19. The shape of a graph: extrema and concavity

Although, for some purposes, we can describe a function accurately enough by identifying its extrema and where it increases or decreases, for other purposes it is useful also to know *how* it increases or decreases. Moreover, although we have known since Lecture 1 how to locate the extrema graphically—just look at the graph—we have yet to consider locating extrema analytically. These are our concerns in this lecture.

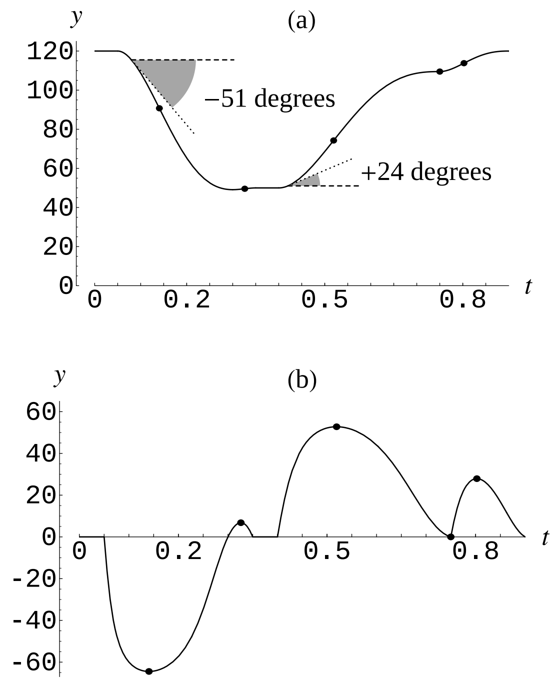

Figure 1: (a) Volume of blood in a human left ventricle, $y = V(t)$ ml. (b) Elevation of graph in degrees (i.e., $y = \frac{\pi}{180}\theta(t)$, where $\theta(t)$ is elevation in radians at time t). The dots correspond to inflection points where the concavity changes sign, i.e. where the elevation has a local extremum.

Let's begin once again with ventricular volume in our cardiac cycle (Figure 1a). If you look carefully, you will see that although V increases throughout [0.4, 0.75], its increase on [0.4, 0.52] is different from that on [0.52, 0.75]. Why? Imagine that your graph is a narrow tunnel and that you are a long and skinny worm who slinks along from left to right, always looking straight ahead into the tunnel. You are also a very clever worm with a penchant for mathematics, and so as you travel from left to right you record a trace of your elevation, i.e., the angle between your line of sight (shown dotted in Figure 1a) and the horizontal (shown dashed); elevation is counted positively when your line of sight is above horizontal but negatively when it is below, so your elevation always lies between $\pm 90°$. Your trace of elevation is sketched in Figure 1b, directly below the graph of V. Observe that you travel horizontally for 0.05 seconds until, at $t = 0.05$s, your elevation dips below zero and you start to slither downhill. Your elevation continues to decrease until, at $t = 0.14$s, it reaches a minimum of $-64°$; thereafter, your elevation increases, but you are still going down. Your descent does not end until $t = 0.3$, when your elevation reaches zero again. Thereafter, your elevation increases to a maximum of 7° at $t = 0.33$, decreases to zero at $t = 0.35$, and then remains zero as you glide horizontally through the tunnel's isovolumetric relaxation section. At $t = 0.4$, your elevation begins to rise sharply as you climb uphill toward a maximum elevation of 53° at $t = 0.52$; thereafter, your elevation decreases, but you are still going up. Your line of sight is momentarily level at $t = 0.75$, but your elevation then increases once more as you resume your upward climb; it achieves its final local maximum of 28° at $t = 0.8$, then decreases to zero at $t = 0.9$ as you approach the ventricular maximum at the end of the cardiac cycle.

In more abstract terms, a graph is *concave up* if its elevation is increasing but *concave down* if its elevation is decreasing. Moreover, a graph has an *inflection point* wherever its concavity changes from up to down or vice versa—i.e., where its elevation changes from increasing to decreasing or vice versa. Accordingly, V is concave down on [0.05, 0.14], concave up on [0.14, 0.33], concave down on [0.33, 0.35], concave up on [0.4, 0.52], concave down on [0.52, 0.75], concave up on [0.75, 0.8] and concave down on [0.8, 0.9] with inflection points at $t = 0.14$, $t = 0.33$, $t = 0.52$, $t = 0.75$ and $t = 0.8$ (as indicated by the dots in Figure 1). Whenever a graph is perfectly straight (not necessarily flat), it is said to have no concavity; for example, V has no concavity on [0, 0.05] or [0.35, 0.4].

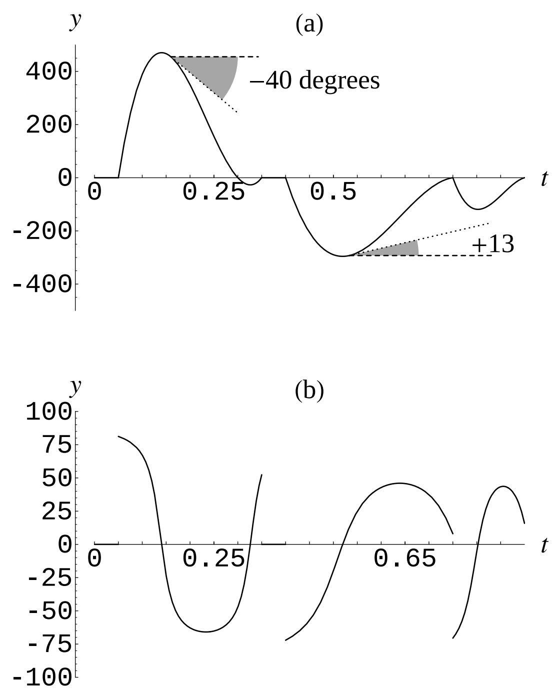

Figure 2: (a) Outflow of blood from a human left ventricle, $y = f(t)$ ml/s. (b) Elevation of graph in degrees (i.e., $y = \frac{\pi}{180}\theta(t)$, where $\theta(t)$ is elevation in radians at time t).

Similar considerations apply to ventricular outflow f (defined by $f(t) = -V'(t)$, whose elevation is plotted in Figure 2b, directly below the graph of f itself in Figure 2a.

Notice how elevation increases abruptly from 0° to 81° at $t = 0.05$ and decreases abruptly from 52° to 0° at t$= 0.35$, from 0° to -72° at $t = 0.4$, and from 8° to -71° at $t = 0.75$. That is, elevation is discontinuous at $t = 0.05$, $t = 0.35$, $t = 0.4$ and $t = 0.75$ with discontinuities (or jumps) of 81, -52, -72 and -79 (= -71 - 8) degrees, respectively; each discontinuity of elevation corresponds to a sharp turn in the tunnel or, more abstractly, to a corner in the graph of f itself (thus f has corners at $t = 0.05$, $t = 0.35$, $t = 0.4$ and $t = 0.75$). A corner can be an inflection point; for example, f has an inflection point at $t = 0.75$, where its concavity changes abruptly from down to up (because elevation changes discontinuously from decreasing to increasing). Incidentally, note from inspection of Figure 4 of Lecture 1 that the concavities of f and $-f$ are always opposite (except where f has no concavity). This result is general: it applies to any (ordinary) function.

So far, so good—but our perspective is still entirely graphical. Now we must collect our thoughts together and turn them into analytical statements. Let us agree to denote the elevation of y in radians at time t by $\theta(t)$, so that $\frac{\pi}{180}\theta(t)$ is the elevation in degrees as plotted in Figures 1-2. Then, from the above discussion of Figures 1-2, y is increasing or decreasing with respect to t according to whether θ is positive or negative and

$$y \text{ has a local extremum at } t = c \Longleftrightarrow \theta(t) \text{ changes sign at } t = c \tag{1a}$$
$$y \text{ concave up} \Longleftrightarrow \theta(t) \text{ increasing with respect to } t \tag{1b}$$
$$y \text{ concave down} \Longleftrightarrow \theta(t) \text{ decreasing with respect to } t. \tag{1c}$$

Note that there are two ways in which $\theta(t)$ can change sign at $t = c$. The first and more frequent case is where $\theta(c) = 0$, e.g., for $c = 0.3$ in Figure 1 or for $c = 0.14$, $c = 0.33$, $c = 0.52$ and $c = 0.8$ in Figure 2; in this case the extremum is said to be *smooth*. In the second case, the extremum is at a *corner*: $\theta(c)$ is undefined because $\theta(t)$ is discontinous at $t = c$ but $\theta(c^-)$ and $\theta(c^+)$ have opposite signs. An example occurs for $c = 0.75$ in Figure 2, where elevation jumps discontinuously from positive to negative with $\frac{\pi}{180}\theta(c^-) = 8^\circ$ and $\frac{\pi}{180}\theta(c^+) = -71^\circ$. These considerations enable us to rewrite (1) as

$$y \text{ has a local extremum at } t = c \Longleftrightarrow \text{EITHER } \theta(c) = 0 \text{ OR } \theta(c^+)\theta(c^-) < 0 \tag{2a}$$
$$y \text{ concave up} \Longleftrightarrow \theta'(t) > 0 \tag{2b}$$
$$y \text{ concave down} \Longleftrightarrow \theta'(t) < 0. \tag{2c}$$

There are two kinds of smooth local extremum, however: a smooth local maximum where elevation is decreasing, and a smooth local minimum where elevation is increasing. Likewise, there are two kinds of corner local extremum: a corner local maximum where $\theta(c^-) > 0 > \theta(c^+)$, and a corner local minimum where $\theta(c^+) > 0 > \theta(c^-)$. So we can replace (2) by

$$y \text{ has a local maximum at } t = c \Longleftrightarrow \begin{cases} \text{EITHER } \theta(c) = 0 \text{ AND } \theta'(c) < 0 \\ \text{OR } \theta(c^-) > 0 > \theta(c^+) \end{cases} \tag{3a}$$

$$y \text{ has a local minimum at } t = c \Longleftrightarrow \begin{cases} \text{EITHER } \theta(c) = 0 \text{ AND } \theta'(c) > 0 \\ \text{OR } \theta(c^+) > 0 > \theta(c^-) \end{cases} \tag{3b}$$

$$y \text{ concave up} \Longleftrightarrow \theta'(t) > 0 \tag{3c}$$
$$y \text{ concave down} \Longleftrightarrow \theta'(t) < 0. \tag{3d}$$

But the tangent of the angle of elevation is always the slope of the curve, i.e.,

$$\tan(\theta) \;=\; \frac{dy}{dt} \tag{4}$$

(see Figure 3). Differentiating with respect to t:

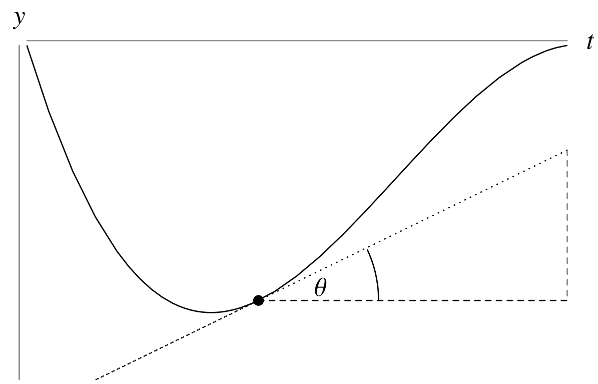

Figure 3: Pictorial version of (4)

$$\frac{d}{dt}\left\{\tan(\theta)\right\} \;=\; \frac{d}{dt}\left\{\frac{dy}{dt}\right\}. \tag{5}$$

The right-hand side of this equation is the (label assigned by the) derivative of the derivative of y or *second derivative* of y with respect to t, for which we adopt the standard notation $\frac{d^2y}{dt^2}$.* The chain rule can be applied to the left-hand side of (5). Thus

$$\frac{d}{d\theta}\left\{\tan(\theta)\right\}\frac{d\theta}{dt} \;=\; \frac{d^2y}{dt^2} \tag{6}$$

or

$$\sec^2(\theta)\frac{d\theta}{dt} \;=\; \frac{d^2y}{dt^2}. \tag{7}$$

Because $\sec^2(\theta)$ is always positive for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, the sign of $\theta'(t)$ must always agree with the sign of $\frac{d^2y}{dt^2}$; moreover, because the sign of $\tan(\theta)$ in (4) always agrees with that of θ, the sign of θ must also agree with that of $\frac{dy}{dt}$. Hence (assuming that y varies continuously—though not necessarily smoothly—with respect to t) we can replace (3) by

$$y \text{ has a local maximum at } t = c \Longleftrightarrow \begin{cases} \text{EITHER } \left.\frac{dy}{dt}\right|_{t=c} = 0 \text{ AND } \left.\frac{d^2y}{dt^2}\right|_{t=c} < 0 \\ \text{OR } \left.\frac{dy}{dt}\right|_{t=c^-} > 0 > \left.\frac{dy}{dt}\right|_{t=c^+} \end{cases} \tag{8a}$$

$$y \text{ has a local minimum at } t = c \Longleftrightarrow \begin{cases} \text{EITHER } \left.\frac{dy}{dt}\right|_{t=c} = 0 \text{ AND } \left.\frac{d^2y}{dt^2}\right|_{t=c} > 0 \\ \text{OR } \left.\frac{dy}{dt}\right|_{t=c^+} > 0 > \left.\frac{dy}{dt}\right|_{t=c^-} \end{cases} \tag{8b}$$

$$y \text{ concave up} \Longleftrightarrow \frac{d^2y}{dt^2} > 0 \tag{8c}$$

$$y \text{ concave down} \Longleftrightarrow \frac{d^2y}{dt^2} < 0. \tag{8d}$$

*Or $f''(t)$, if $y = f(t)$.

These equivalences, together where appropriate with dominance arguments and information about asymptotes (Lecture 4), suffice to determine the shape of an ordinary function's graph. They also suffice to determine the global minimum or maximum of any continuous function f on any closed interval $[a, b]$—because, as we pointed out in Lecture 1, a global extremizer is always either a local extremizer or an endpoint.

A remark is now in order. Most of the functions we deal with in calculus are smooth on their entire domain. In particular, $\sin$, $\cos$, the exponential function and any polynomial are smooth on $(-\infty, \infty)$, $\ln$ is smooth on $(0, \infty)$, and any rational function (i.e., any ratio of two polynomials) is smooth on its domain (which is almost all of $(-\infty, \infty)$ but excludes points at which the denominator would be zero). Furthermore, any combination of these functions is smooth on its domain. But if you have a smooth function, then you must have smooth extrema. So corner extrema are not especially frequent, and arise in practice only where the components of a join are spliced. A couple of examples will help to make this clear.

Consider first the function f defined on $(-\infty, \infty)$ by

$$f(t) \;=\; \frac{|t|-1}{t^2+1}. \tag{9}$$

It perhaps isn't obvious that this is a join of two smooth components, but it is:

$$f(t) \;=\; \begin{cases} \frac{-t-1}{t^2+1} & \text{if} \quad t \in (-\infty, 0) \\ \frac{t-1}{t^2+1} & \text{if} \quad t \in [0, \infty). \end{cases} \tag{10}$$

The only potential corner extremum is at $t = 0$, so we test for that first. The derivative of f is readily found by the methods of Lecture 7:

$$f'(t) \;=\; \begin{cases} \frac{-1+2t+t^2}{(t^2+1)^2} & \text{if} \quad t \in (-\infty, 0) \\ \frac{1+2t-t^2}{(t^2+1)^2} & \text{if} \quad t \in (0, \infty) \end{cases} \tag{11}$$

from Exercise 1. So we obtain

$$f'(0^-) \;=\; \lim_{t \to 0^-} f'(t) \;=\; -1, \qquad f'(0^+) \;=\; \lim_{t \to 0^+} f'(t) \;=\; 1. \tag{12}$$

Because the derivative crosses zero from negative to positive by jumping from -1 to $+1$, $t = 0$ is a corner local minimizer, and the local minimum is $f(0) = -1$.

Any other local extremum must be a smooth extremum, and hence must satisfy $f'(t) = 0$. From (11), the only candidate extremizers in $(-\infty, 0)$ satisfy $-1 + 2t + t^2 = 0$ or $t = -1 \pm \sqrt{2}$, which implies $t = -1 - \sqrt{2}$ because $-1 + \sqrt{2} \notin (-\infty, 0)$. So $t = -1 - \sqrt{2}$ is the only possible local extremizer in $(-\infty, 0)$. Is it a maximum or a minimum? To answer that, we need $f''(-1 - \sqrt{2})$. Again using the methods of Lecture 8:

$$f''(t) \;=\; \begin{cases} \frac{(1-t)(1+4t+t^2)}{(t^2+1)^3} & \text{if} \quad t \in (-\infty, 0) \\ \frac{(1+t)(1-4t+t^2)}{(t^2+1)^3} & \text{if} \quad t \in (0, \infty) \end{cases} \;=\; \frac{(1+|t|)(1-4|t|+t^2)}{(t^2+1)^3} \tag{13}$$

from Exercise 2. Substituting $t = -1 - \sqrt{2}$ into (13) yields$f''(-1 - \sqrt{2}) = 1 - \frac{3}{2\sqrt{2}}$, which is negative (because $8 < 9$). Hence, from (8a), $t = -1 - \sqrt{2} \approx -2.41$ is a local maximizer, and the local maximum is $f(-1 - \sqrt{2}) = \frac{1}{2}\{\sqrt{2} - 1\} \approx 0.207$.

From (13) and Exercise 3, we see that $f''(t) > 0$ on $(-\infty, 0)$ if $1 + 4t + t^2 > 0$ or $t \in (-\infty, -2-\sqrt{3}) \cup (-2+\sqrt{3}, 0)$, whereas $f''(t) < 0$ on $(-\infty, 0)$ if $1 + 4t + t^2 < 0$ or $t \in (-2-\sqrt{3}, -2+\sqrt{3})$. Thus f is increasing and concave up on $(-\infty, -2-\sqrt{3})$, increasing and concave down on $(-2-\sqrt{3}, -1-\sqrt{2})$, decreasing and concave down on $(-1-\sqrt{2}, -2+\sqrt{3})$ and decreasing and concave up on $(-2+\sqrt{3}, 0)$, with inflection points at $t = -2 \pm \sqrt{3}$. Also, because (9) implies that $f(t) \approx \frac{1}{|t|}$ when $|t|$ is very large, we know that $f(t) \to 0$ as $t \to -\infty$. This information suffices to determine the shape of the graph for $(-\infty, 0)$. The corresponding analysis for $(0, \infty)$ is unnecessary, because f is an even function—i.e., $f(-t) = f(t)$ for all $t \in (-\infty, \infty)$. So we obtain the graph for $(0, \infty)$ from the graph for $(-\infty, 0)$ by reflection in the y-axis.

Furthermore, because $f(0) = -1$ is the only local minimum and $f(0) < f(\pm\infty)$, $f(0)$ is also the global minimum. Likewise, because $f(-1 \pm \sqrt{2}) = \frac{1}{2}\{\sqrt{2} - 1\}$ are the only local maxima and $f(-1 \pm \sqrt{2}) > f(\pm\infty)$, $-1 \pm \sqrt{2}$ are also global maximizers. The entire graph of f is sketched in Figure 4, which confirms that $f(0) = -1$ is the global minimum, that $f(-1 \pm \sqrt{2}) = \frac{1}{2}\{\sqrt{2} - 1\}$ is the global maximum, that 0 is the unique global minimizer and that $-1 \pm \sqrt{2}$ are the non-unique global maximizers. Of course, the global minimum and maxima could have been determined approximately from the graph alone, but the inflection points would have been much harder to locate—even approximately.

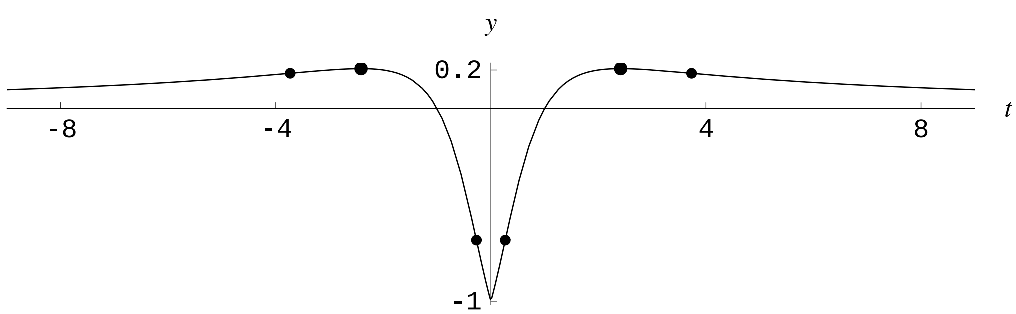

Figure 4: The graph $y = f(t)$ for f defined by (9). The large dots are extrema, the small dots are inflection points.

For a second example, consider the function f defined on $(-\infty, \infty)$ by

$$f(t) \quad = \quad \frac{(t-1)^2}{(1+|t|)(t^2+1)}, \tag{14}$$

which again is a join of two smooth components:

$$f(t) \quad = \quad \begin{cases} \frac{1-t}{t^2+1} & \text{if} \quad t \in (-\infty, 0) \\ \frac{(t-1)^2}{(1+t)(t^2+1)} & \text{if} \quad t \in [0, \infty) \end{cases} \tag{15}$$

The only potential corner extremum is again at $t = 0$, so again we test for that first. From Exercise 4, the derivative of f is given by

$$f'(t) \quad = \quad \begin{cases} \frac{-1-2t+t^2}{(t^2+1)^2} & \text{if} \quad t \in (-\infty, 0) \\ \frac{(1-t)(t^3-3t^2-3t-3)}{(1+t)^2(t^2+1)^2} & \text{if} \quad t \in (0, \infty) \end{cases} \tag{16}$$

from Exercise 1. So we obtain

$$f'(0^-) \;=\; \lim_{t\to 0^-} f'(t) \;=\; -1, \qquad f'(0^+) \;=\; \lim_{t\to 0^+} f'(t) \;=\; -3. \tag{17}$$

Because the derivative does not change sign as it (discontinously) crosses zero, in this case there is no corner local extremum at $t = 0$.

Any other local extremum must be a smooth extremum, and hence must satisfy $f'(t) = 0$. From (17), the only candidate extremizer in $(-\infty, 0)$ satisfies $-1 - 2t + t^2 = 0$ or $t = 1 - \sqrt{2}$ (because $1 + \sqrt{2} \notin (-\infty, 0)$). To determine whether it is a maximum or a minimum, we need $f''(1 - \sqrt{2})$. Again using the methods of Lecture 8, we have

$$f''(t) \;=\; \begin{cases} -\frac{2(1+t)(1-4t+t^2)}{(t^2+1)^3} & \text{if} \quad t \in (-\infty, 0) \\ \frac{2(t^6-6t^5-3t^4+15t^2+6t+3)}{(1+t)^3(t^2+1)^3} & \text{if} \quad t \in (0, \infty) \end{cases} \tag{18}$$

from Exercise 5. Substituting $t = 1 - \sqrt{2}$ into (18) yields$f''(1 - \sqrt{2}) = -1 - \frac{3}{2\sqrt{2}}$, which is negative. Hence, from (8a), $t = 1 - \sqrt{2} \approx -0.41$ is a local maximizer, and the local maximum is $f(1 - \sqrt{2}) = \frac{1}{2}\{\sqrt{2} + 1\} \approx 1.207$. Moreover, because $1 - 4t + t^2 > 0$ when t is negative, (18) implies that the only inflection point on $(-\infty, 0)$ is at $t = -1$: for $t < -1$ we have $f''(t) > 0$, and for $t > -1$ we have $f''(t) < 0$. So the graph of f is increasing and concave up on $(-\infty, -1)$, increasing but concave down on $(-1, 1 - \sqrt{2})$ and decreasing and concave down on $(1 - \sqrt{2}, 0)$. Also, because (14) implies that $f(t) \approx \frac{1}{|t|}$ when $|t|$ is very large, we again know that $f(t) \to 0$ as $t \to -\infty$. The above information suffices to determine the shape of the graph for $(-\infty, 0)$.

In this case, the corresponding analysis for $(0, \infty)$ is necessary, because f is neither an odd nor an even function. From (16), candidates for $f'(t) = 0$ on $(-\infty, 0)$ must satisfy either $t = 1$ or $t^3 - 3t^2 - 3t - 3 = 0$. From (18) we have $f''(1) = \frac{1}{2}$, and so $t = 1$ is a local minimizer. It turns out that $t^3 - 3t^2 - 3t - 3 = 0$ has only one real solution, namely

$$t \;=\; c \;=\; 1 + \sqrt[3]{4 - 2\sqrt{2}} + \sqrt[3]{4 + 2\sqrt{2}} \;\approx\; 3.951 \tag{19}$$

(never mind for the moment how I found c—that would merely distract us from our purpose), and $f''(c) \approx -0.8781 \times 10^{-2}$ is negative. So $t = c$ is a local maximizer. It also turns out that there are only two real solutions of $t^6 - 6t^5 - 3t^4 + 15t^2 + 6t + 3 = 0$, namely (again ignoring how I found them),

$$t \;=\; \xi_1 \;\approx\; 1.460 \quad \text{and} \quad t \;=\; \xi_2 \;\approx\; 6.407. \tag{20}$$

So, from (18), the only inflection points on $(0, \infty)$ are at $t = \xi_1$ and $t = \xi_2$: for $0 < t < \xi_1$ we have $f''(t) > 0$, for $\xi_1 < t < \xi_2$ we have $f''(t) < 0$ and for $\xi_2 < t < \infty$ we have $f''(t) > 0$ again. So the graph of f is decreasing and concave up on $(0, 1)$, increasing and concave up on $(1, \xi_1)$, increasing and concave down on (ξ_1, c), decreasing and concave down on (c, ξ_2) and decreasing and concave up on (ξ_2, ∞). The above information, together with the knowledge that $f(t) \to 0$ as $t \to \infty$ (from (14)), suffices to determine the shape of the graph for $(0, \infty)$.

Furthermore, because $f(1) = 0$ is the only local minimum and $f(t)$ is never negative, 1 must be the global minimizer. But there are two local maximizers, namely, $t = 1 - \sqrt{2}$

and $t = c$, and so the global maximum is the larger of $f(1-\sqrt{2})$ and $f(c)$. We know from above that $f(1-\sqrt{2}) \approx 1.207$, and $f(c) \approx 0.106$. So $t = 1-\sqrt{2}$ is the global maximizer. The entire graph of f is sketched in Figure 5 to confirm our results.

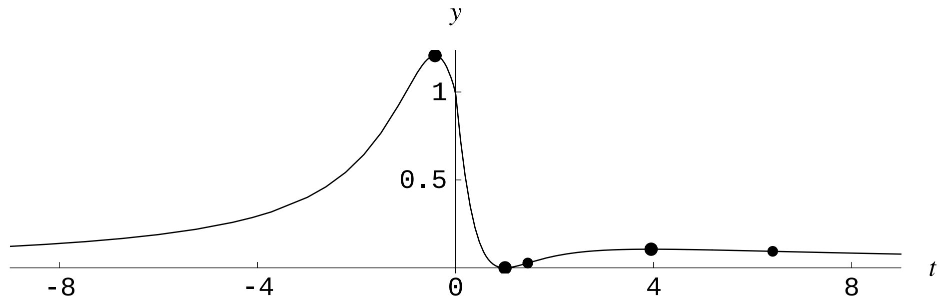

Figure 5: The graph $y = f(t)$ for f defined by (14). The large dots are extrema, the small dots are inflection points.

Now for three closing remarks. First, the technical term for a point at which $\frac{dy}{dt} = f'(t)$ either is zero or does not exist is a *critical point*. So a local extremum always occurs at a critical point, although a critical point need not be a local extremum—because it might be an inflection point at which $\frac{dy}{dt} = f'(t)$ and $\frac{d^2y}{dt^2} = f''(t)$ are both zero. The technical term for such a point is *stationary point*. For example, there is a stationary point for $y = V(t)$ in Figure 1 at $t = 0.8$, where the elevation drops to zero but never actually crosses zero.

Second, as remarked in Lecture 1, global extrema are highly domain-dependent: if you change the domain, then chances are that you change the function's global maximum or minimum. In particular, for $y = f(t)$ in Figure 5, $t = 1$ will not be the global minimizer on any subdomain that excludes $t = 1$ and $t = 1-\sqrt{2}$ will not be the global maximizer on any subdomain that excludes $t = 1-\sqrt{2}$. Furthermore, because a global extremum need not occur at a critical point, you must always check the endpoints. For example, from Figure 5 the maximum of f on $[0,3]$ is $f(0) = 1$, whereas the maximum of f on $[3,6]$ is $f(c) \approx 0.106$.

Third, for any function, an inflection point is always a critical point of the derivative. To see why, think once more in terms of ventricular volume V and inflow v, which are related by $V' = v$. If V has an inflection point at $t = \xi$, then $V''(\xi) = 0$. But $v'(t) = V''(t)$ for any t and so $v'(\xi) = 0$, which makes $t = \xi$ a critical point. See Figure 1, where the dots correspond to inflection points in the upper diagram but to extrema in the lower diagram.

Exercises

1. Verify (11).

2. Verify (13).

3. Show that $1+4t+t^2 \leq 0$ where $-2-\sqrt{3} \leq t \leq -2+\sqrt{3}$ (and otherwise $1+4t+t^2 > 0$).

4. Verify (16).

5. Verify (18).

6. Figure 1 of Lecture 11 shows ventricular inflow $v(t) = \frac{2450}{3} - \frac{192500}{9}t + \frac{980000}{9}t^2 - \frac{1400000}{9}t^3$ at the end of the systolic phase of our cardiac cycle. Find the maximum of v on $[0.28, 0.35]$.

7. A function f is defined on $[0, 5]$ by $f(t) = 17 - 18t + 8t^2 - t^3$.

(a) Find an expression for $f'(t)$.
(b) Hence find all local extrema.
(c) Where is f concave up? Where is f concave down?
(d) Find both the minimum and the maximum of f on $[0, 5]$.

8. A function f is defined on $[0, 4]$ by $f(t) = 3t^3 - 14t^2 + 9t + 8$.

(a) Find an expression for $f'(t)$.
(b) Hence find all local extrema.
(c) Where is f concave up? Where is f concave down?
(d) Find both the minimum and the maximum of f on $[0, 4]$.

9. A function f is defined on $[0, 3]$ by $f(t) = \frac{1}{3}t(9t - 2t^2 - 12)$.

(a) Find an expression for $f'(t)$.
(b) Hence find all local extrema.
(c) Where is f concave up? Where is f concave down?
(d) Find both the minimum and the maximum of f on $[0, 3]$.

10. A function f is defined on $[0, 7]$ by $f(t) = \frac{1}{6}(t - 1)(2t^2 - 19t + 41)$.

(a) Find an expression for $f'(t)$.
(b) Hence find all local extrema.
(c) Where is f concave up? Where is f concave down?
(d) Find both the minimum and the maximum of f on $[0, 7]$.

11. A function f is defined on $[2, 8]$ by $f(t) = t(27t - 2t^2 - 108)$. Where is it concave up? Where is it concave down? Find its global maximum.

Suitable problems from standard calculus texts

Stewart (2003): p. 287, ## 47-62; pp. 304-306, ## 1, 11-20 and 31-52; p. 323, ## 1-52.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

Solutions or hints for selected exercises

6. We have $v'(t) = -\frac{17500}{9}(240t^2 - 112t + 11)$. Let the maximizer be $t = c$. Then $240c^2 - 112c + 11 = 0 \implies c = \frac{14\pm\sqrt{31}}{60}$, i.e., $c \approx 0.326$ or $c \approx 0.141$. But $0.141 \notin [0.28, 0.35]$, so $c = \frac{14+\sqrt{31}}{60}$ is the only possibility; and $v''(t) = -\frac{17500}{9}(480t - 112) \implies v''(c) = -\frac{1400000}{9}\sqrt{31} < 0$, confirming that c is the maximizer.

11. Note that $f'(t) = 6(t-3)(6-t)$ and $f''(t) = 6(9-2t)$. So f is concave up on $[2, \frac{9}{2}]$ and concave down on $[\frac{9}{2}, 8]$ with $\max(f, 2, 8) = f(6) = -108$.