10. Finite sums and infinite series

You already know what a sequence is, and from any sequence you can induce another by adding the first \( n \) terms. Usually, we use a lower-case letter for the original sequence and an upper-case letter for the induced sequence, i.e., the sequence of finite sums; thus, if the original sequence is \( \{s_k\} \), then the sequence of finite sums will be \( \{S_n\} \) defined by

\[
S_n = s_1 + s_2 + \ldots + s_{n-1} + s_n = \sum_{k=1}^{n} s_k
\]  

(1)

(Using the summation notation introduced at the end of Lecture 3). Often there’s a trick for obtaining a simple expression for \( S_n \), in other words, a formula for the finite sum. For example, if \( s_k = k \) then we can write the sum twice as follows, first forwards, then backwards:

\[
S_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = \sum_{k=1}^{n} k
\]

(2)

\[
S_n = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 = \sum_{k=1}^{n} \{n - (k-1)\}.
\]

Now, if we add each term to the one directly above, then we find that the result is \( n + 1 \) in all \( n \) cases. So twice \( S_n \) must sum to \( n \) times \( n + 1 \). That is, \( 2S_n = n(n + 1) \) or

\[
S_n = 1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1).
\]

(3)

For example, \( 1 + 2 + 3 + 4 + \ldots + 99 + 100 = \frac{1}{2} \times 100 \times 101 = 5050 \). But that particular trick in essence works only for the example on which we have used it.

A trick that works much more often is to rewrite \( s_k \) as the difference between the \( k \)-th and the \( (k - 1) \)-th term of a judiciously chosen different sequence, say \( \{p_k\} \). For if

\[
s_k = p_k - p_{k-1}
\]

(4)

for all \( k = 1, \ldots, n \) then*  

\[
S_n = s_n + s_{n-1} + s_{n-2} + s_{n-3} + \ldots + s_4 + s_3 + s_2 + s_1 = (p_n - p_{n-1}) + (p_{n-1} - p_{n-2}) + (p_{n-2} - p_{n-3}) + \ldots
\]

\[
+ (p_4 - p_3) + (p_3 - p_2) + (p_2 - p_1) + (p_1 - p_0)
\]

\[
= p_n + (-p_{n-1} + p_{n-1}) + (-p_{n-2} + p_{n-2}) + (-p_{n-3} + p_{n-3}) + \ldots
\]

\[
+ (-p_3 + p_3) + (-p_2 + p_2) + (-p_1 + p_1) - p_0
\]

\[
= p_n - p_0
\]

(5)

because all terms cancel in pairs, except for the very first and last.

*Note that (4) implies \( \{p_k\} = \{p_k \mid k \geq 0\} \), whereas \( \{s_k\} = \{s_k \mid k \geq 1\} \)
Suppose, for example, that we wish to calculate the finite sum
\[
\frac{1}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots + \left(\frac{2}{3}\right)^n = \sum_{k=1}^{n} \left(\frac{2}{3}\right)^k,
\]
for which \(s_k = \left(\frac{1}{3}\right)^k\). Here a judicious choice is
\[
p_k = -\frac{3}{2} \left(\frac{1}{3}\right)^{k+1},
\]
because now
\[
p_k - p_{k-1} = -\frac{3}{2} \left(\frac{1}{3}\right)^{k+1} + \frac{3}{2} \left(\frac{1}{3}\right)^k = \frac{3}{2} \cdot \left\{ -\frac{1}{3} + 1 \right\} \left(\frac{1}{3}\right)^k
\]
\[= \frac{3}{2} \cdot 2 \cdot \left(\frac{1}{3}\right)^k = \left(\frac{1}{3}\right)^{k} = s_k
\]
and (5) implies
\[
S_n = p_n - p_0 = -\frac{3}{2} \left(\frac{1}{3}\right)^{n+1} + \frac{3}{2} \left(\frac{1}{3}\right)^{0+1} = \frac{1}{2} \left\{1 - \left(\frac{1}{3}\right)^n\right\}.
\]
In other words,
\[
\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k = \frac{1}{2} \left\{1 - \left(\frac{1}{3}\right)^n\right\}.
\]
For example, \(\frac{1}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \left(\frac{2}{3}\right)^7 = \frac{1}{2} \left\{1 - \left(\frac{1}{3}\right)^7\right\} = \frac{1093}{2187}\).
An even more judicious choice, namely,
\[
p_k = \frac{ar^k}{r - 1}
\]
yields the standard formula for the sum of a finite geometric sum of \(n\) terms with first term \(a\) and ratio \(r\), where \(r \neq 1\). For (11) implies
\[
p_k - p_{k-1} = \frac{ar^k}{r - 1} - \frac{ar^{k-1}}{r - 1} = \frac{ar^k - ar^{k-1}}{r - 1} = \frac{ar^{k-1}(r - 1)}{r - 1} = ar^{k-1}.
\]
So, on setting
\[
s_k = ar^{k-1}
\]
in (5) we obtain
\[
S_n = p_n - p_0 = \frac{ar^n}{r - 1} - \frac{a r^0}{r - 1} = \frac{a(r^n - 1)}{r - 1}.
\]
In other words,
\[
\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} = \frac{a(r^n - 1)}{r - 1},
\]
For \(r = 1\) we need no tricks:
\[
\sum_{k=1}^{n} a = a + a + a + \ldots + a + a = na.
\]
\[
\begin{align*}
\sum_{k=1}^{n} k &= 1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1) \\
\sum_{k=1}^{n} k^2 &= 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \\
\sum_{k=1}^{n} k^3 &= 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n + 1)^2 \\
\sum_{k=1}^{n} k^4 &= 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{1}{30}n(n + 1)(2n + 1)(3n^2 + 3n - 1) \\
\sum_{k=1}^{n} k^5 &= 1^5 + 2^5 + 3^5 + \ldots + n^5 = \frac{1}{12}n^2(n + 1)^2(2n^2 + 2n - 1) \\
\sum_{k=1}^{n} k^6 &= 1^6 + 2^6 + 3^6 + \ldots + n^6 = \frac{1}{42}n(n + 1)(2n + 1)(3n^4 + 6n^3 - 3n + 1)
\end{align*}
\]

Table 1: The sums of the powers of the first \(n\) integers for the first six integer exponents.

In sum:

\[
\sum_{k=1}^{n} a r^{k-1} = \begin{cases} 
\frac{a(r^n-1)}{r-1} & \text{if } r \neq 1 \\
\frac{na}{1-r} & \text{if } r = 1.
\end{cases}
\]

(17)

Of course, (10) is the special case for which \(n = 7\) and \(a = r = \frac{1}{3}\).

The only thing about the trick that is the slightest bit, well, tricky is judiciously guessing \(p_k\). But for all of the results in Table 1—some of which are needed in Lecture 12—the judicious choice of \(p_k\) always turns out to be just \(S_k\) itself . . . which you know, because you know \(S_n\) from the table. Now you know everything you need to know to complete Exercise 1 by yourself.

In many of the above cases, \(s_k\) increases with \(k\), and so \(\{s_k\}\) and \(\{S_n\}\) both diverge. If \(s_k \to 0\) as \(k \to \infty\), however, it is possible for the sequence \(\{S_n\}\) to converge to a limit, which we denote by \(S_\infty\). That is,

\[
S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} s_k = \sum_{k=1}^{\infty} s_k.
\]

(18)

Then the right-hand side of (18) is called an infinite series and is usually written as

\[
\sum_{k=1}^{\infty} s_k \quad \text{or} \quad s_1 + s_2 + s_3 + s_4 + \ldots
\]

(19)

The left-hand side of (18) is the sum to which this infinite series converges. For example, because\(^\dagger\)

\[
\lim_{n \to \infty} r^n = 0 \quad \text{whenever } |r| < 1,
\]

(20)

\(^\dagger\)See Equation (22) of Lecture 3.
(15) implies that
\[ \sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \ldots \]
\[ = \lim_{n \to \infty} \frac{a(r^n - 1)}{r - 1} = \frac{a(0 - 1)}{r - 1} = \frac{a}{1 - r} \quad \text{whenever} \ |r| < 1 \]  
(21)

(on using the limit combination rule). Thus the infinite geometric series with first term \(a\) and ratio \(r\) converges to the sum \(\frac{a}{1-r}\) whenever \(|r| < 1\) (but diverges whenever \(|r| \geq 1\)).

In particular, for \(a = r = \frac{1}{3}\) we obtain
\[ \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \ldots = \frac{1}{2}. \]  
(22)

As a further example, consider the infinite series
\[ \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)}. \]  
(23)

Here \(s_k = \frac{1}{k(k+1)} = \frac{k-1}{k} = \frac{k+1 - k}{k} \). So with \(p_k = \frac{k}{k+1}\) in (4)-(5) we obtain
\[ S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = p_n - p_0 = \frac{n}{n+1} - 0 = \frac{n}{n+1} \quad \text{(24)} \]

and thus deduce from (18)-(19) that
\[ S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = \frac{1}{1 + 0} = 1. \]  
(25)

In other words, \(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \ldots \) (forever) = 1.

**Exercises**

1. Verify Table 1.

2. Show that
\[ \frac{1}{1.2.4} + \frac{1}{2.3.5} + \frac{1}{3.4.6} + \frac{1}{4.5.7} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+3)} = \frac{7}{36}. \]

**Hint:** A judicious choice is \(p_k = \frac{k(7k^2 + 42k + 59)}{36(k+1)(k+2)(k+3)}\).

**Suitable problems from standard calculus texts**

Stewart (2003): p. 720, #14, 15, 17-20 and 26 (for which use \(p_n = \frac{n(5n+13)}{6(n+2)(n+3)}\)).

**Reference**