9. Properties of the exponential and logarithm

To derive further properties of the exponential and logarithm, which as you know are inverse functions, we begin by noting that the chain rule yields an exceedingly simple and useful result when applied to a composition of two inverses. Recall from Lecture 8 that if x is related to y according to $y = U(x)$ and z is in turn related to y according to $z = P(y)$ then the implied relationship between x and z satisfies

$$\frac{dz}{dx} \;=\; \frac{dz}{dy}\,\frac{dy}{dx}. \tag{1}$$

If U and P are inverse functions, however, i.e., if $P = U^{-1} \Leftrightarrow U = P^{-1}$, then $y = U(x) \Longrightarrow x = P(y) \Longrightarrow z = x$, and so (1) implies $\frac{dx}{dx} = \frac{dx}{dy}\frac{dy}{dx}$. But $\frac{dx}{dx} = 1$. So $1 = \frac{dx}{dy}\frac{dy}{dx}$, implying

$$\frac{dx}{dy} \;=\; \left\{\frac{dy}{dx}\right\}^{-1}. \tag{2}$$

So it isn't necessary to find the derivative of both the exponential and the logarithm: if we obtain the derivative of the exponential, then that of the logarithm will follow from (2).

To find the derivative of the exponential, we first recall from Lecture 7 that

$$\exp(x) \;=\; \lim_{n\to\infty} \omega_n(x) \tag{3}$$

where

$$\omega_n(x) \;=\; \left(1 + \frac{x}{n}\right)^n. \tag{4}$$

We can differentiate (4) by using the chain rule. Set $y = U(x) = 1 + \frac{x}{n} \Longrightarrow \frac{dy}{dx} = 0 + \frac{1}{n}\frac{dx}{dx} = \frac{1}{n}$ and $z = P(y) = y^n \Longrightarrow \frac{dz}{dy} = ny^{n-1}$. Then $\omega_n(x) = z$,

$$\frac{d}{dx}\{\omega_n(x)\} \;=\; \frac{dz}{dx} \;=\; \frac{dz}{dy}\,\frac{dy}{dx} \;=\; ny^{n-1}\cdot\frac{1}{n} \;=\; \left(1+\frac{x}{n}\right)^{n-1} \;=\; \frac{\omega_n(x)}{1+\frac{x}{n}} \tag{5}$$

and, assuming that the derivative of a function's limit equals the limit of the function's derivative,* we have

$$\begin{aligned}\frac{d}{dx}\{\exp(x)\} \;&=\; \frac{d}{dx}\left\{\lim_{n\to\infty}\omega_n(x)\right\} \;=\; \lim_{n\to\infty}\frac{d}{dx}\{\omega_n(x)\} \;=\; \lim_{n\to\infty}\frac{\omega_n(x)}{1+\dfrac{x}{n}}\\ &=\; \frac{\lim\limits_{n\to\infty}\omega_n(x)}{1 + x\lim\limits_{n\to\infty}\frac{1}{n}} \;=\; \frac{\exp(x)}{1+x\cdot 0} \;=\; \exp(x)\end{aligned} \tag{6}$$

by the combination rule. So the exponential is its own derivative:

$$y \;=\; \exp(x) \Longrightarrow \frac{dy}{dx} \;=\; \exp(x). \tag{7}$$

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*Which it does, but we'll skip the details; however, see Exercises 1-2.

It follows immediately from (2) that the derivative of the inverse function is given by

$$\frac{dx}{dy} \;=\; \left\{\frac{dy}{dx}\right\}^{-1} \;=\; \frac{1}{\exp(x)} \;=\; \frac{1}{y} \tag{8}$$

for all $y > 0$. That is,

$$x \;=\; \ln(y) \Longrightarrow \frac{dx}{dy} \;=\; \frac{1}{y}. \tag{9}$$

Needless to say, (9) means not only that $\frac{d}{dy}\ln(y) = 1/y$, but also $\frac{d}{dx}\ln(x) = 1/x$, because these are just two different ways of stating precisely the same result.

We now establish four important properties of the logarithmic function, namely,

$$\ln(AB) \;=\; \ln(A) + \ln(B) \qquad \text{for any positive } A \text{ and } B \tag{10a}$$

$$\ln\Big(\frac{A}{B}\Big) \;=\; \ln(A) - \ln(B) \qquad \text{for any positive } A \text{ and } B \tag{10b}$$

$$\ln\Big(\frac{1}{A}\Big) \;=\; -\ln(A) \qquad \text{for any positive } A \tag{10c}$$

$$\ln\left(A^r\right) \;=\; r\,\ln(A) \qquad \text{for any positive } A \text{ and any rational } r. \tag{10d}$$

All of these properties follow from exploiting the observation that a constant function is the only function whose derivative is precisely zero everywhere on its domain—because only a horizontal straight line can have zero slope at every point. For example, to obtain (10a) we proceed as follows. First we define

$$f(x) \;=\; \ln(bx) - \ln(x) - \ln(b) \tag{11}$$

for $x > 0$, where b is any positive number. From the chain rule with $y = bx$ and $z = \ln(y)$,

$$\frac{d}{dx}\{\ln(bx)\} \;=\; \frac{d}{dx}\{\ln(y)\} \;=\; \frac{dz}{dx} \;=\; \frac{dz}{dy}\frac{dy}{dx} \;=\; \frac{1}{y}\cdot b \;=\; \frac{b}{y} \;=\; \frac{1}{x} \tag{12}$$

and so $f'(x) \;=$

$$\frac{d}{dx}\{\ln(bx) - \ln(x) - \ln(b)\} \;=\; \frac{d}{dx}\{\ln(bx)\} - \frac{d}{dx}\{\ln(x)\} - \frac{d}{dx}\{\ln(b)\} \;=\; \frac{1}{x} - \frac{1}{x} - 0 \;=\; 0$$

for all $x > 0$, so that $f(x)$ must be constant. Constant means independent of x. Therefore,

$$f(x) \;=\; f(1) \tag{13}$$

But $f(1) = \ln(b) - \ln(1) - \ln(b) = -\ln(1) = 0$. Therefore $f(x) = 0$ for all $x > 0$, implying $\ln(x) = \ln(x) + \ln(b)$ for any positive x. In particular, the result must hold for $x = A$ and $b = B$, implying (10a). Both (10b) and (10c) now follow readily, because each is a special case of (10a); see Exercise 4.

To obtain (10d) we set

$$f(x) \;=\; \ln\left(x^r\right) - r\ln(x), \quad x > 0 \tag{14}$$

where r is any rational number and recall from Lecture 7 that now

$$\frac{d}{dx}\{x^r\} = r\,x^{r-1}. \tag{15}$$

From the chain rule with $y = x^r$ and $z = \ln(y)$, we obtain

$$\frac{d}{dx}\{\ln(x^r)\} = \frac{d}{dx}\{\ln(y)\} = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{1}{y}\cdot rx^{r-1} = \frac{rx^{r-1}}{x^r} = \frac{r}{x}. \tag{16}$$

So $f'(x) =$

$$\frac{d}{dx}\{\ln(x^r) - r\ln(x)\} = \frac{d}{dx}\{\ln(x^r)\} - r\frac{d}{dx}\{\ln(x)\} = \frac{r}{x} - r\cdot\frac{1}{x} = 0$$

for all $x > 0$, so that again $f(x)$ must be constant, implying $f(x) = f(1)$. But $f(1) = \ln(1^r) - r\ln(1) = \ln(1) - r\ln(1) = (1-r)\ln(1) = (1-r)\cdot 0 = 0$. Therefore $\ln(x^r) = r\ln(x)$ for any positive x, implying (10d).

Because the exponential and logarithm are inverse functions, however, i.e., because $y = \exp(x) \Longleftrightarrow x = \ln(y)$, to the four logarithmic properties (10) there must correspond the following four exponential properties:

$$\exp(a+b) = \exp(a)\,\exp(b) \qquad \text{for any } a \text{ and } b \tag{17a}$$

$$\exp(a-b) = \frac{\exp(a)}{\exp(b)} \qquad \text{for any } a \text{ and } b \tag{17b}$$

$$\frac{1}{\exp(a)} = \exp(-a) \quad \text{for any } a \tag{17c}$$

$$\{\exp(a)\}^r = \exp(ar) \quad \text{for any } a \text{ and any rational } r. \tag{17d}$$

For example, if $\exp(a) = A$ and $\exp(b) = B$ then $a = \ln(A)$ and $b = \ln(B)$, so that $a + b = \ln(A)+\ln(B)$, implying $a+b = \ln(AB)$, and hence $AB = \exp(a+b)$. So we have shown that $\exp(a)\,\exp(b) = \exp(a+b)$, or (17a). The other properties follow similarly; see Exercise 5.

You already know that if a and b are integers or rational numbers and $c > 0$ then

$$c^{-a} = \frac{1}{c^a}, \quad c^0 = 1, \quad c^{a+b} = c^a\,c^b. \tag{18}$$

But we have now established (here and in Lecture 7) that

$$\exp(-a) = \frac{1}{\exp(a)}, \quad \exp(0) = 1, \quad \exp(a+b) = \exp(a)\,\exp(b). \tag{19}$$

Thus the effect of $\exp$, acting on a, is to raise some number, say e, to the power of a. Whatever number e is, if we raise it to the power of 1 then it must yield itself. So

$$e = \exp(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.718281828 \tag{20}$$

as in (28) of Lecture 3, and in place of $\exp(x)$ we write e^x henceforth. So (7) becomes

$$y = e^x \Longrightarrow \frac{dy}{dx} = e^x. \tag{21}$$

Now we know why $\exp$ is called the exponential function.

Given these properties, for any x and $c > 0$ we *define* c to the power of x by

$$c^x = e^{x\ln(c)}. \tag{22}$$

With this definition, properties of exponents are extended at once from rational numbers to all real numbers. For example, we have $(c^x)^w = (c^w)^x$ because

$$\begin{aligned} (c^x)^w &= \left(e^{x\ln(c)}\right)^w = e^{w\ln(e^{x\ln(c)})} = e^{w\,x\ln(c)} \\ &= e^{x\,w\ln(c)} = e^{x\ln(e^{w\ln(c)})} = \left(e^{w\ln(c)}\right)^x = (c^w)^x. \end{aligned} \tag{23}$$

Moreover, derivatives for (22) and any power function follow at once from the chain rule:

$$\frac{d}{dx}\{c^x\} = \frac{d}{dx}\left\{e^{x\ln(c)}\right\} = e^{x\ln(c)}\frac{d}{dx}\{x\ln(c)\} = e^{x\ln(c)}\ln(c) = c^x\ln(c) \tag{24}$$

and, for an arbitrary real (but not necessarily rational) exponent r we have

$$\frac{d}{dx}\{x^r\} = \frac{d}{dx}\left\{e^{r\ln(x)}\right\} = e^{r\ln(x)}\frac{d}{dx}\{r\ln(x)\} = x^r\frac{d}{dx}\{r\ln(x)\} = x^r r\frac{1}{x} = rx^{r-1}. \tag{25}$$

The last result extends (15) from the rationals to the reals.

Exercises

1. To obtain the derivative of the exponential function, we had to assume that the derivative of a function's limit (as $n \to \infty$) equals the limit of the function's derivative. Although this result is true (for a very broad class of functions including all those which interest us this semester), its proof is well beyond our scope. Nevertheless, we can at least convince ourselves that the result is highly plausible by testing it on functions whose derivatives we already know. In this spirit, show that

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} = \lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\}$$

for the function sequence $\{f_n(x)\}$ defined by

$$f_n(x) = \frac{nx^2+1}{n+x^2+1}$$

2. Similarly, show that

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} = \lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\}$$

for the function sequence $\{f_n(x)\}$ defined by

$$f_n(x) = \frac{n^2x+n+x^2}{n^2x^2+2nx+10x^3}$$

3. Similarly, show that

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} \;=\; \lim_{n\to\infty} \frac{d}{dx}\{f_n(x)\}$$

for the function sequence $\{f_n(x)\}$ defined by

$$f_n(x) \;=\; \sin\left(\frac{nx^2+1}{n+x}\right)$$

4. Obtain (10b)-(10c).
Hint: First set $B = 1/A$ in (10a) to obtain (10c), and then deduce (10b).

5. Obtain (17b)-(17d).
Hint: With $\exp(a) = A$ and $\exp(b) = B$, the method for (17b) is virtually identical to that for (17a); for (17c), first use the inverse property to rewrite (10c) as $1/A = \exp(-\ln(A))$; and for (17c), first use the inverse property to rewrite (10d) as $A^r = \exp(r\ln(A))$.

6. Exploit the observation that $f'(x) = 0 \implies f(x) =$ constant to prove that $\sin^2(x) + \cos^2(x) \;=\; 1$ (without using Pythagoras' theorem or anything like that).

7. Exploit the observation that $f'(x) = 0 \implies f(x) =$ constant in a clever way to prove both that $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ and that $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.

Suitable problems from standard calculus texts

Stewart (2003): p. 191, ## 11, 12, 14, 17, 28, 32, 33, 39; p. 197, ## 12-18, 25, 26, 29, 33, 42; p. 216, ## 8, 21-24; pp. 224-225, ## 5, 6, 15, 16, 21-24, 28, 31, 36, 42-47(a); and p. 249, ##1-48.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

2. Because $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to\infty}\frac{1}{n^2} = 0$ we have

$$\lim_{n\to\infty} f_n(x) \;=\; \lim_{n\to\infty}\frac{n^2x+n+x^2}{n^2x^2+2nx+10x^3} \;=\; \lim_{n\to\infty}\frac{x+\frac{1}{n}+x^2\cdot\frac{1}{n^2}}{x^2+2x\cdot\frac{1}{n}+10x^3\cdot\frac{1}{n^2}}$$
$$=\; \frac{x+0+x^2\cdot 0}{x^2+2x\cdot 0+10x^3\cdot 0} \;=\; \frac{x}{x^2} \;=\; \frac{1}{x}$$

on using the combination rule. Hence

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} \;=\; \frac{d}{dx}\left\{\frac{1}{x}\right\} \;=\; \frac{-1}{x^2}.$$

On the other hand, by the quotient rule,

$$\begin{aligned}
\frac{d}{dx}\{f_n(x)\} \;&=\; \frac{d}{dx}\left\{\frac{n^2x+n+x^2}{n^2x^2+2nx+10x^3}\right\} \\
&= \frac{\frac{d}{dx}\{n^2x+n+x^2\}(n^2x^2+2nx+10x^3)-(n^2x+n+x^2)\frac{d}{dx}\{n^2x^2+2nx+10x^3\}}{(n^2x^2+2nx+10x^3)^2} \\
&= \frac{(n^2+0+2x)(n^2x^2+2nx+10x^3)-(n^2x+n+x^2)(2n^2x+2n+30x^2\}}{x^2(n^2x+2n+10x^2)^2} \\
&= \frac{-n^4x^2-2xn^3-2(1+20x^3)n^2-28x^2n-10x^4}{x^2(n^4x^2+4xn^3+4(1+5x^3)n^2+40x^2n+100x^4)} \\
&= \frac{-x^2-2x\cdot\frac{1}{n}-2(1+20x^3)\cdot\frac{1}{n^2}-28x^2\cdot\frac{1}{n^3}-10x^4\cdot\frac{1}{n^4}}{x^2(x^2+4x\cdot\frac{1}{n}+4(1+5x^3)\cdot\frac{1}{n^2}+40x^2\cdot\frac{1}{n^3}+100x^4\cdot\frac{1}{n^4})}
\end{aligned}$$

Hence, on using the combination rule, we have

$$\lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\} \;=\; \frac{-x^2-2x\cdot 0-2(1+20x^3)\cdot 0-28x^2\cdot 0-10x^4\cdot 0}{x^2(x^2+4x\cdot 0+4(1+5x^3)\cdot 0+40x^2\cdot 0+100x^4\cdot 0)} \;=\; \frac{-1}{x^2}$$

So

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} \;=\; \lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\}$$

as required.

3. In this case we have

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) \;&=\; \lim_{n\to\infty}\sin\left(\frac{nx^2+1}{n+x}\right) \;=\; \lim_{n\to\infty}\sin\left(\frac{x^2+\frac{1}{n}}{1+x\cdot\frac{1}{n}}\right) \\
&=\; \sin\left(\frac{x^2+0}{1+x\cdot 0}\right) \;=\; \sin(x^2)
\end{aligned}$$

on using the combination rule. Hence

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} \;=\; \frac{d}{dx}\left\{\sin(x^2)\right\} \;=\; 2x\cos(x^2)$$

by the chain rule. On the other hand, also by the chain rule,

$$\begin{aligned}
\frac{d}{dx}\{f_n(x)\} &= \frac{d}{dx}\left\{\sin\left(\frac{nx^2+1}{n+x}\right)\right\} = \cos\left(\frac{nx^2+1}{n+x}\right)\frac{d}{dx}\left\{\frac{nx^2+1}{n+x}\right\} \\
&= \cos\left(\frac{nx^2+1}{n+x}\right)\frac{\frac{d}{dx}\{nx^2+1\}(n+x)-(nx^2+1)\frac{d}{dx}\{n+x\}}{(n+x)^2} \\
&= \cos\left(\frac{nx^2+1}{n+x}\right)\frac{(2nx+0)(n+x)-(nx^2+1)(0+1)}{(n+x)^2} \\
&= \cos\left(\frac{nx^2+1}{n+x}\right)\frac{2xn^2+x^2n-1}{n^2+2xn+x^2} \\
&= \cos\left(\frac{x^2+\frac{1}{n}}{1+x\cdot\frac{1}{n}}\right)\frac{2x+x^2\cdot\frac{1}{n}-\frac{1}{n^2}}{1+2x\cdot\frac{1}{n}+x^2\cdot\frac{1}{n^2}}
\end{aligned}$$

Hence, on using the combination rule, we have

$$\lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\} = \cos\left(\frac{x^2+0}{1+x\cdot 0}\right)\frac{2x+x^2\cdot 0-0}{1+2x\cdot 0+x^2\cdot 0} = 2x\cos(x^2)$$

So

$$\frac{d}{dx}\left\{\lim_{n\to\infty} f_n(x)\right\} = \lim_{n\to\infty}\frac{d}{dx}\{f_n(x)\}$$

as required.

7. Define $y = f(x) = u^2 + v^2$ where

$$\begin{aligned}
u &= \sin(x+b) - \{\sin(x)\cos(b) + \cos(x)\sin(b)\} \\
v &= \cos(x+b) - \{\cos(x)\cos(b) - \sin(x)\sin(b)\}
\end{aligned}$$

implying

$$\begin{aligned}
\frac{du}{dx} &= \cos(x+b) - \{\cos(x)\cos(b) - \sin(x)\sin(b)\} = v \\
\frac{dv}{dx} &= -\sin(x+b) + \{\sin(x)\cos(b) + \cos(x)\sin(b)\} = -u
\end{aligned}$$

after use of the chain rule and simplification. Then, after routine algebraic manipulations, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\{u^2+v^2\} = \frac{d}{dx}\{u^2\} + \frac{d}{dx}\{v^2\} \\
&= 2u\frac{du}{dx} + 2v\frac{dv}{dx} = 2uv + 2v\{-u\} = 0
\end{aligned}$$

for all values of x, implying that y must be a constant, whose value must be $f(0) = \{\sin(0+b)-\sin(0)\cos(b)-\cos(0)\sin(b)\}^2+\{\cos(0+b)-\cos(0)\cos(b)+\sin(0)\sin(b)\}^2 = \{\sin(b)-\sin(b)\}^2+\{\cos(b)-\cos(b)\}^2 = 0$. So $y = 0$, implying $u^2+v^2 = 0$. But the only way for the sum of two squared terms to equal zero is for each of those terms to be zero. Hence $u = 0 = v$. The desired results follow from setting $x = A$ and $b = B$.