11. The definite integral

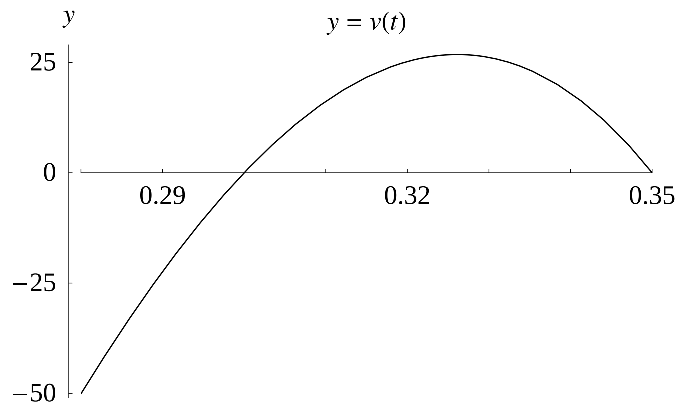


Figure 1: Ventricular inflow at end of the systolic phase. The graph shows the restriction to $[0.28, 0.35]$ of that sketched in Figure 3b of Lecture 1; here $v(t) = \frac{350}{9}(1 - 20t)(3 - 10t)(7 - 20t)$.

Figure 1 shows ventricular inflow at the end of the ejection phase in our cardiac cycle, as defined by the function v introduced in Figure 3 of Lecture 1. On [0.28, 0.3) we have $v(t) < 0$, corresponding to arterial outflow: the ventricle is still draining. On (0.3, 0.35) we have $v(t) > 0$, corresponding to the arterial backflow that closes the aortic valve: the ventricle refills very slightly. In 0.07 seconds, inflow increases from $v(0.28) = -50.1$ ml/s to $v(0.3) = 0$ to $v(0.326) = 26.8$ ml/s before decreasing again to zero at $t = 0.35$s. For the first fiftieth of a second, blood flows out of the ventricle; but for the next twentieth of a second, blood flows back in. So how much blood flows in or out, overall? In other words, what is net transport of blood by the flow? The purpose of this lecture is to answer that question by introducing the concept of definite integral, and to establish some of the definite integral's properties.

We begin by asking what it really means for the inflow to be -50.1ml/s at $t = 0.28$s. It means that *if* flow continued at precisely this rate for the next 0.01s then $-50.1 \times 0.01 = -0.501$ml of blood would flow into the ventricle. In other words, 0.501ml of blood would be discharged into the aorta. This volume of discharge is numerically equal to the area of the shaded rectangle below [0.28, 0.29] in Figure 9a. If we regard shaded area as positive or negative according to whether it is above or below the horizontal axis, then the signed area of the rectangle is -0.501 and represents the (negative) volume of blood that would be transported into the ventricle—if flow continued at -50.1ml/s for 0.01s.

But it doesn't, of course, because by $t = 0.29$s it has already increased to $v(0.29) = -22.4$ml/s. If this higher (i.e., less negative) rate had instead been maintained for the same hundredth of a second, then the ventricular recharge would instead have been $-22.4 \times 0.01 = -0.224$ml. In other words, 0.224ml of blood would have been discharged into the aorta; this volume is numerically equal to the area of the shaded rectangle below [0.28, 0.29] in Figure 9b, and the corresponding signed area is -0.224. The true volume of blood transported into the ventricle during [0.28, 0.29] must be somewhere in between: it is underestimated by -0.501 ml, but it is overestimated by -0.224.

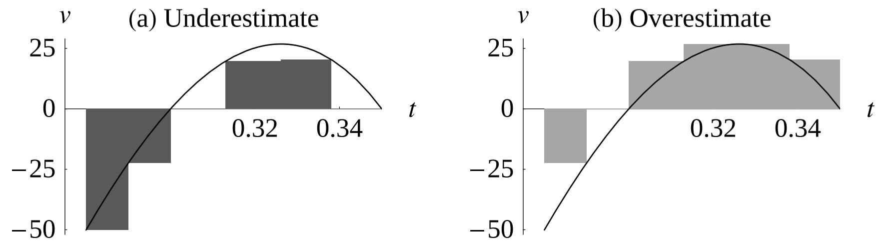


Figure 2: Net amount of blood transported into the ventricle during $[0.28, 0.35]$ is underestimated by the sum of the six signed areas of the darker shaded rectangles (which is -0.2239) and overestimated by the sum of the six signed areas of the lighter shaded rectangles (which is 0.9467) .

-0.501	$<$	NET TRANSPORT OF BLOOD DURING [0.28, 0.29]	$<$	-0.224
-0.224	$<$	NET TRANSPORT OF BLOOD DURING [0.29, 0.3]	$<$	0
0	$<$	NET TRANSPORT OF BLOOD DURING [0.3, 0.313]	$<$	0.258
0.258	$<$	NET TRANSPORT OF BLOOD DURING [0.313, 0.326]	$<$	0.35
0.243	$<$	NET TRANSPORT OF BLOOD DURING [0.326, 0.338]	$<$	0.32
0	$<$	NET TRANSPORT OF BLOOD DURING [0.338, 0.35]	$<$	0.243
-0.22	$<$	NET TRANSPORT OF BLOOD DURING [0.28, 0.35]	$<$	0.95

Table 1: Crude under- and overestimates of ventricular recharge at end of the systolic phase. Note that $[0.3, 0.35] = [0.3, \frac{1}{2}\{0.3 + c\}] \cup [\frac{1}{2}\{0.3 + c\}, c] \cup [c, \frac{1}{2}\{c + 0.35\}] \cup [\frac{1}{2}\{c + 0.35\}, 0.35]$ where $c \approx 0.326129$ is the global maximizer on $[0.28, 0.35]$; but numbers have been rounded to three significant figures. Thus, e.g., [0.313, 0.326] rounds $[\frac{1}{2}\{0.3+c\}, c]$, and the bounds in the fourth line of the table round $\frac{1}{2}\{c-0.3\}v\left(\frac{1}{2}\{0.3 + c\}\right) \approx 0.257982$ and $\frac{1}{2}\{c-0.3\}v(c) \approx 0.350018$, respectively.

A similar analysis applies to [0.29, 0.3], on which flow increases from $v(0.29) = -22.4$ml/s to $v(0.3) = 0$ml/s: recharge is underestimated by $0.01 \times v(0.29) = -0.224$ml (signed area of shaded rectangle under [0.29, 0.3] in Figure 9a) but overestimated by $0.01 \times v(0.3) = 0$ml (no shaded rectangle over [0.29, 0.3] in Figure 9b). Thus net recharge during [0.28, 0.3], on which v increases with respect to time but flow is never positive, is greater than $-0.501 - 0.224 = -0.725$ml but less than $-0.224 + 0 = -0.224$ml. Continuing in this manner, we readily obtain the results in Table 1. Summing, we find that the net volume of blood transported into the ventricle during [0.28, 0.35] exceeds -0.22ml (sum of signed areas of darker shaded rectangles in Figure 9a) but is less than 0.95ml (sum of signed areas of lighter shaded rectangles in Figure 9b).

Needless to say, these bounds are perfectly useless, but we obtained them by dividing [0.28, 0.35] into only six subintervals. We can improve our estimates by dividing each subinterval into two equal pieces, but otherwise proceeding as before. The result is shown in Figure 3. We can obtain even more accurate estimates if we double the number of subintervals again, from 12 to 24, as shown in Figure 4. In fact, we can improve the accuracy indefinitely, by continually doubling the number of subintervals as illustrated by Figure 5. At each doubling, net transport of blood into the ventricle during [0.28, 0.35]

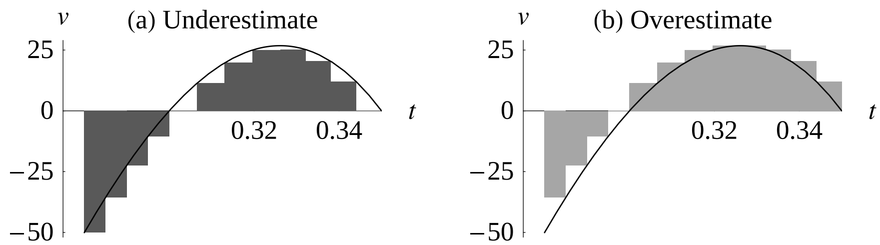


Figure 3: Net amount of blood transported into the ventricle during $[0.28, 0.35]$ is underestimated by the sum of the 12 signed areas of the darker shaded rectangles (which is 0.1169) and overestimated by the sum of the 12 signed areas of the lighter shaded rectangles (which is 0.7022) .

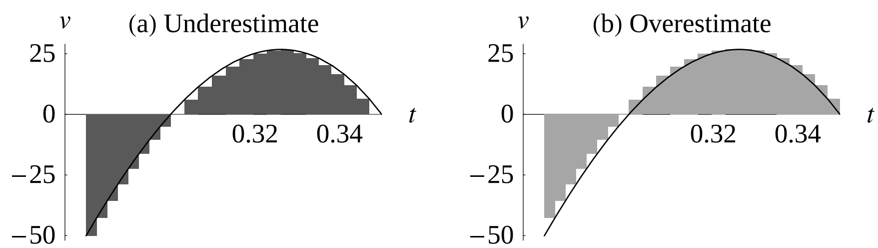


Figure 4: Net amount of blood transported into the ventricle during $[0.28, 0.35]$ is underestimated by the sum of the 24 signed areas of the darker shaded rectangles (which is 0.2752) and overestimated by the sum of the 24 signed areas of the lighter shaded rectangles (which is 0.5679).

is underestimated by the total signed area of the darker shaded rectangles but overestimated by the total signed area of the lighter shaded rectangles. In the limit as the number of doublings approaches infinity, however, the two signed areas must coincide, with true net transport sandwiched in between them. Thus true net transport of blood is the common limit—say L—of both an increasing sequence of underestimates and a decreasing sequence of overestimates. Whatever the value of L, from Figure 5 (with $n = 7$) we have $0.4164 < L < 0.4347$; and by the end of the following lecture, we will be able to show that

$$L \;=\; \tfrac{22981}{54000} \;\approx\; 0.4256. \tag{1}$$

It now appears that total signed area is a sufficiently important quantity to deserve its own notation, and a good notation will take account of everything that quantity can possibly depend on. The total signed area between the horizontal axis and the graph of f on subdomain $[a, b]$—counted positively above the axis, and negatively below it—depends on both f and $[a, b]$. But it does not depend on anything else. Moreover, for reasons that will become apparent later, the accepted name for total signed area associated with f on domain $[a, b]$ is "definite integral of f from a to b," and the standard symbol for integral is "$\int$." So it appears that a good notation would be $\int(f, a, b)$; that is, we should

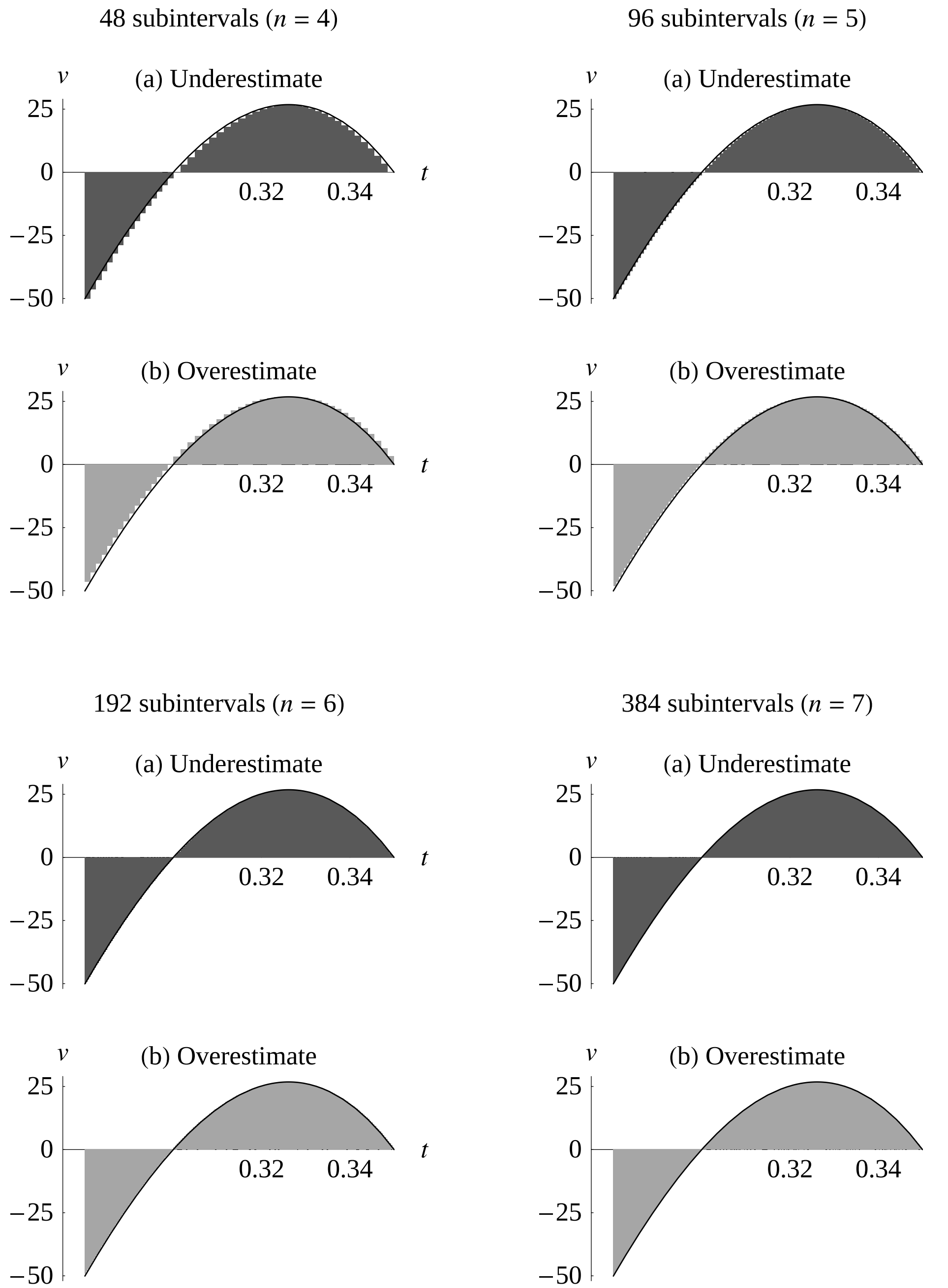


Figure 5: For each value of n, net amount of blood transported into the ventricle during $[0.28, 0.35]$ is underestimated by the sum of $6 \cdot 2^{n-1}$ signed areas of darker shaded rectangles (0.3514, 0.3887, 0.4072 and 0.4164 for $n = 4$, 5, 6 and 7, respectively) and overestimated by the sum of $6 \cdot 2^{n-1}$ signed areas of lighter shaded rectangles (0.4977, 0.4619, 0.4438 and 0.4347 for $n = 4$, 5, 6 and 7, respectively).

write

$$\int (f, a, b) \;=\; \begin{cases} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a, b] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{cases} \tag{2a}$$

Traditionally, however, the above notation is not considered sufficiently evocative of the limiting process through which we reach the definite integral. Through that process, we take the limit of the sum of the areas of enough infinitesimally wide rectangular strips to cover the domain in the limit as their thickness approaches zero (and hence their number approaches infinity). The typical rectangle at station t has thickness δt, height $f(t)$ and hence area $f(t)\,\delta t$. So it would be more evocative to write

$$\lim_{\delta t \to 0} \sum_{t \in [a,b]} f(t)\,\delta t \;=\; \begin{cases} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a, b] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{cases} \tag{2b}$$

Nevertheless, it would also be extremely cumbersome. So the conventional notation is a compromise: in place of either (2a) or (2b) we write

$$\int_a^b f(t)\,dt \;=\; \begin{cases} \text{total signed area between the graph of } f \text{ and} \\ \text{segment } [a, b] \text{ of the horizontal axis, counted} \\ \text{positively above the axis, negatively below it.} \end{cases} \tag{2c}$$

We interpret $\int_a^b f(t)\,dt$ as a single, stand-alone symbol for the common right-hand side of (2a)-(2c). Compared to (2a), it has the distinct *dis*advantage of suggesting that the definite integral from a to b depends on t, which it *absolutely does not*—from that point of view, (2a) is a far superior notation. On the other hand, it has the advantage that the function whose definite integral is being calculated need not be explicitly named (which, with only 26 letters in the alphabet, is not to be sneezed at). In any event, (2c) is the notation we will use, and to use it successfully we must always bear in mind that it depends on f, a and b—but it *does not depend in any way* on the "dummy" variable t we temporarily use to identify the location of a generic infinitesimal strip before taking the limit that defines the definite integral. To put it another way,

$$\int\limits_a^b f(t)\,dt, \qquad \int\limits_a^b f(x)\,dx \qquad \text{and even} \qquad \int\limits_a^b f(\bullet)\,d\bullet$$

are all *precisely* the same thing. For example, from the definition of definite integral in (2), Figure 6 establishes both that

$$\int_a^b 1\,dt \;=\; b - a \;=\; \int_a^b 1\,dx \tag{3}$$

and that

$$\int_a^b t\,dt \;=\; \tfrac{1}{2}(b^2 - a^2) \;=\; \int_a^b x\,dx. \tag{4}$$

By interpreting $\int_a^b v(t)\,dt$ as the net recharge due to the inflow v we can deduce some important properties of the definite integral. First, if v is net inflow, then $-v$ is net outflow

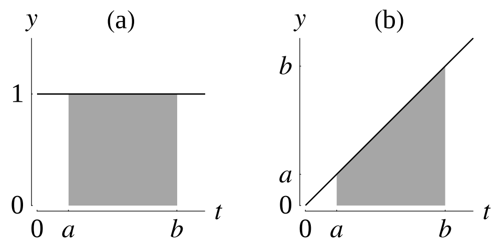


Figure 6: (a) The shaded area between $y = 1$ and the horizontal axis is that of rectangle with area $(b-a)\cdot 1$. (b) The shaded area between $y = t$ and the horizontal axis is that of a trapezium with area $\frac{1}{2}(b-a)(b+a)$.

(see Figure 4 of Lecture 1). Hence $\int_a^b \{-v(t)\}\,dt$ is net discharge from the ventricle during $[a, b]$. But net discharge is simply the negative of net recharge; hence

$$\int_a^b \{-v(t)\}\,dt \;=\; -\int_a^b v(t)\,dt. \tag{5}$$

Second, because, e.g., doubling or tripling an inflow doubles or triples associated net recharge, $\int_a^b \{2v(t)\}\,dt = 2\int_a^b \{v(t)\}\,dt$, $\int_a^b \{3v(t)\}\,dt = 3\int_a^b \{v(t)\}\,dt$, and so on. In general, changing the flow by a factor of k will change the associated recharge by a factor of k or

$$\int_a^b \{k\,v(t)\}\,dt \;=\; k\int_a^b v(t)\,dt. \tag{6}$$

Third, suppose that two venules converge at C to form a vein, as cartooned in Figure 7. At time t, let $u(t)$ ml/s and $v(t)$ ml/s be the outflows from the venules at C. Then total flow into the vein at C (in ml/s) must be $u(t) + v(t)$, because there is nowhere else for blood to go. For the same reason, discharge into the vein during any interval $[a, b]$ must equal total discharge out of the venule, or

$$\int_a^b \{u(t) + v(t)\}\,dt \;=\; \int_a^b u(t)\,dt \;+\; \int_a^b v(t)\,dt. \tag{7}$$

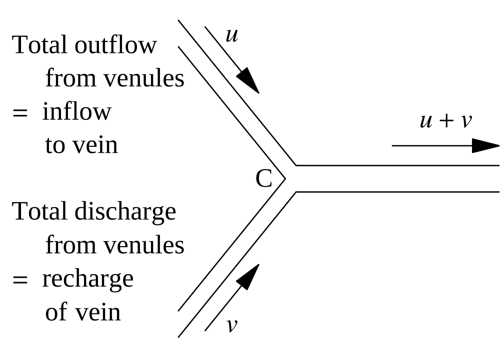


Figure 7: A pictorial proof of (7).

Now we note an important point: even if functions u and v were not realistic as inflows, we could still *pretend* that they were inflows, and none of their properties could thereby change. Thus (5)-(7) do not apply only to inflows: they are general properties of definite integrals. That is, for arbitrary functions f and g, the following always hold:

$$\int_a^b \{-f(t)\}\,dt \;=\; -\int_a^b f(t)\,dt \tag{8a}$$

$$\int_a^b \{k\,f(t)\}\,dt \;=\; k\int_a^b f(t)\,dt \tag{8b}$$

$$\int_a^b \{f(t)+g(t)\}\,dt \;=\; \int_a^b f(t)\,dt \;+\; \int_a^b g(t)\,dt. \tag{8c}$$

Furthermore, we may combine them into a single equation as follows:

$$\int_a^b \{q\,f(t)+k\,g(t)\}\,dt \;=\; q\int_a^b f(t)\,dt \;+\; k\int_a^b g(t)\,dt \tag{8d}$$

where q and k are any real numbers (positive, negative or zero). We recover (7) as the special cases where $q=0$ and $k=-1$, where $q=0$, and where $k=q=1$, respectively. A further general property of definite integrals is that

$$\int_a^b f(t)\,dt \;=\; \int_a^c f(t)\,dt \;+\; \int_c^b f(t)\,dt \tag{9}$$

for any c such that $a \le c \le b$; see Figure 8.

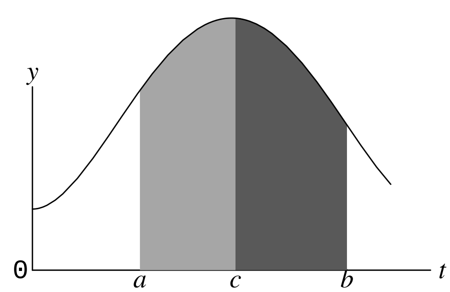


Figure 8: A pictorial proof of (9). The lighter shading represents the signed area $\int_a^c f(t)\,dt$, the darker shading represents the signed area $\int_c^b f(t)\,dt$ and the combined shading represents the signed area $\int_a^b f(t)\,dt$; all signed areas are drawn positive, but the result clearly holds in general.

In Lecture 12 we will use the above properties to establish (1), i.e., that

$$\int_{0.28}^{0.35} v(t)\,dt \;=\; = \;\tfrac{22981}{54000} \;\approx\; 0.4256, \tag{10}$$

where $v(t)$ is defined by the caption to Figure 1. But we aren't quite ready to establish (10) yet. Therefore, to illustrate the usefulness of (8)-(9), we suppose that $v(t)$ is defined not as

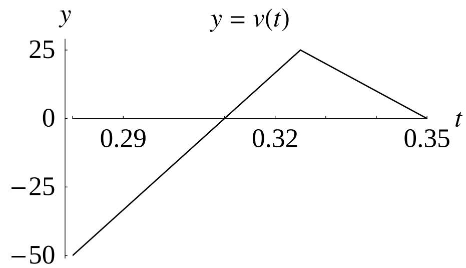


Figure 9: Ventricular inflow at end of the systolic phase according to (11).

in Figure 1, but instead by the join whose graph is sketched in Figure 9 (and for which no realism is claimed):

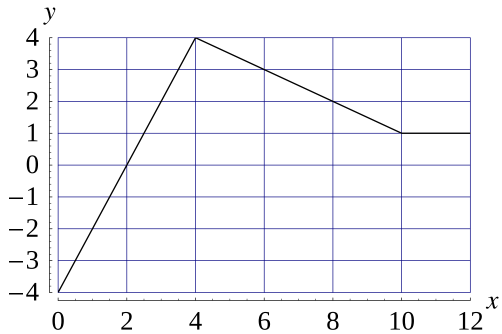
$$v(t) \;=\; \begin{cases} \frac{5000}{3}t - \frac{1550}{3} & \text{if } \ 0.28 \le t < 0.325 \\ 350 - 1000t & \text{if } \ 0.325 \le t \le 0.35 \end{cases} \tag{11}$$

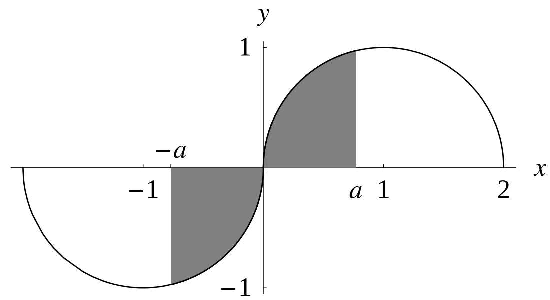
Then, from (9) and successive applications of (8) and (3)-(4), the net recharge is

$$\begin{aligned} \int_{0.28}^{0.35} v(t)\,dt &= \int_{0.28}^{0.325} v(t)\,dt + \int_{0.325}^{0.35} v(t)\,dt \\ &= \int_{0.28}^{0.325} \left\{\tfrac{5000}{3}t - \tfrac{1550}{3}\cdot 1\right\}\,dt + \int_{0.325}^{0.35} \{350\cdot 1 - 1000t\}\,dt \\ &= \tfrac{5000}{3}\int_{0.28}^{0.325} t\,dt - \tfrac{1550}{3}\int_{0.28}^{0.325} 1\,dt + 350\int_{0.325}^{0.35} 1\,dt - 1000\int_{0.325}^{0.35} t\,dt \\ &= \tfrac{5000}{3}\cdot\tfrac{(0.325)^2-(0.28)^2}{2} - \tfrac{1550}{3}\cdot\{0.325 - 0.28\} \\ &\qquad + 350\cdot\{0.35 - 0.325\} - 1000\cdot\tfrac{(0.35)^2-(0.325)^2}{2} \\ &= -0.25\text{ml} \end{aligned} \tag{12}$$

(equivalent to a net discharge of 0.25ml).

Exercises





1. A piecewise-linear join g is defined by the graph $y = g(x)$ shown above on the left.

(a) Find an explicit expression for $g(x)$ for all $x \in [0, 12]$

(b) Calculate $\int_0^4 g(x)\,dx$

(c) Calculate $\int_0^{12} g(x)\,dx$

(d) It is known that $4 < b < 10$. Calculate $\int_0^b g(x)\,dx$.

2. A continuous join f is defined on $[-2, 2]$ by

$$f(x) \;=\; \begin{cases} -\sqrt{-x(2+x)} & \text{if} \quad -2 \le x < 0 \\ \sqrt{x(2-x)} & \text{if} \quad 0 \le x \le 2. \end{cases}$$

Because $y = \sqrt{-x(2+x)} \implies (x+1)^2 + y^2 = 1^2$, which is the equation of a circle of radius 1 with center (-1,0), and because $y = \sqrt{x(2-x)} \implies (x-1)^2 + y^2 = 1^2$, which is the equation of a circle of radius 1 with center (1,0), the graph of f consists of two semi-circles, as illustrated above on the right. Calculate each of the following definite integrals:

(a) $\int_{-2}^{-1} f(x)\,dx$

(b) $\int_{-2}^{0} f(x)\,dx$

(c) $\int_{-2}^{1} f(x)\,dx$

(d) $\int_{-1}^{2} f(x)\,dx$

(e) $\int_0^1 f(x)\,dx$

(f) $\int_0^2 f(x)\,dx$

(g) $\int_{-a}^{a} f(x)\,dx$ for any $a \in [0, 1]$.

3. A piecewise-linear function f is defined on $[0, 6]$ by

$$f(x) \;=\; \begin{cases} 2 & \text{if} \quad 0 \le x < 1 \\ 2x & \text{if} \quad 1 \le x < 2 \\ 8 - 2x & \text{if} \quad 2 \le x < 4 \\ 0 & \text{if} \quad 4 \le x \le 6. \end{cases}$$

Show that $\int_0^3 f(x)\,dx = 8$.

4. A piecewise-linear function f is defined on $[0, 7]$ by

$$f(x) = \begin{cases} 3+2x & \text{if} \quad 0 \le x < 2 \\ 13-3x & \text{if} \quad 2 \le x < 4 \\ x-3 & \text{if} \quad 4 \le x < 5 \\ 2 & \text{if} \quad 5 \le x \le 7. \end{cases}$$

Show that $\int_0^7 f(x)\,dx = \frac{47}{2}$.

Suitable problems from standard calculus texts

Stewart (2003): p. 392, ## 33-40, 43 and 50.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

1. **(a)**

$$f(x) = \begin{cases} 2(x-2) & \text{if} \quad 0 \le x < 4 \\ 6-\frac{1}{2}x & \text{if} \quad 4 \le x < 10 \\ 1 & \text{if} \quad 10 \le x \le 12. \end{cases}$$

(b) 0 **(c)** 17 **(d)** $6b - \frac{1}{4}b^2 - 20$.

2. **(a)** $-\frac{1}{4}\pi$ **(b)** $-\frac{1}{2}\pi$ **(c)** $-\frac{1}{4}\pi$ **(d)** $\frac{1}{4}\pi$ **(e)** $\frac{1}{4}\pi$ **(f)** $\frac{1}{2}\pi$ **(g)** 0.