21. Work

In the simplest possible circumstances, the (mathematical) definition of work is simply

$$\text{WORK} \;=\; \text{FORCE} \times \text{DISTANCE}. \tag{1}$$

To be more precise, the work W done on a *particle*—i.e., a body whose entire mass is concentrated at a point—by a *constant* force F that moves the particle from A to B is the component of force in that direction times the distance, say s. For example, suppose that a ballbearing of mass m kilograms falls a distance h meters under the gravitational force $F = mg$ newtons (where g ms^{-2} is the acceleration that gravity induces on the particle). Then, because the direction of motion is identical to the direction in which the force acts,

$$W \;=\; F \cdot s \;=\; mgh \tag{2}$$

newton-meters or *joules*, from (1). If instead the ballbearing rolls a distance s down an inclined plane under the force of gravity, however, then less work is done by the force. For suppose that the plane is inclined to the horizontal at an angle α. Then the force of gravity can be resolved into two components, $F\cos(\alpha) = mg\cos(\alpha)$ newtons perpendicular to (and into) the plane and $F\sin(\alpha) = mg\sin(\alpha)$ newtons parallel to (and down) the plane. The normal component $mg\cos(\alpha)$ does no work on the ballbearing, because it cannot move the ballbearing perpendicular to the plane. So the only work is done by the tangential component $mg\sin(\alpha)$, and that work is $W = mg\sin(\alpha)s$ joules.

Nevertheless, as in life, so with work: the simplest possible circumstances almost never arise. If *either* the force is not constant *or* the mass of the body on which it acts is distributed (along a line or over an area or through a volume) as opposed to being concentrated in a point, then (1) must be replaced by an equation of the form

$$\delta W \;=\; w(x)\,\delta x + o(\delta x), \qquad a < x < b \tag{3}$$

for an infinitesimal amount of work or "worklet" δW, so that the total amount of work is

$$W \;=\; \int_a^b w(x)\,dx. \tag{4}$$

Here a and b are simply the least and greatest values of the relevant independent variable: perhaps a particle moves from $x = a$ to $x = b$ under a force that varies with x, or perhaps relevant mass is distributed between $x = a$ and $x = b$. As usual, examples serve best to introduce the method. We assume in the following that the units of measurement are consistently SI (kilograms for mass, meters for length, newtons for force and joules for work), so that we never need to mention them explicitly.

Accordingly, consider a uniform rope of length L and mass M, whose *line density*

$$\rho \;=\; \frac{M}{L} \tag{5}$$

is constant by assumption. In Figure 1a this rope is lying horizontally on a table, with its left-hand end tucked beneath a horizontal rod. Now suppose that the left-hand end of the

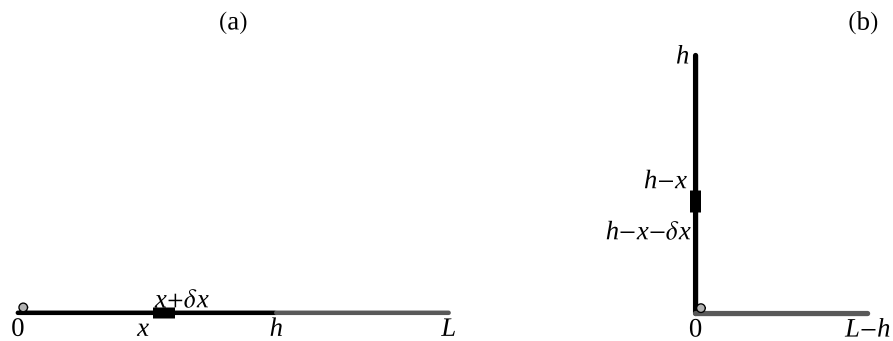

Figure 1: The (a) initial and (b) final configuration of a frictionless rope constrained by rod.

rope is raised vertically through a distance h (where $h < L$), so that the right-hand end of the rope comes to rest on the table at a distance $L - h$ from the rod (Figure 1b). What is the work done against gravity in moving the rope?

To answer this question, we first of all note that no work is done against gravity by raising the grey part of the rope, because it never leaves the table; thus the only relevant values of x (measured horizontally from the rod towards the right) are from $x = 0$ to $x = h$. So consider an infinitesimal piece of rope that initially lies on the table between distance x and distance $x + \delta x$ from the rod, where $0 < x < h$ (Figure 1a). The mass of this "ropelet" is simply its length times its line density, or $\delta x \times \rho = \rho\delta x$. Because the rope moves a distance h, the left-hand end of the ropelet comes to rest at a distance $h - x$ above the table, whereas the right-hand comes to rest a distance $h - x - \delta x$ above the table. So, because no work is done against gravity until the ropelet passes underneath the rod, if all of its mass were concentrated at its left-hand end, then the work done against gravity would be $\text{FORCE} \times \text{DISTANCE} = \rho\delta x\, g \times (h - x)$, by (1); whereas if all of its mass were instead concentrated at its right-hand end, then the work done against gravity would instead be $\text{FORCE} \times \text{DISTANCE} = \rho\delta x\, g \times (h - x - \delta x)$. But the mass is not concentrated at either of its ends; rather, it is uniformly distributed between them. So the worklet δW required to move the ropelet must lie between the two extremes. That is, $\rho\delta x\, g \times (h - x - \delta x) < \delta W < \rho\delta x\, g \times (h - x)$ or $\rho g(h - x)\delta x - \rho g\delta x^2 < \delta W < \rho g(h - x)\delta x$, implying

$$\delta W \;=\; \rho g(h - x)\,\delta x + o(\delta x), \qquad 0 < x < h. \tag{6}$$

In terms of (4), $a = 0$, $b = h$ and $w(x) = \rho g(h - x)$. The total work done in raising the rope is now obtained by summing all the worklets done by raising all the ropelets in the limit as the number of ropelets (and hence the number of worklets) tends to infinity and the error in (6) approaches zero—in other words, by integration:

$$\begin{aligned} W \;&=\; \lim_{\delta W \to 0} \sum_{x \in [0,h]} \delta W \;=\; \lim_{\delta x \to 0} \sum_{x \in [0,h]} \{\rho g(h - x)\delta x + o(\delta x)\} \;=\; \int_0^h \rho g(h - x)\,dx \\ &=\; \rho g \int_0^h (h - x)\,dx \;=\; \rho g \left\{-\tfrac{1}{2}(h - x)^2\right\}\Big|_0^h \;=\; \rho g\left\{0 + \tfrac{1}{2}(h - 0)^2\right\} \;=\; \tfrac{1}{2}\rho g h^2. \end{aligned} \tag{7}$$

It may seem surprising at first that the answer is independent of L; however, by considering only work done against gravity, we totally ignore any work done against friction. In these circumstances, the answer must be independent of L, because lengthening the rope cannot add to the work done in raising part of it.

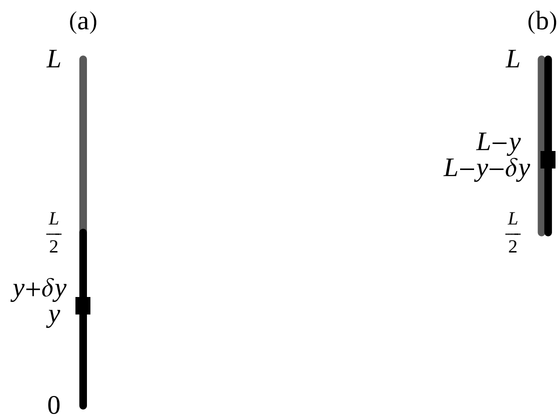

Figure 2: The (a) initial and (b) final configuration of a hanging rope.

By contrast, if one end of the rope is hanging from the ceiling and the other end is raised to the same position, then the work done against gravity must clearly depend on L. In Figure 2 the part of the rope on which no work is done (because it doesn't move) is again shown grey, with distances measured vertically from the bottom of the initial configuration. Consider an infinitesmal ropelet of length δy and hence mass $\rho\delta y$ whose lower end is initially at height y, and whose upper end is at height $y + \delta y$ (Figure 2a). In the final configuration, the ends of the ropelet are at heights $L - y$ and $L - y - \delta y$, respectively. So the distance s moved by any point of the ropelet lies between $L - 2y - 2\delta y$ and $L - 2y$: in other words, $s = L - 2y + O(\delta y)$. Because the ropelet has mass $\rho\delta y$, it follows that the element of work done in moving it from its initial to its final position is

$$\delta W \;=\; \rho\delta y\, s \;=\; \rho g(L-2y)\delta y + \delta y\, O(\delta y) \;=\; \rho g(L-2y)\delta y + o(\delta y). \tag{8}$$

Note that $\delta y\, O(\delta y)$ is $o(\delta y)$ because $\delta y\, O(\delta y)$ is so small that even after you divide it by δy, the result still tends to zero as $\delta y \to 0$ (because $= O(\delta y) \to 0$ as $\delta y \to 0$). Again, the total work done to raise the rope is obtained by summing all the worklets in the limit as the number of ropelets (and hence the number of worklets) tends to infinity and the error in (8) approaches zero—in other words, by integration:

$$\begin{aligned} W \;&=\; \lim_{\delta W \to 0} \sum_{y \in [0, \frac{L}{2}]} \delta W \;=\; \lim_{\delta y \to 0} \sum_{y \in [0, \frac{L}{2}]} \{\rho g(L-2y)\delta y + o(\delta y)\} \;=\; \int_0^{\frac{L}{2}} \rho g(L-2y)\, dy \\ &=\; \rho g \int_0^{\frac{L}{2}} (L-2y)\, dy \;=\; \rho g \left\{-\tfrac{1}{4}(L-2y)^2\right\}\Big|_0^{\frac{L}{2}} \;=\; -\tfrac{1}{4}0^2 + \tfrac{1}{4}(L-0)^2 \;=\; \tfrac{1}{4}\rho g L^2. \end{aligned} \tag{9}$$

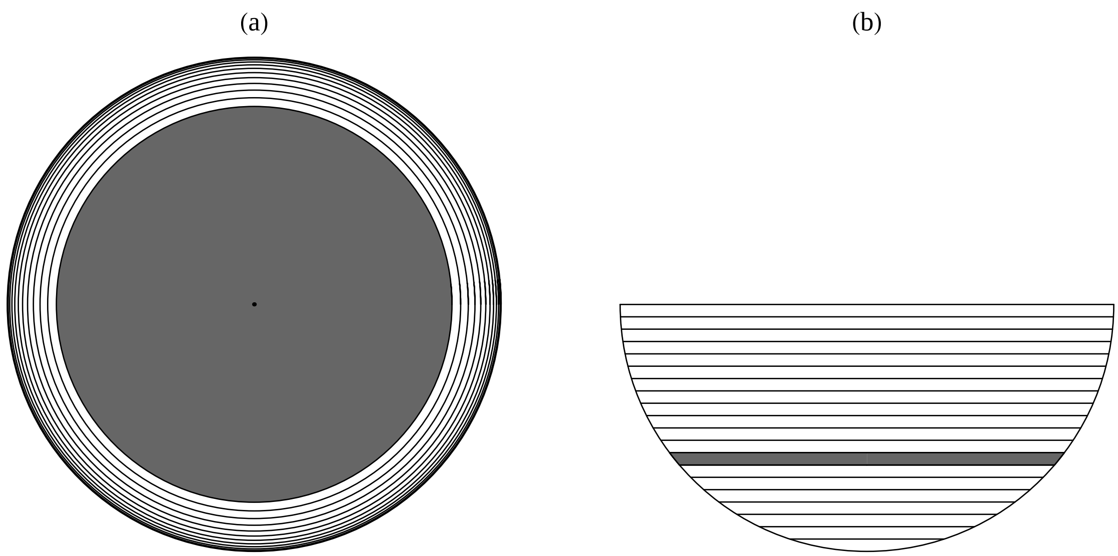

Figure 3: Using coaxial circular disks with sloping rims to partition the volume of liquid in a hemispherical tank. (a) The view from above with a typical elementary volume shaded. (b) Any vertical cross-section through the axis of symmetry, with the same elementary volume shaded.

For our next example, we will calculate the work done against gravity to pump out a hemispherical tank. Let the tank have radius r. Then any vertical cross-section is a semi-circle of radius r; and so, if we place the origin of coordinates at the center of the surface of the liquid when the tank is full, then the equation of the semi-circle is

$$x^2 + y^2 \; = \; r^2, \qquad -r \le y \le 0. \tag{10}$$

Furthermore, any horizontal cross-section of the tank is a circle, and the radius of this circle at depth $|y| = -y$ below the surface is

$$x \; = \; \sqrt{r^2 - y^2} \tag{11}$$

(from (10)). So if we slice the liquid horizontally into infinitesimal circular disks of thickness δy (of which, of course, there are infinitely many, though only twenty are indicated in Figure 3) then the volume of such an elementary disk or droplet of liquid is

$$\delta V \; = \; \pi x^2 \, \delta y + o(\delta y) \; = \; \pi\{r^2 - y^2\} \, \delta y + o(\delta y). \tag{12}$$

Let the liquid inside the tank have constant density—i.e., mass per unit volume—σ, so that the total mass M of the liquid is σ times the volume of a hemisphere of radius r, or

$$M \; = \; \sigma \cdot \tfrac{1}{2} \cdot \tfrac{4}{3}\pi r^3 \; = \; \tfrac{2}{3}\pi r^3 \sigma. \tag{13}$$

With circular symmetry, the mass of the liquid is distributed in two directions—along and perpendicular to the axis of symmetry, or vertically and horizontally; however, the horizontal distribution has no effect on the work done against gravity in raising a droplet, because every particle of liquid in a thin horizontal disk is at the same height. That is, if the (upper surface of the) droplet is at depth $|y|$, then the entire droplet must be raised a distance $|y| + O(\delta y)$; and because its mass is $\sigma\delta V$, it follows that the worklet done against gravity in raising the droplet to the top of the tank is

$$\delta W \;=\; \sigma\delta V g\{|y| + O(\delta y)\} \;=\; \sigma\pi g|y|\{r^2 - y^2\}\,\delta y + o(\delta y), \tag{14}$$

on using (12). As usual, the total work done in pumping all the liquid out is the sum of all the worklets in the limit as $\delta W \to 0$ or

$$\begin{aligned} W \;&=\; \lim_{\delta W \to 0} \sum_{y \in [-r,0]} \delta W \;=\; \lim_{\delta y \to 0} \sum_{y \in [-r,0]} \{\sigma\pi g|y|\{r^2 - y^2\}\,\delta y + o(\delta y)\} \\ &=\; \int_{-r}^{0} \sigma\pi g|y|\{r^2 - y^2\}\,dy \;=\; \sigma\pi g \int_{-r}^{0} (-y)(r^2 - y^2)\,dy \\ &=\; \sigma\pi g \int_{-r}^{0} \{y^3 - r^2 y\}\,dy \;=\; \sigma\pi g\left\{\tfrac{1}{4}y^4 - \tfrac{1}{2}r^2y^2\right\}\Big|_{-r}^{0} \\ &=\; \sigma\pi g\left\{0 - \tfrac{1}{4}(-r)^4 + \tfrac{1}{2}r^2(-r)^2\right\} \;=\; \tfrac{1}{4}\sigma\pi g r^4 \;=\; \tfrac{3}{8}Mgr \end{aligned} \tag{15}$$

joules from (13), because $|y| = -y$ when $y \le 0$.

Now, we said at the outset that WORK = FORCE $\times$ DISTANCE must be replaced by an infinitesimal analog whenever either the force is not constant or the mass of the body on which it acts is distributed, but we have so far considered only the second of these two possibilities. The classic example of the first possibility concerns the tension or compression of a spring. If the natural length of the spring is L and it is extended or compressed by an amount x, where $x > 0$ for extension but $x < 0$ for compression, then there results a force of magnitude $k|x|$ that attempts to restore the spring to its natural length (and succeeds unless the motion of the spring is constrained); $k(>0)$ is called the spring constant.* If the spring has been extended then the restoring force attempts to shorten it, whereas if the spring has been compressed then the restoring force attempts to lengthen it; but in either case, the worklet done in perturbing the spring an infinitesimal distance δx is

$$\delta W \;=\; k|x|\delta x + o(\delta x). \tag{16}$$

Hence the work required to increase the length of the spring from its natural length L to $L + h$ (i.e., to increase its extension from 0 to h) is

$$W \;=\; \int_0^h k|x|\,dx \;=\; \int_0^h kx\,dx \;=\; \tfrac{1}{2}kx^2\Big|_0^h \;=\; \tfrac{1}{2}kh^2 - \tfrac{1}{2}k\cdot 0^2 \;=\; \tfrac{1}{2}kh^2. \tag{17}$$

Similarly, the work required to shorten the length of the spring from its natural length L to $L - h$ (i.e., to decrease its extension to $-h$ from 0) is

$$W \;=\; \int_{-h}^{0} k|x|\,dx \;=\; \int_{-h}^{0} k(-x)\,dx \;=\; -\tfrac{1}{2}kx^2\Big|_{-h}^{0} \;=\; -\tfrac{1}{2}k\cdot 0^2 + \tfrac{1}{2}k\,(-h)^2 \;=\; \tfrac{1}{2}kh^2. \tag{18}$$

*Note that $|x|$ must not be too large, or the theory breaks down.

All of our examples so far have led to easy integrations, but there are various reasons why that need not happen, the most obvious being that density may vary with height or depth. Suppose, for example, that the (line) density of the rope in Figure 1 is no longer uniform but rather increases from ρ_0 at the end initially underneath the rod (the thin end) to $\rho_0 \ln(e + L)$ at the other end (the thick end) in such a way that the density at distance x from the thin end is given by

$$\rho \;=\; \rho_0 \ln(e + x). \tag{19}$$

We could even suppose that the rope can be moved only a distance h because it is then too thick to slide any further beneath the rod. Then the method that yielded

$$W \;=\; \int_0^h \rho g(h - x)\, dx \tag{20}$$

in (7) remains valid, but now

$$W \;=\; \rho_0 g \int_0^h (h - x) \ln(e + x)\, dx \tag{21}$$

instead. It is not so obvious how to find a suitable anti-derivative of w in this case.[†] Neverthless, from Exercise 1 we discover that

$$q(x) \;=\; x(x - 4h - 2e) + 2(e + 2h - x)(e + x) \ln(e + x) \tag{22}$$

satisfies

$$q'(x) \;=\; (h - x) \ln(e + x), \tag{23}$$

and hence that q is the requisite anti-derivative. So, from (21)-(23),

$$W \;=\; \rho_0 g \int_0^h q'(x)\, dx \;=\; \rho_0 g q(x)\big|_0^h \;=\; \rho_0 g\{q(h) - q(0)\}. \tag{24}$$

In other words, from Exercise 2,

$$W \;=\; \tfrac{1}{4}\rho_0 g \left\{2(e + h)^2 \ln(e + h) - 3h^2 - 6eh - 2e^2\right\}. \tag{25}$$

Exercise

1. Verify (23).

2. Verify (25).

Suitable problems from standard calculus texts

Stewart (2003): pp. 463-464, ## 11-26; p. 469, ## 27-29.

Reference

Stewart, J. 2003 *Calculus: early transcendentals*. Belmont, California: Brooks/Cole, 5th edn.

[†] Although see Calculus II!