12. More on the definite integral

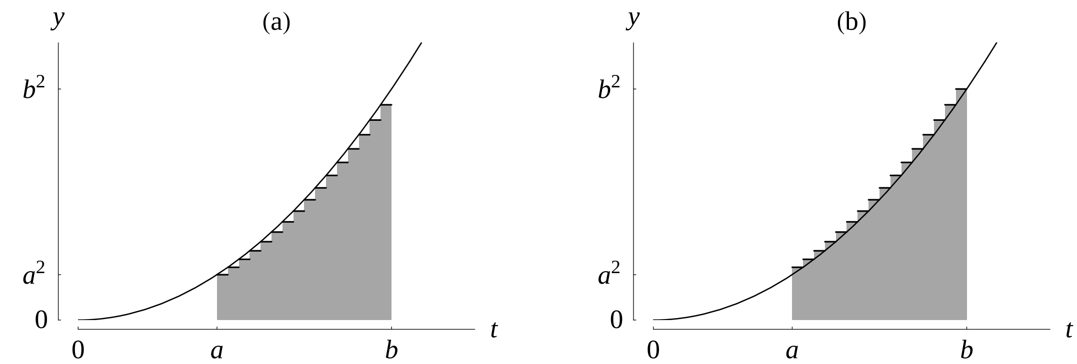

Figure 1: (a) Left-hand and (b) right-hand approximation to $\int_a^b f(t)\,dt$ for $n = 16$.

When f is neither a constant nor a linear function, the key to evaluating $\int_a^b f(t)\,dt$ is to observe that piecewise-constant functions can approximate f as accurately as we please for the purpose of calculating its definite integral. The method we describe is perfectly general, but for the sake of definiteness we suppose that f is the function defined by

$$f(t) \;=\; t^2. \tag{1}$$

Figure 1 shows two piecewise-constant approximations to f on the interval $[a, b]$. The one on the left is the approximation f_{16} obtained by dividing $[a, b]$ into sixteen equal subintervals and insisting that the approximation is actually correct at the left-hand end of each subinterval; for that reason, we call f_{16} a left-hand (piecewise-constant) approximation. The one on the right is the approximation ϕ_{16} obtained by again dividing $[a, b]$ into sixteen equal subintervals but instead insisting that the approximation is actually correct at the right-hand end of each subinterval; for that reason, we call ϕ_{16} a (surprise, surprise!) right-hand approximation. More generally, for any f and with n equal subintervals, the n-th left-hand and n-th right-hand approximations f_n and ϕ_n are defined by

$$f_n(t) \;=\; f\left(a + \tfrac{(i-1)(b-a)}{n}\right) \quad \text{when} \quad a + \tfrac{(i-1)(b-a)}{n} \le t < a + \tfrac{i(b-a)}{n} \quad \text{for} \quad i = 1, \ldots, n \tag{2}$$

$$\phi_n(t) \;=\; f\left(a + \tfrac{i(b-a)}{n}\right) \quad \text{when} \quad a + \tfrac{(i-1)(b-a)}{n} \le t < a + \tfrac{i(b-a)}{n} \quad \text{for} \quad i = 1, \ldots, n. \tag{3}$$

Now here's the thing: for any f, both $\int_a^b f_n(t)\,dt$ and $\int_a^b \phi_n(t)\,dt$ are easy to calculate, because the region between segment $[a, b]$ of the horizontal axis and the graph of any piecewise-constant function is always piecewise-rectangular. In fact, because the constituent rectangles of f_n and ϕ_n all have precisely the same width, namely, $(b - a)/n$, we readily obtain the n-th left-hand approximation

$$\int_a^b f_n(t)\,dt \;=\; \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \tfrac{(i-1)(b-a)}{n}\right) \;=\; \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{(i-1)(b-a)}{n}\right) \tag{4}$$

and the n-th right-hand approximation

$$\int_a^b \phi_n(t)\,dt \;=\; \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + \tfrac{i(b-a)}{n}\right) \;=\; \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{i(b-a)}{n}\right) \tag{5}$$

to $\int_a^b f(t)\,dt$. Note how (because $b > a$) the sign of f will always guarantee that area is counted positively above the axis and negatively below it (although in this particular case, it is always counted positively).

As illustrated by Figure 1, for any increasing function f we have

$$\int_a^b f_n(t)\,dt \;<\; \int_a^b f(t)\,dt \;<\; \int_a^b \phi_n(t)\,dt \tag{6}$$

for all values of n, no matter how large (and regardless of the sign of f, although in Figure 1 f is always positive). The greater the value of n, the greater the accuracy of both overestimate and underestimate, with $\int_a^b f(t)\,dt$ always sandwiched between. In the limit as $n \to \infty$, inequalities (6) weaken, as $\int_a^b f_n(t)\,dt$ and $\int_a^b \phi_n(t)\,dt$ coalesce. That is, we have

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt \;\le\; \int_a^b f(t)\,dt \;\le\; \lim_{n\to\infty} \int_a^b \phi_n(t)\,dt \tag{7}$$

and

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt \;=\; \lim_{n\to\infty} \int_a^b \phi_n(t)\,dt, \tag{8}$$

so that of necessity

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt \;=\; \int_a^b f(t)\,dt \;=\; \lim_{n\to\infty} \int_a^b \phi_n(t)\,dt. \tag{9}$$

In other words, $\int_a^b f(t)\,dt$ is the limit as $n \to \infty$ of *either* $\int_a^b f_n(t)\,dt$ or $\int_a^b \phi_n(t)\,dt$. Furthermore, (9) is always true, even if f is not an increasing function: if f is decreasing, then because $\int_a^b \phi_n(t)\,dt < \int_a^b f(t)\,dt < \int_a^b f_n(t)\,dt$ for all n, and more generally because $[a, b]$ can always be subdivided into intervals on which f is either increasing, decreasing or constant. Indeed the common limit as $n \to \infty$ of $\int_a^b f_n(t)\,dt$ and $\int_a^b \phi_n(t)\,dt$ is the legal definition of the definite integral of f from a to b; however, we continue to think of the definite integral intuitively as merely signed area.

Because either $\int_a^b f_n(t)\,dt$ or $\int_a^b \phi_n(t)\,dt$ will yield $\int_a^b f(t)\,dt$ in the limit as $n \to \infty$, for $f(t) = t^2$ we are free to calculate only the second. Substituting into (5), we obtain

$$\int_a^b \phi_n(t)\,dt \;=\; \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \tfrac{i(b-a)}{n}\right) \;=\; \frac{b-a}{n} \sum_{i=1}^{n} \left\{a + \tfrac{i(b-a)}{n}\right\}^2. \tag{10}$$

But

$$\begin{aligned}\sum_{i=1}^{n}\left\{a+\tfrac{i(b-a)}{n}\right\}^2 &= \sum_{i=1}^{n}\left\{a^2\cdot 1+\tfrac{2a(b-a)i}{n}+\tfrac{(b-a)^2i^2}{n^2}\right\}\\ &= \sum_{i=1}^{n}a^2\cdot 1+\sum_{i=1}^{n}\tfrac{2a(b-a)i}{n}+\sum_{i=1}^{n}\tfrac{(b-a)^2i^2}{n^2}\\ &= a^2\sum_{i=1}^{n}1+\tfrac{2a(b-a)}{n}\sum_{i=1}^{n}i+\tfrac{(b-a)^2}{n^2}\sum_{i=1}^{n}i^2\\ &= a^2\cdot n+\tfrac{2a(b-a)}{n}\cdot\tfrac{1}{2}n(n+1)+\tfrac{(b-a)^2}{n^2}\cdot\tfrac{1}{6}n(n+1)(2n+1)\end{aligned} \tag{11}$$

on using Table 1 from Lecture 10, and because $1+1+1+\ldots+1$ (n times) $= n$. Substituting back into (10), we obtain

$$\begin{aligned}\int_a^b \phi_n(t)\,dt &= \tfrac{b-a}{n}\cdot a^2\cdot n+\tfrac{b-a}{n}\cdot\tfrac{2a(b-a)}{n}\cdot\tfrac{1}{2}n(n+1)+\tfrac{b-a}{n}\cdot\tfrac{(b-a)^2}{n^2}\cdot\tfrac{1}{6}n(n+1)(2n+1)\\ &= (b-a)\left\{a^2+a(b-a)\left(1+\tfrac{1}{n}\right)+\tfrac{1}{6}(b-a)^2\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\right\}\end{aligned}$$

after simplification. So

$$\begin{aligned}\int_a^b f(t)\,dt &= \lim_{n\to\infty}\int_a^b \phi_n(t)\,dt\\ &= (b-a)\lim_{n\to\infty}\left\{a^2+a(b-a)\left(1+\tfrac{1}{n}\right)+\tfrac{1}{6}(b-a)^2\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\right\}\\ &= (b-a)\left\{a^2+a(b-a)\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)+\tfrac{1}{6}(b-a)^2\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)\lim_{n\to\infty}\left(2+\tfrac{1}{n}\right)\right\}\\ &= (b-a)\left\{a^2+a(b-a)(1+0)+\tfrac{1}{6}(b-a)^2(1+0)(2+0)\right\}\\ &= (b-a)\left\{a^2+a(b-a)+\tfrac{1}{3}(b-a)^2\right\} = \tfrac{1}{3}(b-a)(a^2+ab+b^2) = \tfrac{1}{3}(b^3-a^3)\end{aligned} \tag{12}$$

after simplification. We have thus established that

$$\int_a^b t^2\,dt = \tfrac{1}{3}(b^3-a^3). \tag{13}$$

A very similar calculation shows that

$$\int_a^b t^3\,dt = \tfrac{1}{4}(b^4-a^4) \tag{14}$$

(Exercise 4). Furthermore, from Lecture 11, we already know that

$$\int_a^b 1\,dt = b-a, \qquad \int_a^b t\,dt = \tfrac{1}{2}(b^2-a^2). \tag{15}$$

Now we have all of the results we need to calculate the recharge associated with the ventricular inflow $v(t)=\frac{350}{9}(1-20t)(3-10t)(7-20t)$ sketched in Figure 1 of Lecture 11 for $t\in[0.28,0.35]$. First we rewrite $v(t)$ as a polynomial:

$$v(t) = \tfrac{2450}{3}-\tfrac{192500}{9}t+\tfrac{980000}{9}t^2-\tfrac{1400000}{9}t^3, \qquad \tfrac{7}{25}\le t\le\tfrac{7}{20}. \tag{16}$$

Then with $a = 0.28$ and $b = 0.35$ in (13)-(15) we obtain

$$\begin{aligned}
\int\limits_{0.28}^{0.35} v(t)\,dt &= \int\limits_{0.28}^{0.35} \left\{ \tfrac{2450}{3}\cdot 1 - \tfrac{192500}{9}\,t + \tfrac{980000}{9}\,t^2 - \tfrac{1400000}{9}\,t^3 \right\} dt \\
&= \tfrac{2450}{3}\int\limits_{0.28}^{0.35} 1\,dt - \tfrac{192500}{9}\int\limits_{0.28}^{0.35} t\,dt + \tfrac{980000}{9}\int\limits_{0.28}^{0.35} t^2\,dt - \tfrac{1400000}{9}\int\limits_{0.28}^{0.35} t^3\,dt \\
&= \tfrac{2450}{3}\cdot\tfrac{7}{100} - \tfrac{192500}{9}\cdot\tfrac{441}{20000} + \tfrac{980000}{9}\cdot\tfrac{20923}{3000000} - \tfrac{1400000}{9}\cdot\tfrac{885969}{400000000} \;=\; \tfrac{22981}{54000},
\end{aligned} \tag{17}$$

upholding the claim we made for L near the beginning of Lecture 11.

Finally, in discussing the definite integral $\int_a^b f(t)\,dt$, we have assumed throughout (at least implicitly) that $b > a$. So the question arises: does $\int_a^b f(t)\,dt$ mean anything if $a > b$? The (not so obvious) answer is yes, because

$$\int_b^a f(t)\,dt \;=\; -\int_a^b f(t)\,dt \tag{18}$$

is a general property of the definite integral, and is easiest to obtain directly from the legal definition. First we note that

$$\sum_{i=1}^{n} \omega_i \;=\; \omega_1 + \omega_2 + \ldots + \omega_n \;=\; \omega_n + \omega_{n-1} + \ldots + \omega_1 \;=\; \sum_{i=1}^{n} \omega_{n+1-i}$$

for any ω_i, in particular for $\omega_i = f\big(b + \frac{(i-1)(a-b)}{n}\big)$. Then, from (4)-(5) and (9), we obtain

$$\begin{aligned}
\int_b^a f(t)\,dt &= \lim_{n\to\infty}\int_b^a f_n(t)\,dt \;=\; \lim_{n\to\infty}\frac{a-b}{n}\sum_{i=1}^{n} f\left(b + \tfrac{(i-1)(a-b)}{n}\right) \\
&= -\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(b + \tfrac{(i-1)(a-b)}{n}\right) \\
&= -\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(b + \tfrac{(\{n+1-i\}-1)(a-b)}{n}\right) \\
&= -\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(b + \tfrac{(n-i)(a-b)}{n}\right) \\
&= -\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(a + \tfrac{i(b-a)}{n}\right) \\
&= -\lim_{n\to\infty}\int_a^b \phi_n(t)\,dt \;=\; -\int_a^b f(t)\,dt
\end{aligned} \tag{19}$$

as required. Note that for $a = b$ the result reduces to

$$\int_a^a f(t)\,dt \;=\; 0. \tag{20}$$

Exercises

1. Calculate $\int_1^3 g(t)\,dt$ for g defined on $[1,3]$ by

$$g(t) \quad = \quad \begin{cases} 4t^3 + 52 & \text{if} \quad 1 \leq t < 2 \\ 3t^2 + 36t & \text{if} \quad 2 \leq t \leq 3. \end{cases}$$

Also, verify that g is continuous.
Hint: Use (13)-(15) in conjunction with (8)-(9) from Lecture 11.

2. Calculate $\int_0^2 g(t)\,dt$ for g defined on $[0,2]$ by

$$g(t) \quad = \quad \begin{cases} 3t^2 - 4t & \text{if} \quad 0 \leq t < 1 \\ 4t^3 - 5t^2 & \text{if} \quad 1 \leq t \leq 2. \end{cases}$$

Also, verify that g is continuous.
Hint: Use (13)-(15) in conjunction with (8)-(9) from Lecture 11.

3. Calculate $\int_2^4 g(t)\,dt$ for g defined on $[2,4]$ by

$$g(t) \quad = \quad \begin{cases} 4t^3 + 6t^2 + 2t + 240 & \text{if} \quad 2 \leq t < 3 \\ 3t^2 + 128t - 3 & \text{if} \quad 3 \leq t \leq 4. \end{cases}$$

Is g is continuous?
Hint: Use (13)-(15) in conjunction with (8)-(9) from Lecture 11.

4. Obtain (14).
Hint: Follow the method that yielded (13), using Table 1 of Lecture 10 where necessary.

5. Evaluate $\int_a^b t^4\,dt$.
Hint: Follow the method that yielded (13), using Table 1 of Lecture 10 where necessary.

6. Evaluate $\int_a^b t^5\,dt$.
Hint: Follow the method that yielded (13), using Table 1 of Lecture 10 where necessary.

7. Evaluate $\int_a^b t^6\,dt$.
Hint: Follow the method that yielded (13), using Table 1 of Lecture 10 where necessary.

8. Evaluate $\int_{-1}^1 e^x - 1\,dx$.
Hint: Assume that $e^{\delta x} = 1 + \delta x + o(\delta x)$. We will obtain this result in Lecture 20, where it appears as equation (11). Otherwise follow the method that yielded (13), using equation (11) of Lecture 10 where necessary.

Suitable problems from standard calculus texts

Stewart (2003): pp. 391-392, ## 21-25 and 41-42. For the first five problems use (13)-(15) in conjunction with (8) from Lecture 11.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

1. 176.

2. $\frac{7}{3}$.

3. 830.

8. From (4), i.e., using a left-handed approach, we have

$$\begin{aligned}
\int_{-1}^{0} e^x - 1\,dx &= \lim_{n\to\infty} \frac{0-(-1)}{n} \sum_{i=1}^{n} f\left(-1 + \tfrac{(i-1)(0-(-1))}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(-1 + \tfrac{i-1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\{e^{-1+(i-1)/n} - 1\right\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left\{e^{-1}e^{i/n}e^{-1/n} - 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{\sum_{i=1}^{n} e^{-1}e^{i/n}e^{-1/n} - \sum_{i=1}^{n} 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-1}e^{-1/n} \sum_{i=1}^{n} e^{i/n} - \{1+1+\ldots+1\} \quad (n \text{ times})\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-(1+1/n)} \sum_{i=1}^{n} \left(e^{1/n}\right)^i - n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-(1+1/n)} e^{1/n} \frac{1-\left(e^{1/n}\right)^n}{1-e^{1/n}} - n\right\} \\
&= \lim_{n\to\infty} \frac{1}{n} \left\{e^{-1}e^{-1/n}e^{1/n} \frac{1-e^1}{1-e^{1/n}} - n\right\} \\
&= \lim_{n\to\infty} \left\{\frac{1}{n} e^{-1} \frac{1-e}{1-e^{1/n}} - 1\right\} \\
&= \lim_{n\to\infty} \frac{e^{-1}-1}{n(1-e^{1/n})} - 1 = \lim_{\delta x\to 0+} \frac{\delta x(e^{-1}-1)}{1-e^{\delta x}} - 1 \\
&= \lim_{\delta x\to 0+} \frac{(1-e^{-1})\delta x}{e^{\delta x}-1} - 1 = (1-e^{-1}) \lim_{\delta x\to 0+} \frac{\delta x}{e^{\delta x}-1} - 1 \\
&= (1-e^{-1}) \lim_{\delta x\to 0+} \frac{\delta x}{1+\delta x+o(\delta x)-1} - 1 \\
&= (1-e^{-1}) \lim_{\delta x\to 0+} \frac{1}{1+\frac{o(\delta x)}{\delta x}} - 1 = (1-e^{-1})\frac{1}{1+0} - 1 \\
&= -e^{-1},
\end{aligned}$$

in which we have used equation (11) of Lecture 10 with $a = 1$ and $r = e^{1/n}$. Similarly, using a right-handed approach (for a bit of variety), i.e., from (5):

$$\begin{aligned}
\int\limits_0^1 e^x - 1\,dx &= \lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^{n} f\left(0 + \tfrac{i(1-0)}{n}\right) \quad = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\tfrac{i}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left\{e^{i/n} \quad - \quad 1\right\} \quad = \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{i=1}^{n}\left(e^{1/n}\right)^{i} \quad - \quad \sum_{i=1}^{n} 1\right\} \\
&= \lim_{n\to\infty} \frac{1}{n}\left\{e^{1/n}\frac{1-\left(e^{1/n}\right)^{n}}{1-e^{1/n}} \quad - \quad n\right\} \quad = \lim_{n\to\infty}\left\{\frac{1}{n}e^{1/n}\frac{1-e}{1-e^{1/n}} \quad - \quad 1\right\} \\
&= (1-e)\lim_{n\to\infty}\frac{1}{n}\frac{e^{1/n}}{1-e^{1/n}} \quad - \quad 1 \quad = (e-1)\lim_{n\to\infty}\frac{1}{n}\frac{e^{1/n}}{e^{1/n}-1} \quad - \quad 1 \\
&= (e-1)\lim_{\delta x\to 0+}\frac{\delta x e^{\delta x}}{e^{\delta x}-1} \quad - \quad 1 \\
&= (e-1)\lim_{\delta x\to 0+}\frac{\delta x\{1+\delta x+o(\delta x)\}}{\delta x + o(\delta x)} \quad - \quad 1 \\
&= (e-1)\lim_{\delta x\to 0+}\frac{1+\delta x+o(\delta x)}{1+\frac{o(\delta x)}{\delta x}} \quad - \quad 1 \\
&= (e-1)\frac{1+0+0}{1+0} \quad - \quad 1 \quad = (e-1)\cdot 1 - 1 \quad = e-2.
\end{aligned}$$

Now

$$\int\limits_{-1}^{1} e^x - 1\,dx = \int\limits_{-1}^{0} e^x - 1\,dx + \int\limits_{0}^{1} e^x - 1\,dx = -e^{-1} + e - 2 = e - \frac{1}{e} - 2 \approx 0.3504.$$