3. Smoothness and concavity: a graphical perspective

Although, for some purposes, we can describe a function accurately enough by identifying its extrema and whether it increases or decreases, for other purposes it is useful also to know *how* it increases or decreases. For example, Figure 1(a) shows ventricular volume V in our cardiac cycle. If you look carefully, you will see that although V increases throughout [0.4, 0.75], its increase on [0.4, 0.52] is different from that on [0.52, 0.75]. Why? Imagine that your graph is a narrow tunnel and that you are a long and skinny worm who slinks along from left to right, always looking straight ahead into the tunnel. You are also a very clever worm with a penchant for mathematics, and so as you travel from left to right you record a trace of your **elevation**, i.e., the angle between your line of sight (shown dotted in Figure 1) and the horizontal (shown dashed); note that elevation is counted positively when your line of sight is above horizontal but negatively when it is below, so that elevation always lies between ± 90°. Your trace of elevation is sketched in Figure 1(b), directly below the graph of V. Observe that you travel horizontally for 0.05 seconds until, at $t = 0.05$, your elevation dips below zero and you start to slither downhill. Your elevation continues to decrease until, at $t = 0.14$, it reaches a minimum of -64°; thereafter, your elevation increases, but you are still going down. Your descent does not end until $t = 0.3$, when your elevation reaches zero again. Thereafter, your elevation increases to a maximum of 7° at $t = 0.33$, decreases to zero at $t = 0.35$, and then remains zero as you glide horizontally through the tunnel's isovolumetric relaxation section. At $t = 0.4$, your elevation begins to rise sharply as you climb uphill toward a maximum elevation of 53° at $t = 0.52$; thereafter, your elevation decreases, but you are still going up. Your line of sight is momentarily level at $t = 0.75$, but your elevation then increases once more as you resume your upward climb; it achieves its final local maximum of 28° at $t = 0.8$, then decreases to zero at $t = 0.9$ as you approach the ventricular maximum at the end of the cardiac cycle.

In more abstract terms, a graph is **concave up** if its elevation is increasing but **concave down** if its elevation is decreasing. Moreover, a graph has an **inflection** point wherever its concavity changes from up to down, or vice versa (in other words, wherever there is a local extremum of elevation). Accordingly, V is concave down on [0.05, 0.14], concave up on [0.14, 0.33], concave down on [0.33, 0.35], concave up on [0.4, 0.52], concave down on [0.52, 0.75], concave up on [0.75, 0.8] and concave down on [0.8, 0.9] with inflection points at $t = 0.14$, $t = 0.33$, $t = 0.52$, $t = 0.75$ and $t = 0.8$ (as indicated by the dots in Figure 5). Whenever a graph is perfectly straight (not necessarily flat), it is said to have no concavity; for example, V has no concavity on [0, 0.05] or [0.35, 0.4].

Similar considerations apply to ventricular outflow, whose elevation is plotted in Figure 2(b), directly below the graph of f itself in Figure 2(a) Notice how elevation increases abruptly from 0° to 81° degrees at $t = 0.05$ and decreases abruptly from 52° to 0° at $t = 0.35$, from 0° to –72° at $t = 0.4$, and from 8° to –71° at $t = 0.75$. We say that elevation is **discontinuous** at $t = 0.05$, $t = 0.35$, $t = 0.4$ and $t = 0.75$ with **discontinuities** of 81, –52, –72 and –79 (= –71 – 8), respectively; each discontinuity of elevation corresponds to a sharp turn in the tunnel or, more abstractly, to a **corner** in the graph of f itself (thus f has corners at $t = 0.05$, $t = 0.35$, $t = 0.4$ and $t = 0.75$). A corner can be an inflection point; for example, f has an inflection point at $t = 0.75$ where its concavity changes abruptly from down to up. Incidentally, note from inspection of Figure 1.4 that the concavities of f and –f are always opposite (except where f has no concavity). This result is general: it applies to any (ordinary) function.

A function without discontinuities on a given domain is said to be **continuous** on that domain. A continuous function without corners on a given domain is **smooth** on that domain; or, if you prefer, a function is smooth if its elevation is continuous. Thus V is a smooth (and therefore continuous) function, whereas f is continuous (but not smooth). Note that corners cannot exist at endpoints. Thus f is smooth on [0.4, 0.75], even though it is smooth on neither [0.4, 0.76] nor [0.39, 0.75]. In other words, smoothness is a property of both function and domain (like invertibility and extremum).

Nothing in nature really changes discontinuously, however; on the contrary, almost everything changes smoothly. When, for example, silence is broken by a thunderbolt, pressure is a smooth function of time throughout, but pressure rises so steeply over such a short interval that, for all practical purposes, no essential information is lost by assuming the change to be discontinuous. Nevertheless, mathematics is useful in understanding biology not because it replicates nature but rather because it models nature, that is, because it captures the essentials of biological phenomena in abstract form while ignoring inessential, and hence distracting, details. Thus a function with corners – or even a discontinuous one – can often, in practice, be the most useful model of a variable that in principle changes smoothly.

DATE OF READINGS	SAMPLE SIZE	MEAN WIDTH (mm)	DATE OF READINGS	SAMPLE SIZE	MEAN WIDTH (mm)
November 16, 1967	15	2.0	June 19, 1968	16	2.5
December 18, 1967	13	4.0	July 10, 1968	15	2.2
January 19, 1968	18	5.7	August 19, 1968	15	2.3
February 16, 1968	18	7.0	September 18 1968	15	2.6
March 18 1968	18	5.8	October 17 1968	15	3.9
April 18, 1968	16	4.9	November 13, 1968	22	5.9
May 17, 1968	16	3.7			

Table 3.1 Mean testicular size in the European starling

To illustrate this point, we consider size of gonads, which varies seasonally in birds. For example, Schwab (1971, p. 436) has measured testicular size in European starlings, *Sturnus vulgaris,* subjected to a constant 12-hour light/dark cycle. Table 1 records his measurements of mean testicular width over a 12-month period. Because readings were taken approximately a month apart, we can regard 30 days as a unit of time, with t = 0 at the beginning of the period and t = 12 to the end. Under these assumptions, Schwab's data are plotted on the left-hand side of Figure 3. Also shown are graphs of functions S and W that lie close to Schwab's data points; W is the piecewise-cubic join defined on [0, 12] by

$$W(t) = \begin{cases} 1.995 + 2.195t - 0.175t^2 & \text{if } 0 \le t \le c \\ 8.86788 + 0.00934343t - 0.272727t^2 + 0.0209596t^3 & \text{if } c \le t \le 12 \end{cases} \qquad (3.1)$$

where c = 3.0 is the global maximizer, and S the piecewise-cubic join defined on [0, 12] by

$$S(t) = \begin{cases} 1.995 + 2.195t - 0.175t^2 & \text{if } 0 \le t < 3 \\ H(t) & \text{if } 3 \le t < 3.002 \\ 8.86788 + 0.00934343t - 0.272727t^2 + 0.0209596t^3 & \text{if } 3.002 \le t \le 12 \end{cases} \qquad (3.2)$$

where

$$H(t) = 445147.2481 - 446647.4823t + 149384.5255t^2 - 16654.01947t^3. \qquad (3.3)$$

Either S or W provides an excellent approximation to Schwab's data; hence either S or W is a reasonable representation of mean testicular width. Can you tell the difference between them, from the left-hand side of Fig 3? I can't. But from the right-hand side of Figure 3, where the graphs have been magnified a thousandfold or so, you can see that S is smooth on [0, 12] whereas W has a corner at $t = 3.0$. Does this mean that S is a truer model, because mean testicular width changes smoothly? Perhaps. But if one insists on using S in place of W, then one pays an exorbitant price in terms of complexity: not only does S have an extra component H, but its formula is also very unwieldy. Moreover, S and W differ by much less than the standard deviation in Schwab's data. So W is a better model than S. In approximating real data, there is always a tradeoff between smoothness and simplicity.

Why, in defining W, do we write $c = 3.0$ instead of $c = 3$? We have already observed in Lecture 2 that any join of the form

$$W(t) = \begin{cases} F(t) & \text{if} \quad a \le t \le c \\ G(t) & \text{if} \quad c \le t \le b \end{cases} \tag{3.4}$$

must satisfy $F(c) = G(c)$ or $Z(c) = 0$, where Z is defined by

$$Z(c) = F(c) - G(c). \tag{3.5}$$

In the case of gonad size, (1) yields $F(c) = 1.995 + 2.195c - 0.175c^2$ and $G(c) = 8.86788 + 0.00934343c - 0.272727c^2 + 0.0209596c^3$, so that (5) implies

$$Z(c) = 6.87288 - 2.18566c - 0.0977273c^2 + 0.0209596c^3. \tag{3.6}$$

In particular, $Z(3) = -0.00227$, which is not zero (although the difference is small). Because $Z(c) = 0$, we deduce from $Z(3) \neq 0$ that $c \neq 3$ (although again the difference is small). To determine the exact value of c, we first rewrite the cubic polynomial Z as a product of linear functions (Exercise 10):

$$Z(c) = 0.0209596(c + 9.6552)(c - 3.00103)(c - 11.3168). \tag{3.7}$$

We already know from Figure 3 that c is approximately 3. So it now follows at once that

$$c = 3.00103 \tag{3.8}$$

Our question is answered: We write $c = 3.0$ because it is true, and we do not write $c = 3$ because it is false.

Reference

Schwab, Robert G. (1971) Circannian Testicular Periodicity in the European Starling in the Absence of Photoperiodic Change. In: Menaker, Michael (ed), Biochronometry, pp. 428-445. National Academy of Sciences, Washington, D.C.

Exercises 3

3.1 Find the global maximum and minimum of the function W graphed in Figure 3. Describe its concavity, and find all subdomains on which it is invertible.

3.2 If a function is increasing and concave up, what kind of function is its inverse?

3.3* A smooth function f with domain [0, 10] and range [-1, 5] has global minimizer $t = 0$, global maximizer $t = 3$, local minimizer $t = 8$, an inflection point at $t = 5$ and $f(10) = 4$. Sketch a possible graph of f. Where is f increasing? Where is f decreasing? Where is f concave up? Where is f concave down?

3.4 A smooth function f is concave up throughout its domain [0, 10] with a global minimum at $t = 4$. Its range is [-3, 5]. Sketch a possible graph of f. Find all possible subdomains on which f is invertible.

3.5 A continuous function f with domain [0, 10] has corners at $t = 2$ and $t = 6$. It is concave up between $t = 2$ and $t = 6$ but otherwise concave down. It has a local maximum at $t = 1$, a local minimum at $t = 2$, a global maximum at $t = 6$ and a global minimum at $t = 10$. Sketch a possible graph of f.

3.6 A strictly increasing smooth function f with domain [0, 10] and range [1, 7] has an inflection point at $t = 3$. Sketch a possible graph of its inverse.

3.7 A continuous function f with domain [0, 10] and range [1, 7] has local minima at $t = 6$ and $t = 8$, local maxima at $t = 5$ and $t = 7$, a corner at $t = 2$, and a global maximum at $t = 10$. If $f(0) = 4$, sketch a possible graph of f. Where is f concave up? Where is f concave down?

3.8 A function f is defined in Exercise 2.8. Is f concave up, concave down, or neither?

3.9 Verify (to 5 s.f.) that (2.14) holds for S defined by (2)-(3).

3.10 Verify that (6) and (7) are identical.

Appendix 3: Functions introduced in Lecture 3 as joins and products of polynomials

POLYNOMIAL REPRESENTATION

$$P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

FUNCTION P NAME	SUBDOMAIN	ORDER m	COEFFICIENTS c_0	c_1	c_2	c_3	c_4
P = W	[0, 3.001]	2	1.995	2.195	–0.175	0	0
	[3.001, 12]	3	8.86788	0.00934343	–0.272727	0.0209596	0
P = S	[0, 3]	2	1.995	2.195	–0.175	0	0
	[3, 3.002]	3	445147	–446647	149385	16654	0
	[3.002, 12]	3	8.86788	0.00934343	–0.272727	0.0209596	0

PRODUCT REPRESENTATION

FUNCTION NAME	SUBDOMAIN	DEFINITION
W	[0, 3.001]	$W(t) = 0.175(13.394 - t)(0.851128 + t)$
	[3.001, 12]	$W(t) = 0.0209596(4.85391 + t)(87.1656 - 17.866t + t^2)$
S	[0, 3]	$S(t) = 0.175(13.394 - t)(0.851128 + t)$
	[3, 3.002]	$S(t) = 16654.0(3.06638 - t)(8.71684 - 5.9035t + t^2)$
	[3.002, 12]	$S(t) = 0.0209596(4.85391 + t)(87.1656 - 17.866t + t^2)$

Answers and Hints for Selected Exercises

3.1 The global maximum is $W(c) = 7.0$. There is a local minimum at $t = c^*$, where $c^* = 8.7$, approximately; however, $W(c^*) = 2.1$, which exceeds $W(0) = 2.0$. So the global minimum is 2.0. W is invertible on [0, c], [c, c*] and [c*, 12]. The concavity of W is down on $[0, \bar{c}]$ and up on $[\bar{c}, 12]$, where it appears from the diagram that $\bar{c} = c$; but we will show in a later lecture that $\bar{c} = 4.3$, approximately.

3.2 Increasing and concave down.

3.3 In the simplest case, the function f is increasing on [0, 3], decreasing on [3, 8], increasing on [8, 10], concave up on [0, 5] and concave down on [5, 10].

3.4 All possible answers have $f(4) = -3$, with either $f(0) = 5$ or $f(10) = 5$ (or both); and f is invertible on [0, 4], where it is decreasing, and [4, 10], where it is increasing.

3.8 The function is concave up throughout its domain (see Exercise 16.3).