7. Function sequences. Compositions. The exponential and logarithm

In Lectures 5-6 we thought of sequences as functions, with sets of integers for domains, and sets of numbers for ranges. For example, with f_k defined by Table 5.1 and (5.1), leaf thickness frequency has domain [1...15] and range $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}\}$; and with f_k defined by Table 5.2 and (5.2), clutch size frequency has domain [1...7] and range $\{f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$. More generally, a sequence $\{f_k\}$ is defined on [L...M] and has range $\{f_L, f_{L+1}, f_{L+2}, \ldots, f_{M-2}, f_{M-1}, f_M\}$: the range is an ordered list of labels, with first term f_L and last term f_M. Furthermore, the labels tell us everything there is to know about the sequence. Thus an alternative view of sequences is that they are ordered lists. When we think of a sequence as an ordered list, however, there is no special reason for the list to be a list of numbers. In particular, it could instead be a list of functions. The sequence is then a **function sequence**.

The only difference between an ordinary sequence $\{o_n\}$ and a function sequence $\{f_n\}$ is that o_n is a number, whereas f_n is a function. Because f_n is a function, we need to know its domain, [a, b]. Because $\{f_n\}$ is a sequence, we need to know [L...M]. Therefore, in principle, we should use $\{f_n(x) \mid L \le n \le M, a \le x \le b\}$ to denote a function sequence. In practice, however, the values of a, b, L (usually 0 or 1) and M (usually ∞) are obvious from context. So in practice we can denote a function sequence simply by $\{f_n(x)\}$. We think of $\{f_n(x)\}$ as an ordered list of functions.

n	y_n	a_n	$u_n = u_n(x)$
0	1	0	1
1	0	1	1
2	x	1	1+x
3	x	1+x	1+2x
4	$x(1+x)$	$1+2x$	$1+3x+x^2$
5	$x(1+2x)$	$1+3x+x^2$	$1+4x+3x^2$
6	$x(1+3x+x^2)$	$1+4x+3x^2$	$1+5x+6x^2+x^3$
7	$x(1+4x+3x^2)$	$1+5x+6x^2+x^3$	$1+6x+10x^2+4x^3$
8	$x(1+5x+6x^2+x^3)$	$1+6x+10x^2+4x^3$	$1+7x+15x^2+10x^3+x^4$
9	$x(1+6x+10x^2+4x^3)$	$1+7x+15x^2+10x^3+x^4$	$1+8x+21x^2+20x^3+5x^4$
10	$x(1+7x+15x^2+10x^3+x^4)$	$1+8x+21x^2+20x^3+5x^4$	$1+9x+28x^2+35x^3+15x^4+x^5$

Table 7.1 Fibonacci polynomials

Suppose, for example, that reproduction in a Fibonacci population is not as perfect as Fibonacci supposed. Specifically, it is no longer true that every pair of rabbits reproduces itself with certainty every month; rather, it reproduces with probability x (and so fails to reproduce itself with probability $1 - x$), where $0 \le x \le 1$. It is now no longer true that the initial pair contributes a pair of newborns by the end of month 2; in terms of Lecture 5, $y_2 \neq 2$. But the *expected* number of newborn pairs at the end of month 2 is x (in the sense that a very large number, N, of identical but independent Fibonacci breeding experiments would yield Nx newborn pairs by the end of February), and so the *expected* total of rabbit pairs at the end of February is $1 + x$. Thus, if we reinterpret y_k as expected number of young pairs at the end of month k, a_k as expected number of adult pairs at the end of month k and u_k as expected total number of pairs at the end of month k, then $y_2 = x$, $a_2 = 1$ and $u_2 = 1 + x$. See Table 1.

More generally, when reproduction is uncertain, it isn't true that the a_{n-1} adults at the end of month n–1 produce a_{n-1} young at the end of month n. Nevertheless, if we multiply a_{n-1} by the probability that a pair reproduces, which is x, we find that the *expected* number of young at the end of month n is

$$y_n = xa_{n-1}, \tag{7.1}$$

which agrees with (5.9) when x = 1. We interpret (1) as saying that a_{n-1} adults produce xa_{n-1} newborns on average (where the average is taken over a large number of independent Fibonacci breeding experiments). Our model continues to exclude mortality: a young rabbit still becomes an adult after a month has elapsed. So expected number of adults at the end of month n still equals expected number of young at the end of month n–1 plus expected number of adults at the end of month n–1:

$$a_n = y_{n-1} + a_{n-1}. \tag{7.2}$$

In other words, (5.10) still holds, except that we now interpret a_n and y_n as averages (over a very large number of Fibonacci experiments). With the same reinterpretation, (5.4) also still holds, i.e., total expected number of rabbits at the end of month n is

$$u_n = a_n + y_n. \tag{7.3}$$

From (1) and (2), we 5 find that

$$y_{n+1} = xa_n \tag{7.4a}$$
$$a_{n+1} = y_n + a_n \tag{7.4b}$$

replaces (5.11), implying (Exercise 1)

$$u_{n+1} = u_n + xu_{n-1} \tag{7.5}$$

in place of (5.12). Thus expected total of rabbit pairs at time n is defined implicitly by

$$u_0 = 1 \tag{7.6a}$$
$$u_1 = 1 \tag{7.6b}$$
$$u_{n+1} = u_n + xu_{n-1} \quad \text{if} \quad n \geq 1. \tag{7.6c}$$

For example, because $u_2 = 1 + x$, we have $u_3 = u_2 + xu_1 = 1 + x + x = 1 + 2x$, $u_4 = u_3 + xu_2 = 1 + 2x + x(1+x) = 1 + 3x + x^2$, and so on; see Table 1.

If we set $u_n = f_n(x)$, then the expected totals at the end of each month define a function sequence $\{f_n(x)\}$ in which each term f_n has domain [0, 1]. For example, because $u_0 = 1$, $u_1 = 1$, $u_2 = 1 + x$, $u_3 = 1 + 2x$ and $u_4 = 1 + 3x + x^2$, the first five functions in the sequence, namely, f_0, f_1, f_2, f_3 and f_4, are defined by $f_0(x) = 1$, $f_1(x) = 1$, $f_2(x) = 1 + x$, $f_3(x) = 1 + 2x$ and $f_4(x) = 1 + 3x + x^2$. We will call these functions the Fibonacci polynomials. In defining these polynomials, we have been careful to distinguish between f_n (which is a function) and u_n (which is a label in the function's range) by using a different notation for each. As remarked at the end of Lecture 2, however, it is convenient in practice to use a single notation for both function and label, because it is obvious from context which meaning is intended. Henceforward, therefore, we use u in place of f and write $u_n = u_n(x)$ for the (n+1)-th Fibonacci polynomial, as in Table 1.

A more interesting function sequence compares expected number of rabbit pairs at the end of a month with expected number at the end of the previous month. By analogy with Lecture 5, we define the function sequence $\{\phi_n(x)\}$ by

$$\phi_n(x) = \frac{u_n(x)}{u_{n-1}(x)}, \quad n \geq 1. \tag{7.7}$$

Each ϕ_n has the same domain as u_n, namely, [0, 1]. Alternatively, dividing (6) by u_n and proceeding as in Lecture 5, we can define $\{\phi_n(x)\}$ recursively by

$$\phi_1 = 1 \tag{7.8a}$$

$$\phi_{n+1} = 1 + \frac{x}{\phi_n} \quad \text{if } n \geq 1 \tag{7.8b}$$

(Exercise 1). For example, $\phi_2 = 1 + x/\phi_1 = 1 + x/1 = 1 + x$, $\phi_3 = 1 + x/\phi_2 = 1 + x/(1+x) = (1 + 2x)/(1 + x)$, and so on (see Table 2 and Exercise 3). Thus the first three functions of the sequence $\{\phi_n(x)\}$, namely, ϕ_1, ϕ_2 and ϕ_3, are defined by $\phi_1(x) = 1$, $\phi_2(x) = 1 + x$ and $\phi_3(x) = (1 + 2x)/(1 + x)$. Each ϕ_n is a rational function on [0, 1], and so we refer to ϕ_n as a Fibonacci rational function. (Again, it is strictly speaking an abuse of notation to write $\phi_n = \phi_n(x)$ for the n-th Fibonacci rational function, but it is convenient in practice because it is obvious from context whether ϕ_n means a function or a label in its range.)

n	$\phi_n = \phi_n(x)$
1	1
2	$1+x$
3	$(1+2x)/(1+x)$
4	$(1+3x+x^2)/(1+2x)$
5	$(1+4x+3x^2)/(1+3x+x^2)$
6	$(1+5x+6x^2+x^3)/(1+4x+3x^2)$
7	$(1+6x+10x^2+4x^3)/(1+5x+6x^2+x^3)$
8	$(1+7x+15x^2+10x^3+x^4)/(1+6x+10x^2+4x^3)$
9	$(1+8x+21x^2+20x^3+5x^4)/(1+7x+15x^2+10x^3+x^4)$
10	$(1+9x+28x^2+35x^3+15x^4+x^5)/(1+8x+21x^2+20x^3+5x^4)$

Table 7.2 Fibonacci rational functions

The graph of ϕ_n, i.e., the curve $y = \phi_n(x)$, is shown in Figure 1 as a solid curve for $n = 1, ..., 6$. The dashed curve is $y = \phi_\infty(x)$, where ϕ_∞ is defined on [0, 1] by

$$\phi_\infty(x) = \frac{1}{2}\left\{1 + \sqrt{1+4x}\right\}. \tag{7.9}$$

Notice that, as n gets larger and larger, the solid curves appear to converge toward the dashed curve, with $y = \phi_n(x)$ above or below the dashed curve according to whether n is even or odd. In other words, it appears that

$$\lim_{n \to \infty} \phi_n(x) = \phi_\infty(x) \tag{7.10}$$

and that convergence is oscillatory. We prove these results in Appendix 7.

The upshot of all this is that the limit of a convergent function sequence is yet another function. But convergence is a two-sided coin. Its other side is that a function can be defined as the limit of a function sequence. Suppose, for example, that your savings account earns compound interest at an annual rate of x (usually quoted as 100x %, e.g., a rate of 6% means $x = 0.06$). If you deposit a dollar today, how much will it be worth a year from now?

The answer depends on how often the interest is compounded. If the interest is compounded only once, at the end of the year, then your dollar is worth only $1 + x$. If the interest is compounded twice, once after six months and again at year's end, then the dollar is worth $1 + x/2$ after six months, and whatever you have after six months is worth $1 + x/2$ times as much at year's end. In other words, at the end of the year your dollar is worth $(1 + x/2)(1 + x/2) = (1+\frac{x}{2})^2$. Similarly, if the interest is compounded quarterly, then after three months your dollar is worth $1 + x/4$, and at year's end it is worth $(1+\frac{x}{4})^4$.

This argument is readily generalized. Let $\phi_n(x)$ be how much your dollar is worth at year's end if interest is compounded n times a year. Then $\{\phi_n(x)\}$ is a function sequence, defined by

$$\phi_n(x) = \left(1+\frac{x}{n}\right)^n, \quad n \geq 1, \ 0 \leq x < \infty. \qquad (7.11)$$

The sequence is graphed in Figures 2-3, where $y = \phi_n(x)$ is shown as a solid curve for n = 1, ..., 6 in Figure 2 and for $n = 2^m$, where m = 1, ..., 6, in Figure 3. Note that the solid curves converge from below toward the dashed curve, which we denote by $y = \phi_\infty(x)$. Because $\phi_\infty(x)$ is the limit of $\phi_n(x)$ as $n \to \infty$, ϕ_∞ tells you how much your dollar would be worth at year's end, at interest rate x, if interest were compounded continuously from the moment you put your dollar in the bank. It is such an important function in mathematics that we give it a special name, the **exponential** function, and we denote it by the symbol exp. Thus exp is defined on $[0, \infty)$ by

$$\exp(x) = \phi_\infty(x) = \lim_{n\to\infty} \phi_n(x) \qquad (7.12)$$

(although only its restriction to [0, 1] is graphed in Figures 2-3). Note in particular that

$$\exp(0) = 1 \qquad (7.13)$$

(because $\phi_n(0) = 1$ for every value of n, it remains so in the limit as $n \to \infty$).

Observe that exp is increasing and concave upward on $[0, \infty)$. Thus, from (13) and Exercise 1.1, the range of exp is $[\exp(0), \exp(\infty)) = [1, \infty)$. Because exp is increasing, it is invertible; and, again from Exercise 1.1, the inverse function has domain $[1, \infty)$ and range $[0, \infty)$. Moreover, because exp is concave upward on $[0, \infty)$, the inverse function is concave downward on $[1, \infty)$, by Exercise 3.2.

This inverse function is just as important in mathematicas as the exponential function, and so it also has a special name. We call it the **logarithmic** function and denote it by the symbol ln (for natural logarithm). Thus, in particular, (13) implies that

$$\ln(1) = 0. \qquad (7.14)$$

The graphs of exp and ln are sketched in Figure 4. Further properties will be discussed in Lectures 20 and 22.

Other functions, e.g., polynomials, can be combined with exp or ln to form sums, products, quotients or joins. For example, because exp(x) is always positive on $[0, \infty)$, we can define a quotient q on $[0, \infty)$ by

$$q(x) = \frac{1}{\exp(x)}. \qquad (7.15)$$

By (13), $q(0) = 1$. Moreover, because exp(x) gets larger and larger as x increases, q(x) gets smaller and smaller as x increases until eventually it approaches zero. In other words, q is strictly decreasing on $[0, \infty)$ with range $(0, 1]$ and

$$q(\infty) = \lim_{x \to \infty} q(x) = 0, \tag{7.16}$$

so that y = 0 is a horizontal asymptote to the graph y = q(x). Furthermore, because q is strictly decreasing, it has an inverse, say r, with domain $(0, 1]$ and range $[0, \infty)$, and x = 0 is a vertical asymptote to the graph x = r(y). See Figure 5 and Exercise 6.

Not every combination, however, is a sum,product, quotient or join. Our fifth, and final, category of combination is composition, which we now define. Accordingly, let U be a function with domain [a, b], and let Q be another function whose domain is the *range* of U. Suppose $a \le x \le b$. Then U(x) lies in the range of x, which means that U(x) lies in the domain of Q, which in turn means that Q(U(x)) is well defined. So

$$R(x) = Q(U(x)), \qquad a \le x \le b \tag{7.17}$$

defines a function R whose domain and range coincide with those of U and Q, respectively. Furthermore, if U is increasing on [a, b] and Q is increasing on the range of U then R is increasing on [a, b]. We call R the **composition** of U and Q and write R = QU. Note, incidentally, that Q(U(x)) is not the same thing as Q(x)U(x), which is why we use we use QU for composition and $Q \bullet U$ for product.

Suppose for example, that $[a, b] = [0, \infty)$ and

$$U(x) = 1 + x^2, \tag{7.18}$$

so that the range of U is $[1, \infty)$. Then a legitimate Q for composition is any function defined on $[1, \infty)$, e.g., Q defined by

$$Q(y) = \ln(y). \tag{7.19}$$

Now, from (17)-(19),

$$R(x) = Q(U(x)) = \ln(U(x)) = \ln(1 + x^2) \tag{7.20}$$

defines a composition whose domain and range are both $[0, \infty)$, because $[0, \infty)$ is both the domain of U and the range of Q. Note that R is strictly increasing, because both U and Q are strictly increasing.

For another example, suppose that $[a, b] = [0, \infty)$ and that U is the nonnegative integer power function defined by

$$U(x) = Ax^m, \quad A > 0, \quad m \ge 1. \tag{7.21}$$

For any A or m, the range of U is $[0, \infty)$, and so a legitimate Q for composition is any function defined on $[0, \infty)$, e.g., Q defined by

$$Q(y) = \exp(y). \tag{7.22}$$

Now, from (17)-(19), the composition R is defined by

$$R(x) = Q(U(x)) = \exp(U(x)) = \exp(Ax^m). \tag{7.23}$$

Again, both domain and range of R are $[0, \infty)$; and R is strictly increasing, because U and Q are both strictly increasing.

Once a composition has been defined, it is just like any other function, and so it can be combined with many other functions to form sums, products, quotients, joins or further compositions. For example, because R is strictly increasing, R(x) must exceed R(0) = 1; and so, in particular, R(x) is never zero. Thus we can define a quotient q on $[0, \infty)$ by

$$q(x) = \frac{1}{R(x)} = \frac{1}{\exp(Ax^m)}. \tag{7.24}$$

Furthermore, because R is strictly increasing, q must be strictly decreasing. Of course, (15) is the special case of (24) in which $A = 1 = m$.

Exercises 7

7.1 Verify (5) and (8).

7.2* Obtain explicit expressions for $u_5(x)$, $u_6(x)$, $u_7(x)$, $u_8(x)$, $u_9(x)$ and $u_{10}(x)$, defined by (6). In other words, verify Table 1 for the first eleven Fibonacci polynomials. Verify that your results are consistent with those of Lecture 5 when $x = 1$.

7.3* Obtain explicit expressions for $\phi_4(x)$, $\phi_5(x)$, ..., $\phi_{10}(x)$, defined by (8). In other words, verify Table 2 for the first ten Fibonacci rational functions. Verify that your results are consistent with those of Lecture 5 when $x = 1$.

7.4 The function sequence $\{s_n(x) \mid n \geq 0, 0 \leq x \leq 4\}$ is defined recursively by

$$s_0 = 1$$

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{x}{s_n}\right), \quad n \geq 0.$$

(i) Find rational-function expressions for $s_1(x)$, $s_2(x)$, $s_3(x)$ and $s_4(x)$.

(ii) What function s_∞ is defined on [0, 4] by $s_\infty(x) = \lim_{n\to\infty} s_n(x)$?

(iii) Show graphically that the function sequence converges from above (in contrast to Figure 2), in the sense that $s_n(x) \geq s_\infty(x)$ for $n \geq 1$.

Hints: Proceed by analogy with Appendix 4 for (ii), and use Mathematica for (iii).

7.5* The function sequence $\{s_n(x) \mid n \geq 0, 0 \leq x \leq 8\}$ is defined recursively by

$$s_0 = 1$$

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{x}{s_n^2}\right), \quad n \geq 0.$$

(i) Find rational-function expressions for $s_1(x)$, $s_2(x)$ and $s_3(x)$.

(ii) What function s_∞ is defined on [0, 8] by $s_\infty(x) = \lim_{n\to\infty} s_n(x)$?

(iii) Show graphically that the function sequence converges.

7.6 Show that r defined by $r(y) = \ln(1/y)$ is the inverse of q in (15). See Figure 5.

7.7* A sequence $\{H_n(x)\}$ of functions called the Hermite polynomials is defined by the recurrence relation

$$H_0 = 1$$

$$H_1 = 2x$$

$$H_{n+1} = 2(xH_n - nH_{n-1}), \quad n \geq 1.$$

Show that

$$H_4(x) = 4(4x^4 - 12x^2 + 3)$$

and find $H_7(x)$.

7.8* A sequence $\{L_n(x)\}$ of functions called the Laguerre polynomials is defined by the recurrence relation

$$L_0 = 1$$
$$L_1 = 1 - x$$
$$L_{n+1} = (2n+1-x)L_n - n^2 L_{n-1}, \quad n \ge 1.$$

Show that

$$L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$$

and find $L_6(x)$.

7.9* A sequence $\{P_n(x)\}$ of functions called the Legendre polynomials is defined by the recurrence relation

$$P_0 = 1$$
$$P_1 = x$$
$$P_{n+1} = \frac{(2n+1)x}{n+1} P_n - \frac{n}{n+1} P_{n-1}, \quad n \ge 1.$$

Show that

$$P_3(x) = \frac{1}{2}x(5x^2 - 3)$$

and find $P_5(x)$.

7.10 The compositions f and g are defined by $f(x) = H_3(L_2(x))$ and $g(x) = L_2(H_3(x))$, where H_3 and L_2 are defined in Exercises 7-8. Find explicit expressions for f(x) and g(x). What are the orders of these polynomials?

7.11* The compositions f and g are defined by $f(x) = P_2(L_3(x))$ and $g(x) = L_3(P_2(x))$, where L_3 and P_2 are defined in Exercises 8-9. Find explicit expressions for f(x) and g(x). What are the orders of these polynomials?

7.12 If g and h are inverse functions, what are (i) g(h(x)) (ii) h(g(y))? Why?

7.13*

NAME	DOMAIN	ASSIGNMENT RULE
U	[0, 1]	$U(x) = 1 + 4x$
Q	[1, 5]	$Q(y) = \sqrt{y} = y^{1/2}$
R	$[1, \sqrt{5}]$	$R(z) = 1 + z$
S	[0, 1]	$S(x) = 2x$
V	[0, 1]	$V(x) = R(Q(U(x)))$
ρ	[0, 1]	$\rho(x) = \dfrac{S(x)}{V(x)}$

The table above defines three linear functions, namely, U, R and S; a power function, namely, Q; a composition of a composition, namely, V; and a quotient, namely, ρ. Find an explicit expression for ρ(x) and sketch the graph of ρ. Verify that ρ is increasing. What is its global maximum?

7.14 A sequence of functions $\{H_n(x)\}$ called Hermite polynomials is defined in Exercise 4.7, and a sequence of functions $\{P_n(x)\}$ called Legendre polynomials is defined in Exercise 4.9. The compositions f and g are defined by $f(x) = H_3(P_4(x))$ and $g(x) = H_4(P_3(x))$. Find explicit expressions for f(x) and g(x). What are the orders of these polynomials?

Appendix 7: Convergence of the Fibonacci rational function sequence

The purpose of this appendix is to establish the convergence of the sequence $\{\phi_n(x)\}$ defined by (8). Note that (8) is identical to (3.16) when $x = 1$. Hence, from Lecture 3, we already know that $\{\phi_n(1)\}$ converges. Moreover, if $x = 0$ then (8) implies $\phi_n = 1$, so that $\{\phi_n(0)\}$ is again convergent. So assume that $0 < x < 1$.

We first determine what the limit of $\{\phi_n(x)\}$ must be, *if* the sequence converges. Accordingly, suppose that the limit exists, and call it ϕ_∞. As before, if $\phi_n \to \phi_\infty$ as $n \to \infty$ then $\phi_{n+1} \to \phi_\infty$ as $n \to \infty$. Hence, in the limit, (8) implies

$$\phi_\infty = 1 + \frac{x}{\phi_\infty}. \tag{7.A1}$$

This equation is quadratic with only one positive solution, namely,

$$\phi_\infty = \frac{1}{2}\left\{1 + \sqrt{1+4x}\right\}. \tag{7.A2}$$

Now we can prove that $\phi_n(x)$ must converge. Recall that $x < 1$, implying from (8) that $\phi_n > 1$ for $n \geq 2$. Subtracting (A1) from (8) and rearranging, we obtain

$$\phi_{n+1} - \phi_\infty = -\frac{x}{\phi_n\phi_\infty}\left\{\phi_n - \phi_\infty\right\}, \tag{7.A3}$$

so that

$$\begin{aligned} \left|\phi_{n+1} - \phi_\infty\right| &= \frac{x}{\phi_n\phi_\infty}\left|\phi_n - \phi_\infty\right| \\ &< \frac{x}{\phi_\infty}\left|\phi_n - \phi_\infty\right| , \qquad (7.A4) \\ &= \rho(x)\left|\phi_n - \phi_\infty\right|, \end{aligned}$$

where ρ is defined on [0, 1] by

$$\rho(x) = \frac{2x}{1+\sqrt{1+4x}}. \tag{7.A5}$$

You show in Exercise 13, however, that $\rho(x) \leq \rho(1) = 2/\{1+\sqrt{5}\} = 0.618$. Thus (A4)

$$\left|\phi_{n+1} - \phi_\infty\right| < 0.62\left|\phi_n - \phi_\infty\right|, \tag{7.A6}$$

regardless of the value of x. That is, the distance between ϕ_n and ϕ_∞ is reduced by at least 38% at each iteration of the recurrence relation, and must eventually approach zero. Moreover, from (A3), if $\phi_n > \phi_\infty$ then $\phi_{n+1} < \phi_\infty$, and vice versa. That is, the convergence is oscillatory.

Answers and Hints for Selected Exercises

7.4 (ii) $s_\infty(x) = \sqrt{x}$

7.5 (ii) $s_\infty(x) = x^{1/3}$

7.6 If r is the inverse of q, then $y = q(x) \Leftrightarrow x = r(y)$. But $q(x) = 1/\exp(x)$. So $x = r(y) \Leftrightarrow y = 1/\exp(x)$, or $x = r(y) \Leftrightarrow \exp(x) = 1/y$. But ln is the inverse of exp, meaning $\exp(x) = 1/y \Leftrightarrow x = \ln(1/y)$. Therefore $x = r(y) \Leftrightarrow x = \ln(1/y)$, or $r(y) = \ln(1/y)$.

7.7 $H_7(x) = 16x(8x^6 - 84x^4 + 210x^2 - 105)$.

7.8 $L_6(x) = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$.

7.9 $P_5(x) = \frac{1}{8}x(63x^4 - 70x^2 + 15)$

7.10 From Exercises 7-8, we have $H_3(x) = 8x^3 - 12x$ and $L_2(x) = x^2 - 4x + 2$. So $H_3(\bullet) = 8(\bullet)^3 - 12(\bullet)$, for any $\bullet$ whatsoever. In particular,

$$\begin{aligned} H_3(L_2(x)) &= 8\{L_2(x)\}^3 - 12L_2(x) \\ &= 4L_2(x)\left(2\{L_2(x)\}^2 - 3\right) \\ &= 4(x^2 - 4x + 2)\left(2\{x^2 - 4x + 2\}^2 - 3\right) \\ &= 4(x^2 - 4x + 2)\left(2\{x^4 - 8x^3 + 20x^2 - 16x + 4\} - 3\right) \\ &= 4(x^2 - 4x + 2)\left(2x^4 - 16x^3 + 40x^2 - 32x + 5\right) \\ &= 4\{2x^6 - 24x^5 + 108x^4 - 224x^3 + 213x^2 - 84x + 10\}. \end{aligned}$$

Similarly,

$$\begin{aligned} L_2(H_3(x)) &= \{H_3(x)\}^2 - 4H_3(x) + 2 \\ &= (8x^3 - 12x)^2 - 4(8x^3 - 12x) + 2 \\ &= 8^2x^6 - 2\cdot 8\cdot 12\cdot x^3\cdot x + 12^2x^2 - 32x^3 + 48x + 2 \\ &= 2\{32x^6 - 96x^4 - 16x^3 + 72x^2 + 24x + 1\}. \end{aligned}$$

The order of each polynomial is 6.

7.12 (i) x (ii) y

7.13 Because $R(z) = 1 + z$, we have $R(Q(U(x))) = 1 + Q(U(x))$. So $V(x) = 1 + Q(U(x))$. Because $Q(y) = \sqrt{y}$, we have $Q(U(x)) = \sqrt{U(x)}$. So $V(x) = 1 + \sqrt{U(x)} = 1 + \sqrt{1+4x}$, implying

$$\rho(x) = \frac{2x}{1 + \sqrt{1+4x}}$$

Because ρ is increasing, the global maximum is $\rho(1) = 2/\{1+\sqrt{5}\} = 0.618$.

7.14. From Exercise 4.7 we have $H_3(x) = 4x(2x^2-3)$, so $H_3(P_4(x)) = 4P_4(x)\,(2\{P_4(x)\}^2-3)$. From Exercise 4.9 we have $P_4(x) = (35x^4 - 30x^2 + 3)/8$, implying $P_4(H_3(x)) = (35\{H_3(x)\}^4 - 30\{H_3(x)\}^2 + 3)/8$. So

$$\begin{aligned} f(x) &= \frac{1}{2}\left(35x^4 - 30x^2 + 3\right)\left(\frac{(35x^4 - 30x^2 + 3)^2}{32} - 3\right) \\ &= \frac{1}{64}(35x^4 - 30x^2 + 3)(1225x^8 - 2100x^6 + 1110x^4 - 180x^2 - 87) \end{aligned}$$

and

$$\begin{aligned} g(x) &= \frac{1}{8}\left(35\{4x(2x^2-3)\}^4 - 30\{4x(2x^2-3)\}^2 + 3\right) \\ &= 1120x^4(2x^2-3)^4 - 60x^2(2x^2-3)^2 + \frac{3}{8} \\ &= 17920x^{12} - 107520x^{10} + 241920x^8 - 242160x^6 + 91440x^4 - 540x^2 + \frac{3}{8}. \end{aligned}$$

Each polynomial has order 12.