6. Discrete probability distributions. Sums of powers of integers

An important application of sequences is to probability distributions. A probability distribution consists of a set of possible outcomes together with a set of associated probabilities. The set of possible outcomes is called the sample space of a discrete probability distribution. A probability density function or p.d.f. is the nonnegative sequence \( \{p_n\} \) defined by

\[
p_n = \text{Prob}(X = n),
\]

where \( \text{Prob}(X = n) \) denotes the probability that \( n \) is drawn at random from the sample space.

For example, in the case of childbirth, if \( \gamma \approx 0.49 \) is the probability of a girl, then

\[
p_2 = \text{Prob}(X = 2) = \gamma \quad \text{and} \quad p_1 = \text{Prob}(X = 1) = 1 - \gamma.
\]

The easiest way to satisfy (6) is to have \( p_2 = \gamma \) and \( p_1 = 1 - \gamma \). If \( k \) is sufficiently large, say, \( k \geq M \), then

\[
p_k = 0
\]

which is usually written as

\[
\sum_{k=1}^{\infty} p_k = 1
\]

In fact, for the sake of simplicity, we will assume that \( X \) is strictly positive. Then

\[
p_0 = 0
\]

Note that, because \( X < 0 \), we must have

\[
p = 0 X = 0 = \text{Prob}(X = 0) = 0
\]

For example, in the case of childbirth, \( P_n = \text{Prob}(X \leq n) \) is the cumulative distribution function or c.d.f., defined by

\[
P_n = \sum_{i=1}^{n} p_i
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A probability density function whose sample space is a set of integers is said to be a discrete probability distribution. A discrete probability distribution has associated with it to a set of numbers, for example, the experimental value of the random variable. For example, if the experiment is the

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A probability density function whose sample space is a set of integers is said to be a discrete probability distribution. A discrete probability distribution has associated with it to a set of numbers, for example, the experimental value of the random variable. For example, if the experiment is the
distribution if satisfies only two conditions, namely,

More generally, any sequence \( \{p_n\} \) on \([1 \ldots \infty)\) is potentially the p.d.f. of some

\[
\begin{align*}
\frac{6}{5} &= e d \\
\frac{6}{5} &= e d \\
0 &= 0 d \\
0 &= d 0
\end{align*}
\]

and

\[
\begin{align*}
8 \cd 0 &= u d \\
\frac{6}{5} &= e d \\
\frac{6}{5} &= e d \\
0 &= 0 d
\end{align*}
\]

from Hussell's (1972) Lapland Longspur data, then Table 5.2 and (11) imply

\[\text{Table 6.1 Leaf thickness distribution for Dicerandra linearifolia}\]

Exercise 6. \( \text{See Table 1 and Figure 1. It is possible, however, to have } p_x \text{ for all positive } x \text{ see}\)

\[
\frac{6}{5} \begin{align*}
\frac{6}{5} &= \frac{6}{5} \\
\text{FREQUENCY } \frac{684}{169} &= \text{FREQUENCY } \frac{684}{169}
\end{align*}
\]

Then

\[
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\end{align*}
\]
Correspondingly, any sequence \( \{P_n\} \) on \([0 \ldots \infty)\) is potentially the c.d.f. of a distribution if it satisfies only three conditions, namely, \(P_0 = 0\), \(P_n \geq P_{n-1}\), \(1 \leq n < \infty\), and \(P_\infty = \lim_{n \to \infty} P_n = 1\).

We can exploit this equivalence to obtain expressions for sums of powers of positive integers, which we need in Lecture 10. We will obtain an expression for the sum of squares, leaving analogous results for cubes and other powers to the exercises; see Exercises 2-4.

Accordingly, consider the sequence defined on \([0 \ldots \infty)\) by

\[
P_n = \begin{cases} 
\frac{n(n + 1)(2n + 1)}{M(M + 1)(2M + 1)} & \text{if } 0 \leq n \leq M \\
1 & \text{if } M + 1 \leq n < \infty
\end{cases}
\]

You can see by inspection that \(P_0 = 0\), \(P_\infty = 1\), and \(P_n \geq P_{n-1}\). Hence \(\{P_n\}\) is a probability distribution, implying in particular that \(P_n = 1\) for \(n \geq M\), implying \(P_{n-1} = 1\) for \(n \geq M + 1\), so that (11a) implies \(p_n = P_n - P_{n-1} = 1 - 1 = 0\) for \(n \geq M + 1\). Hence (11a) reduces to

\[
\sum_{u=1}^{\infty} u d = \sum_{u=1}^{\infty} u d
\]

Substituting into (12), we find that for \(0 = I - I = 1 - u d - u d = u d\), so

Thus a sequence \(\{u d\}\) that depends neither on conditions (11a) and (13a) than anything else that depends on \(u d\) is a probability distribution, implying in particular that \(I = 0\) if \(u d\) is such a sequence. Correspondingly, any sequence \(\{P_n\}\) of a distribution implies

\[
I = \sum_{u=1}^{\infty} u d\]

Because anything that does not depend on can be brought outside the summation,

\[
I = \sum_{u=1}^{\infty} \frac{(I + \infty)(I + \infty)}{9}
\]

implies

(6.13)

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You can see by inspection that \(I = 0\), \(I = 1\), and \(I = 0\). Hence \(\{P_n\}\) is a probability distribution, implying in particular that \(I = 0\) if \(u d\) is such a sequence. Correspondingly, any sequence \(\{P_n\}\) of a distribution implies

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\[
I = \sum_{u=1}^{\infty} u d
\]

Because anything that does not depend on can be brought outside the summation,
\[ n^2 = 1 + \sum_{m=1}^{M} (M + m)(2M + 1) \] (6.20)

For example,

\[ \frac{1^2 + 2^2 + 3^2}{6} = \frac{3 \cdot (3 + 1)(2 \cdot 3 + 1)}{6} = \frac{3 \cdot 4 \cdot 7}{6} = 14 \]

\[ \frac{1^2 + 2^2 + 3^2 + 4^2}{6} = \frac{4 \cdot (4 + 1)(2 \cdot 4 + 1)}{6} = \frac{4 \cdot 5 \cdot 9}{6} = 30 \]

and so on. We will need (20) and similar results in Lectures 10-11.

References

Exercises 6

6.1 Table 5.2 shows clutch sizes observed among four species of arctic passerine. For each species, produce the analogues of Table 1 and Figure 1.

6.2 Use the c.d.f. defined by

\[ P_n = \begin{cases} \frac{n(n+1)}{M(M+1)} & \text{if } 0 \leq n \leq M+1 \\ 1 & \text{if } M+1 < n < \infty \end{cases} \]

and the method of this lecture to establish that

\[ \sum_{n=1}^{\infty} P_n = \frac{1}{2} \sum_{u=1}^{\infty} u \]  

6.3 Use the c.d.f. defined by

\[ P_n = \begin{cases} \frac{n(n+1)}{2(M+1)^2} & \text{if } 0 \leq n \leq M+1 \\ 1 & \text{if } M+1 < n < \infty \end{cases} \]

and the method of this lecture to establish that

\[ \sum_{n=1}^{\infty} P_n = \frac{1}{4} \sum_{u=1}^{\infty} u \]

6.4 Use the c.d.f. defined by

\[ P_n = \begin{cases} \frac{n(n+1)(2n+1)(3n^2+3n-1)}{M(M+1)(2M+1)(3M^2+3M-1)} & \text{if } 0 \leq n \leq M+1 \\ 1 & \text{if } M+1 < n < \infty \end{cases} \]

and the method of this lecture to establish that

\[ \sum_{n=1}^{\infty} P_n = \frac{1}{30} \sum_{u=1}^{\infty} u \]

6.5 A discrete probability density function is defined by

\[ p_n = \begin{cases} 0 & \text{if } n = 1, 2, \ldots, M \\ 1 & \text{if } n \geq M+1 \end{cases} \]

where \( b \) is a constant. What must be the value of \( b \)?

6.6 A discrete probability density function is defined by

\[ p_n = \frac{\pi}{2} n^2, n \geq 1. \]

(i) Sketch the graph of the c.d.f. \( \{P_n\} \) on subdomain \([0 \ldots 10]\).

(ii) What must be the sum of the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{u=1}^{\infty} \frac{1}{u^2} = \frac{\pi^2}{6} \]

6.7 Each species' produce the analogous of Table 5.2 and Figure 1. Table 5.2 shows clutch sizes observed among four species of arctic passerine. For each species, produce the analogues of Table 1 and Figure 1.
For \( n = M \), we have
\[
\frac{(1 + M)M}{2} = \frac{M^2 (M + 1)}{2}
\]
which implies the result.

From Exercise 2',
\[
b = \frac{2M(M + 1)}{2}.
\]

\[6.6\]

\[
\frac{9}{2}
\]

\[6.3\]

Answers and Hints for Selected Exercises