that of a *function sequence*, which differs from that of an ordinary sequence only in the obvious way: $\{f_n\} = \{f_1, f_2, f_3, \ldots\}$ is an ordinary sequence if it is just a list of numbers, but it is a function sequence if instead it is a list of functions. If we use the notation $\{f_n\}$ for a function sequence as well, however, then we risk confusion between the two; so instead we denote a function sequence by the labels assigned by each function to the independent variable, e.g., $\{f_n(x)\} = \{f_1(x), f_2(x), f_3(x), \ldots\}$ if the independent variable is x, or $\{f_n(t)\} = \{f_1(t), f_2(t), f_3(t), \ldots\}$ if the independent variable is t, and so on.

Suppose, for example, that reproduction in a Fibonacci population is not as perfect as Fibonacci supposed. Specifically, it is no longer true that every pair of rabbits reproduces itself with certainty every month; rather, it reproduces itself with probability x (and so fails to reproduce with probability $1 - x$), where $0 \leq x \leq 1$. It is now no longer true that the initial pair contributes a pair of newborns by the end of month 2; in terms of Lecture 3, $y_2 \neq 2$. But the *expected* number of newborn pairs at the end of month 2 is x (in the sense that a very large number, N, of identical but independent Fibonacci breeding experiments would yield Nx newborn pairs by the end of February), and so the expected total of rabbit pairs at the end of February is $1 + x$. Thus, if we re-interpret y_k as *expected* number of young pairs at the end of month k, a_k as expected number of adult pairs at the end of month k and u_k as total expected number of pairs at the end of month k, then $y_2 = x$, $a_2 = 1$ and $u_2 = 1 + x$; see Table 1.

n	y_n	a_n	$u_n = u_n(x)$
0	1	0	1
1	0	1	1
2	x	1	$1 + x$
3	x	$1 + x$	$1 + 2x$
4	$x(1 + x)$	$1 + 2x$	$1 + 3x + x^2$
5	$x(1 + 2x)$	$1 + 3x + x^2$	$1 + 4x + 3x^2$
6	$x(1 + 3x + x^2)$	$1 + 4x + 3x^2$	$1 + 5x + 6x^2 + x^3$
7	$x(1 + 4x + 3x^2)$	$1 + 5x + 6x^2 + x^3$	$1 + 6x + 10x^2 + 4x^3$
8	$x(1 + 5x + 6x^2 + x^3)$	$1 + 6x + 10x^2 + 4x^3$	$1 + 7x + 15x^2 + 10x^3 + x^4$
9	$x(1 + 6x + 10x^2 + 4x^3)$	$1 + 7x + 15x^2 + 10x^3 + x^4$	$1 + 8x + 21x^2 + 20x^3 + 5x^4 + x^5$
10	$x(1 + 7x + 15x^2 + 10x^3 + x^4)$	$1 + 8x + 21x^2 + 20x^3 + 5x^4$	$1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$

Table 1: The Fibonacci polynomials.

More generally, when reproduction is uncertain, it isn't true that the a_{n-1} adults at the end of month $n-1$ produce a_{n-1} young at the end of month n. Neverthless, if we multiply a_{n-1} by the probability that a pair reproduces, which is x, we find that the *expected* number of young at the end of month n is

$$y_n \;=\; x\, a_{n-1}, \tag{1}$$

which agrees with Lecture 3 for $x = 1$.* We interpret (1) as saying that a_{n-1} adults pro-

*Except, of course, that we are now using n in place of k.

duce xa_{n-1} newborns on average (where the average is taken over a large number of independent Fibonacci breeding experiments). Our model continues to exclude mortality: a young rabbit still becomes an adult after a month has elapsed. So expected number of adults at the end of month n still equals expected number of young at the end of month $n-1$ plus expected number of adults at the end of month $n-1$:

$$a_n = y_{n-1} + a_{n-1}. \tag{2}$$

In other words, the result of Lecture 3 still holds, except that we now interpret a_n and y_n as averages (over a very large number of Fibonacci experiments). With the same reinterpretation, the total expected number of rabbit pairs at the end of month n is still

$$u_n = a_n + y_n. \tag{3}$$

Replacing n by $n+1$ in (1)-(3), we find that

$$y_{n+1} = x\,a_n \tag{4a}$$
$$a_{n+1} = y_n + a_n \tag{4b}$$

and $u_{n+1} = a_{n+1} + y_{n+1} = y_n + a_n + x\,a_n$. So, using (2)-(3), $u_{n+1} = u_n + x(y_{n-1} + a_{n-1})$ or

$$u_{n+1} = u_n + x\,u_{n-1} \tag{5}$$

(from (3) with $n-1$ in place of n). Thus total expected number of rabbit pairs at time n is defined implicitly by

$$u_0 = 1 \tag{6a}$$
$$u_1 = 1 \tag{6b}$$
$$u_{n+1} = u_n + x\,u_{n-1} \quad \text{if} \quad n \geq 1. \tag{6c}$$

For example, because $u_2 = 1 + x$, we have $u_3 = u_2 + xu_1 = 1 + x + x = 1 + 2x$, $u_4 = u_3 + xu_2 = 1 + 2x + x(1+x) = 1 + 3x + x^2$, and so on; see Table 1. The expected totals at the end of each month define a function sequence $\{u_n(x)\}$ in which each term u_n has domain [0, 1]. We will call these functions the Fibonacci polynomials.†

A more interesting function sequence compares expected number of rabbit pairs at the end of a month with expected number at the end of the previous month. By analogy with Lecture 3, we define the function sequence $\{\phi_n(x)\}$ by

$$\phi_n(x) = \frac{u_n(x)}{u_{n-1}(x)}, \quad n \geq 1. \tag{7}$$

Each ϕ_n has the same domain [0, 1] as u_n. Alternatively, dividing (6b) and (6c) by u_0 and u_n, respectively, and proceeding as in Lecture 3, we can define $\{\phi_n(x)\}$ recursively by

$$\phi_1 = 1 \tag{8a}$$
$$\phi_{n+1} = 1 + \frac{x}{\phi_n} \quad \text{if} \quad n \geq 1 \tag{8b}$$

†Note, as discussed at the end of Lecture 1, that we are using a single notation for both function and label, because it is obvious from context which meaning is intended.

n	$\phi_n = \phi_n(x)$
1	1
2	$1+x$
3	$\frac{1+2x}{1+x}$
4	$\frac{1+3x+x^2}{1+2x}$
5	$\frac{1+4x+3x^2}{1+3x+x^2}$
6	$\frac{1+5x+6x^2+x^3}{1+4x+3x^2}$
7	$\frac{1+6x+10x^2+4x^3}{1+5x+6x^2+x^3}$
8	$\frac{1+7x+15x^2+10x^3+x^4}{1+6x+10x^2+4x^3}$
9	$\frac{1+8x+21x^2+20x^3+5x^4}{1+7x+15x^2+10x^3+x^4}$
10	$\frac{1+9x+28x^2+35x^3+15x^4+x^5}{1+8x+21x^2+20x^3+5x^4}$

Table 2: The Fibonacci rational functions.

(Exercise 1). For example, $\phi_2 = 1+x/\phi_1 = 1+x/1 = 1+x$, $\phi_3 = 1+x/\phi_2 = 1+x/(1+x) = (1+2x)/(1+x)$; see Table 2 (and Exercise 2). We will call the functions of the sequence $\{\phi_n(x)\}$ the Fibonacci rational functions.‡

The graph of ϕ_n, i.e., the curve $y = \phi_n(x)$, is shown in Figure 1 as a solid curve for $n = 1, ..., 6$. These graphs lie alternately below and above the dashed curve (which is the same in each diagram), getting closer to it with each successively larger value of n. What function does this dashed curve represent? Let us call it ϕ_∞. Then for large enough n it is impossible to tell the difference between $\phi_n(x)$ and $\phi_{n+1}(x)$, because both are indistinguishable from $\phi_\infty(x)$.Thus (8b) implies

$$\phi_\infty \;=\; 1+\frac{x}{\phi_\infty}, \tag{9}$$

from which it follows readily that

$$\phi_\infty \;=\; \phi_\infty(x) \;=\; \tfrac{1}{2}(1+\sqrt{1+4x}) \tag{10}$$

(Exercise 3). Note that ϕ_∞ is identical to the composition T at the end of Lecture 2.

Let us briefly take stock. On the one hand, Figure 1 very strongly suggests that the function sequence $\{\phi_n(x)\}$ converges as $n \to \infty$, in the sense that for any $x \in [0,1]$, $\phi_n(x)$ can be made as close as we please to whatever value the limiting function assigns to x by allowing n to become sufficiently large. On the other hand, we have shown that if $\{\phi_n(x)\}$ converges, then the limiting function can only be $\phi_\infty = \phi_\infty(x)$ defined by (10) above. Strictly speaking, however, we have not actually *proved* the result that

$$\lim_{n\to\infty} \phi_n(x) \;=\; \phi_\infty(x) \;=\; \tfrac{1}{2}(1+\sqrt{1+4x}) \tag{11}$$

‡Again, we use a single notation for both function and label, because it is obvious from context which meaning is intended.

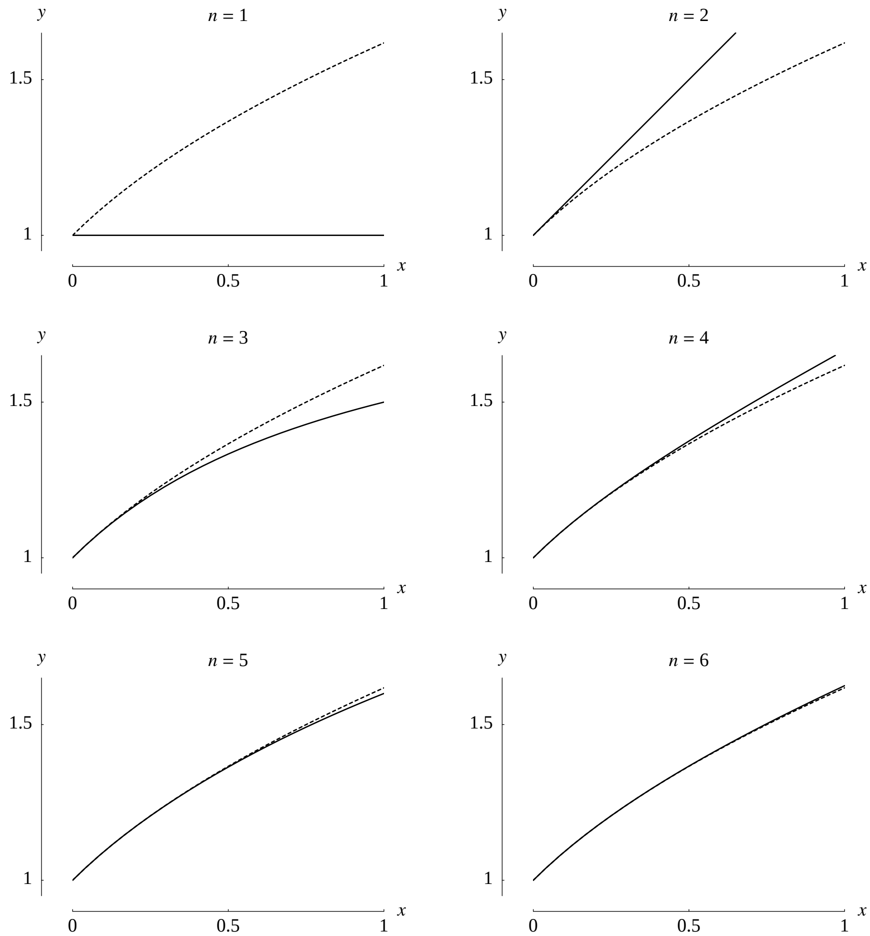


Figure 1: Oscillatory convergence of the Fibonacci rational function sequence. The solid curves show $y = \phi_n(x)$ for $n = 1, \ldots, 6$. The dashed curve, the same in each diagram, is $y = \phi_\infty(x)$.

for $0 \leq x \leq 1$. Nevertheless, the result is true, and a proof appears in the appendix.

The upshot of all the above is that the limit of a convergent function sequence is yet another function. But convergence is a two-sided coin. Its other side is that a function can be defined as the limit of a function sequence.

Suppose, for example, that your savings account earns compound interest at an annual rate of x (usually quoted as $100x$ %, e.g., a rate of 6% means $x = 0.06$). If you deposit a dollar today, how much will it be worth a year from now? The answer depends on how often the interest is compounded. If the interest is compounded only once, at the end of the year, then your dollar is worth only $1 + x$. If the interest is compounded twice, once after six months and again at year's end, then the dollar is worth $1 + \frac{1}{2}x$ dollars after six months, and whatever you have after six months is worth $1 + \frac{1}{2}x$ times as much at year's end. In other words, at the end of the year your dollar is worth $(1+\frac{1}{2}x)(1+\frac{1}{2}x) = (1+\frac{1}{2}x)^2$. Similarly, if the interest is compounded quarterly, then after three months your dollar is worth $1 + \frac{1}{4}x$, and at year's end it is worth $(1 + \frac{1}{4}x)^4$.

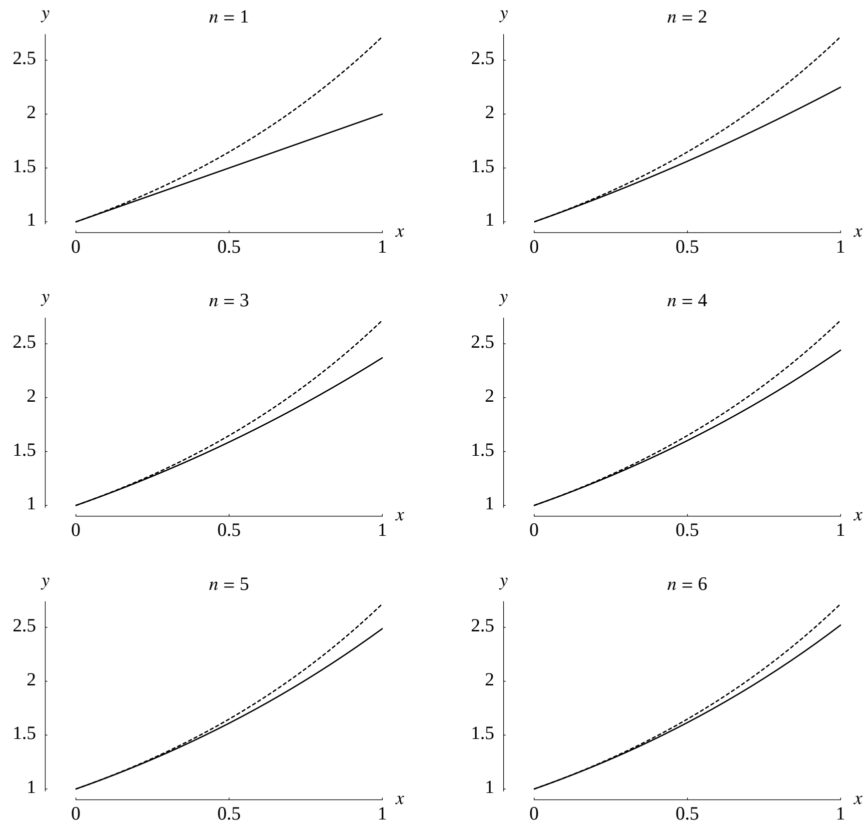


Figure 2: Monotonic convergence of the compound-interest function sequence defined by (12). The solid curves show $y = \omega_n(x)$ for $n = 1, \ldots, 6$. The dashed curve is $y = \exp(x)$.

This argument generalizes. Let $\omega_n(x)$ be how much your dollar is worth at year's end if interest is compounded n times a year. Then $\{\omega_n(x)\}$ is a function sequence defined by

$$\omega_n(x) \;=\; \left(1 + \frac{x}{n}\right)^n . \tag{12}$$

The sequence is graphed in Figures 2-3, where $y = \omega_n(x)$ is shown as a solid curve for $n = 1, ..., 6$ in Figure 2 and for $n = 2^m$, where $m = 1, \ldots, 6$, in Figure 3. Note that the solid curves converge from below toward the dashed curve, which we denote by $y = \omega_\infty(x)$. Because $\omega_\infty(x)$ is the limit of $\omega_n(x)$ as $n \to \infty$, $\omega_\infty(x)$ tells you how much your dollar would be worth at year's end, at interest rate x, if interest were compounded continuously from the moment you put your dollar in the bank. It is such an important function in mathematics that we give it a special name, the *exponential function*, and we denote it by the symbol $\exp$. Thus $\exp$ is defined by

$$\exp(x) \;=\; \omega_\infty(x) \;=\; \lim_{n\to\infty} \omega_n(x). \tag{13}$$

The domain on which this definition is valid turns out to be $(-\infty, \infty)$, although only the restriction of $\exp$ to $[0, 1]$ is graphed in Figures 2-3.§ Note in particular that

$$\exp(0) \;=\; 1 \tag{14}$$

because $\omega_n(0) = 1$ for every value of n—and therefore remains so in the limit as $n \to \infty$.

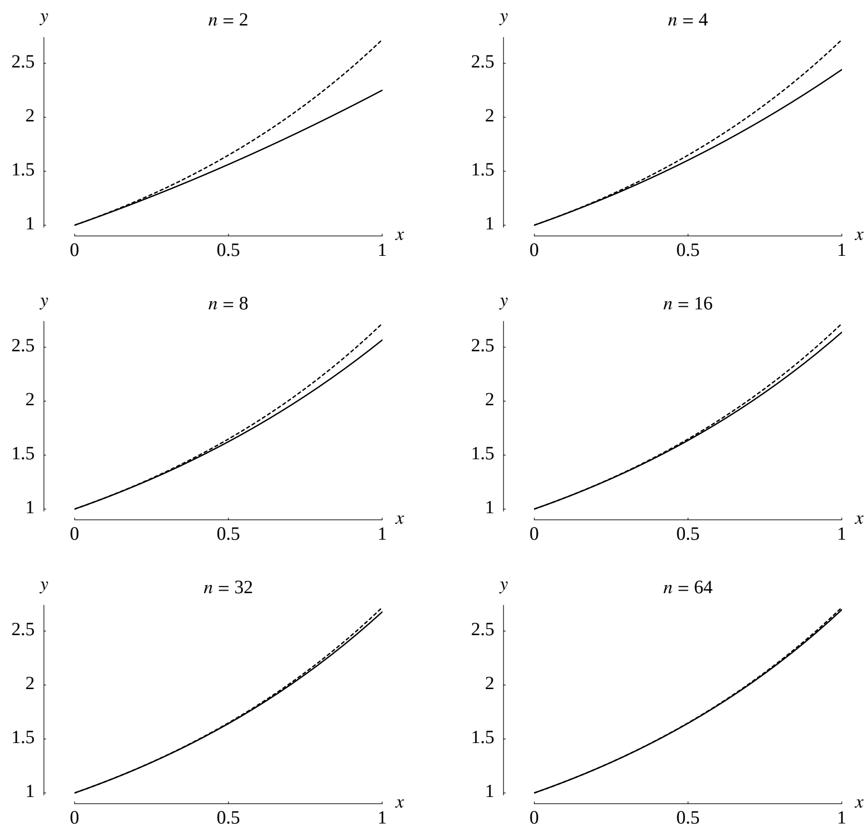


Figure 3: Monotonic convergence of the compound-interest function sequence defined by (12). The solid curves show $y = \omega_n(x)$ for $n = 2^m$, where $m = 1, \ldots, 6$. The dashed curve is $y = \exp(x)$.

It can be shown that $\exp$ is positive and strictly increasing on its domain, from which several things follow at once: its range is $(0, \infty)$; it is invertible; and the inverse function is also strictly increasing, with domain $(0, \infty)$ and range $(-\infty, \infty)$. This inverse is just as important in mathematicas as the exponential itself, and so it also has a special name: we call it the *logarithmic* function and denote it by the symbol $\ln$ (for natural logarithm). The graphs of $\exp$ and $\ln$ are sketched in Figure 4; note that (14) implies

$$\ln(1) \;=\; 0. \tag{15}$$

Further properties of $\exp$ and $\ln$ will be discussed in later lectures, especially Lecture 9.

§Larger values of x are anyhow clearly unrealistic in the case of compound interest . . . I think—but if you know of such a bank, please tell me at once!

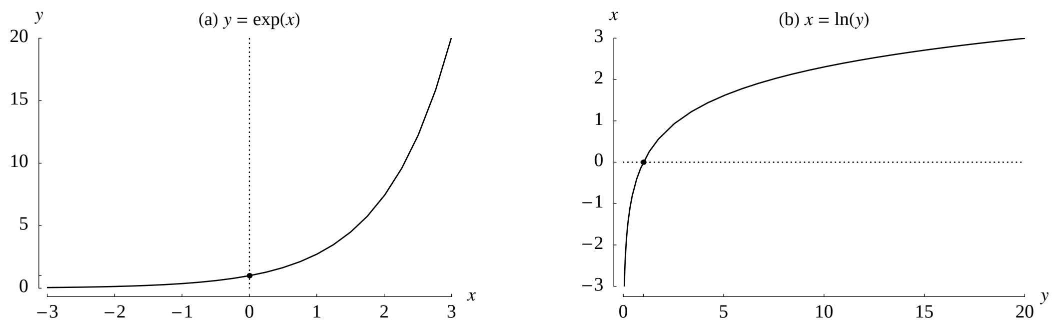


Figure 4: Graphs of (a) the exponential function and (b) its inverse, the logarithmic function.

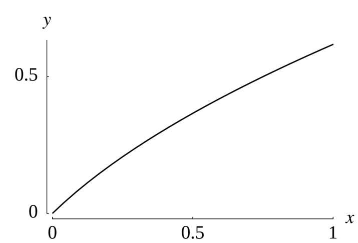


Figure 5: $y = \frac{2x}{1+\sqrt{1+4x}}$.

Appendix: The convergence of the Fibonacci rational function sequence

The purpose of this appendix is to establish the convergence of the sequence $\{\phi_n(x)\}$ defined by (8). Each function in this sequence has domain $[0, 1]$. When $x = 0$, however, (8) implies that $\phi_n = 1$ for all n. So $\{\phi_n(0)\}$ is clearly convergent to $\phi_\infty(0) = 1$, and we can safely assume that $0 < x \leq 1$, so that (8) implies $\phi_n > 1$ for all $n \geq 2$.

Subtracting (9) from (8b), we obtain

$$\phi_{n+1} - \phi_\infty \;=\; -\frac{x}{\phi_n \phi_\infty}\{\phi_n - \phi_\infty\}, \tag{16}$$

so that

$$|\phi_{n+1} - \phi_\infty| \;=\; \frac{x}{\phi_n \phi_\infty}\,|\phi_n - \phi_\infty| \;<\; \frac{x}{\phi_\infty}\,|\phi_n - \phi_\infty| \;=\; \frac{2x}{1+\sqrt{1+4x}}\,|\phi_n - \phi_\infty| \tag{17}$$

for all $n \geq 2$. But $\frac{2x}{1+\sqrt{1+4x}}$ is strictly increasing with respect to x (Figure 5), and so cannot exceed $\frac{2}{1+\sqrt{5}} \approx 0.618$ for any $x \in (0, 1]$. Thus (17) implies

$$|\phi_{n+1} - \phi_\infty| \;<\; 0.62\,|\phi_n - \phi_\infty|\,, \tag{18}$$

regardless of the value of x. That is, the distance between ϕ_n and ϕ_∞ is reduced by at least 38% at each iteration of the recurrence relation, and must eventually approach zero. Moreover, from (16), if $\phi_n > \phi_\infty$ then $\phi_{n+1} < \phi_\infty$, and vice versa; that is, the convergence is oscillatory.

Exercises

1. Verify (8).

2. Verify Table 2.

3. Verify (10).

4. Show that the inverse of g defined by $g(x) = 1/\exp(x)$ is g^{-1} defined by $g^{-1}(y) = \ln(1/y)$.

5. (a) Verify that the function f defined by

$$f(x) \;=\; \frac{1}{\exp(1-4x)}$$

is increasing on $[0,1]$.

(b) What is the range of f?

(c) Find an expression for $f^{-1}(y)$.

6. The function sequence $\{s_n(x) \mid n \geq 0, 0 \leq x \leq 4\}$ is defined recursively by

$$\begin{aligned} s_0 &= 1 \\ s_{n+1} &= \frac{1}{2}\left(s_n + \frac{x}{s_n}\right), \quad \text{for } n \geq 0 \end{aligned}$$

(a) Find rational-function expressions for $s_1(x)$, $s_2(x)$, $s_3(x)$ and $s_4(x)$.

(b) What function s_∞ is defined on $[0,4]$ by $s_\infty(x) = \lim_{n\to\infty} s_n(x)$?

(c) Verify graphically that the function sequence converges from above (in the sense that $s_n(x) \geq s_\infty(x)$ for $n \geq 1$).

7. The function sequence $\{s_n(x) \mid n \geq 0, 0 \leq x \leq 8\}$ is defined recursively by

$$\begin{aligned} s_0 &= 1 \\ s_{n+1} &= \frac{1}{2}\left(s_n + \frac{x}{s_n^{\,2}}\right), \quad \text{for } n \geq 0 \end{aligned}$$

(a) Find rational-function expressions for $s_1(x)$, $s_2(x)$ and $s_3(x)$.

(b) What function s_∞ is defined on $[0,8]$ by $s_\infty(x) = \lim_{n\to\infty} s_n(x)$?

(c) Verify graphically that the function sequence converges.

8. A sequence $\{H_n(x)\}$ of polynomials called the Hermite polynomials is defined by the recurrence relation

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 2x \\ H_{n+1} &= 2(xH_n - nH_{n-1}) \quad \text{if} \quad n \geq 1. \end{aligned}$$

Show that $H_4(x) = 4(4x^4 - 12x^2 + 3)$, and find $H_7(x)$.

9. A sequence $\{L_n(x)\}$ of polynomials called the Laguerre polynomials is defined by the recurrence relation

$$\begin{aligned} L_0 &= 1 \\ L_1 &= 1 - x \\ L_{n+1} &= (2n+1-x)L_n - n^2 L_{n-1} \quad \text{if} \quad n \geq 1. \end{aligned}$$

Show that $L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$, and find $L_6(x)$.

10. A sequence $\{P_n(x)\}$ of polynomials called the Legendre polynomials is defined by the recurrence relation

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x \\ P_{n+1} &= \frac{(2n+1)x}{n+1} P_n - \frac{n}{n+1} P_{n-1} \quad \text{if} \quad n \geq 1. \end{aligned}$$

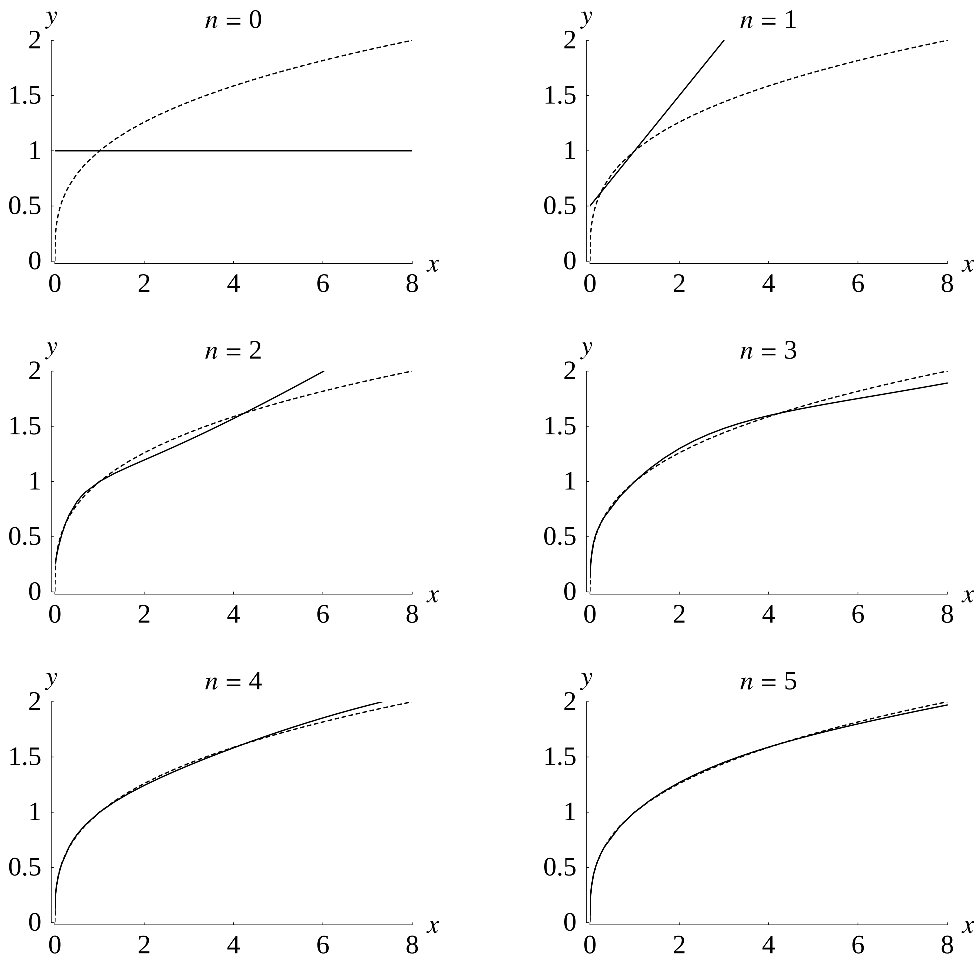
Show that $P_3(x) = \frac{1}{2}x(5x^2 - 3)$, and find $P_5(x)$.

11. The compositions f and g are defined by $f(x) = H_3(L_2(x))$ and $g(x) = L_2(H_3(x))$, where H_3 and L_2 are defined in Exercises 8-9. Find explicit expressions for the polynomials $f(x)$ and $g(x)$. What are their orders?

12. The compositions f and g are defined by $f(x) = P_2(L_3(x))$ and $g(x) = L_3(P_2(x))$, where L_3 and P_2 are defined in Exercises 9-10. Find explicit expressions for the polynomials $f(x)$ and $g(x)$. What are their orders?

Solutions or hints for selected exercises

7. (a) $s_1(x) = \frac{1}{2}(x+1)$, $s_2(x) = \frac{x^3+3x^2+11x+1}{4(x+1)^2}$ and
$s_3(x) = \frac{x^9+9x^8+124x^7+612x^6+1638x^5+2462x^4+2492x^3+756x^2+97x+1}{8(x^4+4x^3+14x^2+12x+1)^2}$.
(b) As $n \to \infty$, $s_n \to s_\infty \Longrightarrow s_{n+1} \to s_\infty$; and so, letting $n \to \infty$ in $s_{n+1} = \frac{1}{2}(s_n + x s_n^{-2})$, we obtain $s_\infty = \frac{1}{2}(s_\infty + x s_\infty^{-2})$, which is readily solved to yield $s_\infty = s_\infty(x) = \sqrt[3]{x}$.
(c) See the diagram below. The solid curves are $y = s_n(x)$ for $n = 0, \ldots, 5$ and the dashed curve is $y = s_\infty(x)$.



8. $H_7(x) = 16x(8x^6 - 84x^4 + 210x^2 - 105)$.

9. $L_6(x) = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$.

10. $P_5(x) = \frac{1}{8}x(63x^4 - 70x^2 + 15)$.