9. From index function to ordinary function. Ventricular recharge

An essential difference between ordinary and index functions is that an ordinary function, say F, yields local properties whereas an index function, say Index, yields global properties. For given t in [a, b], F can ignore every number in the subdomain except t, yet still yield the label F(t). By contrast, Index describes overall properties of F on [a, b] by paying attention to at least two points (e.g., if Index = Diff or Index = DQ) and often the whole interval (e.g., if Index = Max, Min, Area or Int).

Despite this difference, there is an important relationship between index functions and ordinary functions because Index(F, [a, b]) generates an ordinary function if we hold both F and a fixed while varying b. For example, if V(t) denotes ventricular volume at time t in our cardiac cycle, and if $V_{min}(t)$ denotes the lowest such volume achieved since the cycle began, then an ordinary function V_{min} is generated by Index = Min according to

$$V_{min}(t) \quad = \quad \mathrm{Min}(V,[0,t]). \qquad (9.1)$$

The graph of V_{min} is sketched in Figure 1 as the solid curve, with that of V shown dashed for comparison.

Again, we can generate an ordinary function F from the function f graphed in the first three panels of Figure 2 by defining

$$F(t) \quad = \quad \mathrm{Area}(f,[0,t]) \quad = \quad \text{Area of region } 0 \le x \le t,\ 0 \le y \le f(x). \qquad (9.2)$$

Here x denotes a generic THING in the domain of f, whereas t denotes a generic THING in the domain of F; we must use different letters, because the right-hand boundary of the shaded region in Figure 2 is at x = t. The physiological interpretation of F, as we will discover in Lecture 12, is that F(t) is the volume of blood discharged into the aorta during the first t seconds of a cardiac cycle with ventricular outflow defined by Figure 2. Here we focus merely on how to calculate F from f, whose algebraic definition is

$$f(x) \quad = \quad \begin{cases} 0 & \text{if } 0 \le x \le 0.05 \\ 465(20x-1) & \text{if } 0.05 \le x \le 0.1 \\ 465 & \text{if } 0.1 \le x \le 0.15 \\ 310(3-10x) & \text{if } 0.15 \le x \le 0.3. \end{cases} \qquad (9.3)$$

From (3), there are four cases, according to which subdomain of [0, 0.3] contains t. First, the easiest case is when $t \in [0, 0.05]$: because f(x) = 0 for $0 \le x \le 0.05$ and $t \le 0.05$, we have f(x) = 0 for $0 \le x \le t$, and so F(t) = Area(f, [0, t]) = 0. This result holds for all $t \in [0, 0.05]$, including t = 0.05. So, in particular, F(0.05) = 0.

The second case is when $t \in [0.05, 0.1]$. Then, from (8.17) with a = 0, c = 0.05 and b = t, we have

$$\begin{aligned} \mathrm{Area}(f,[0,t]) &= \mathrm{Area}(f,[0,0.05]) + \mathrm{Area}(f,[0.05,t]) \\ &= F(0.05) + \mathrm{Area}(f,[0.05,t]) \qquad (9.4) \\ &= 0 + \mathrm{Area}(f,[0.05,t]) \end{aligned}$$

So, from (2) and (4), for $0.05 \le t \le 1$ we have

$$F(t) \quad = \quad \mathrm{Area}(f,[0.05,t]) \qquad (9.5)$$

From Figure 2(a), however, Area(f, [0.05, t]) is the area of a triangle with base t – 0.05 and height f(t) = 465(20t – 1), by (3). So for $0.05 \le t \le 1$ we have

$$\begin{aligned} F(t) &= 0.5(t-0.05)f(t) = 0.5(t-0.05)465(20t-1) \\ &= 4650t^2 - 465t + 11.625. \end{aligned} \tag{9.6}$$

In particular, $F(0.1) = 11.625$.

The third case to consider is when $t \in [0.1, 0.15]$. Now, from (8.17) with $a = 0$, $c = 0.1$ and $b = t$, we have

$$\begin{aligned} \mathrm{Area}(f,[0,t]) &= \mathrm{Area}(f,[0,0.1]) + \mathrm{Area}(f,[0.1,t]) \\ &= F(0.1) + \mathrm{Area}(f,[0.1,t]) \\ &= 11.625 + \mathrm{Area}(f,[0.1,t]) \end{aligned} \tag{9.7}$$

So, from (2) and (7), for $0.1 \le t \le 0.15$ we have

$$F(t) = 11.625 + \mathrm{Area}(f,[0.1,t]) \tag{9.8}$$

From Figure 2(b), however, Area(f, [0.1, t]) is the area of a rectangle with base $t - 0.1$ and height $f(t) = 465$, by (3). So for $0.1 \le t \le 0.15$ we have

$$\begin{aligned} F(t) &= 11.625 + (t-0.1)f(t) = 11.625 + 465(t-0.1) \\ &= 465t - 34.875 \end{aligned} \tag{9.9}$$

In particular, $F(0.15) = 34.875$.

The last case to consider is when $t \in [0.15, 0.3]$. Now, from (8.17) with $a = 0$, $c = 0.15$ and $b = t$, we have

$$\begin{aligned} \mathrm{Area}(f,[0,t]) &= \mathrm{Area}(f,[0,0.15]) + \mathrm{Area}(f,[0.15,t]) \\ &= F(0.15) + \mathrm{Area}(f,[0.15,t]) \\ &= 34.875 + \mathrm{Area}(f,[0.15,t]) \end{aligned} \tag{9.10}$$

So, from (2) and (10), for $0.15 \le t \le 0.3$ we have

$$F(t) = 34.875 + \mathrm{Area}(f,[0.15,t]) \tag{9.11}$$

From Figure 2(c), however, Area(f, [0.15, t]) is the area of a trapezium of width $t - 0.15$, maximum height $f(0.15) = 465$ and minimum height $f(t) = 310(3-10t)$. We can place two such trapeziums together to form a rectangle of width $t - 0.15$ and height $f(0.15) + f(t)$. The area of each trapezium is half that of the rectangle. So for $0.15 \le t \le 0.3$ we have

$$\begin{aligned} F(t) &= 34.875 + 0.5(t-0.15)\{f(0.15) + f(t)\} \\ &= 34.875 + 0.5(t-0.15)\{465 + 310(3-10t)\} \\ &= 930t - 1550t^2 - 69.75. \end{aligned} \tag{9.12}$$

In particular, $F(0.3) = 69.75$ is the stroke volume. Gathering together (6), (9) and (12), we find that F is the join defined on [0, 0.3] by

$$F(t) = \begin{cases} 0 & \text{if } 0 \le t \le 0.05 \\ 4650t^2 - 465t + 11.625 & \text{if } 0.05 \le t \le 0.1 \\ 465t - 34.875 & \text{if } 0.1 \le t \le 0.15 \\ 930t - 1550t^2 - 69.75 & \text{if } 0.15 \le t \le 0.3. \end{cases} \tag{9.13}$$

Its graph is shown in Figure 2(d). Notice that F is smooth, even though f has corners. For further practice, see Exercises 1-4.

The above calculation can arguably be simplified with the help of two general results about Area. The first result is that the area enclosed by a sum of nonnegative functions, say g and h, equals the sum of the areas enclosed by each in the sense that

$$\text{Area}(g+h,[a,b]) \;=\; \text{Area}(g,[a,b]) \;+\; \text{Area}(h,[a,b]). \qquad (9.14a)$$

The easiest way to obtain this result is from Figure 3. Imagine that Area(g, [a, b]) at top left in Figure 3 has been painted from left to right with a magic brush that tracks the graph of g, so that the width of the brush at x is always g(x) and no paint leaks outside the shaded area. Similarly imagine that Area(h, [a, b]) at top right in Figure 3 has been painted with a brush that tracks the graph of h. The area at bottom left, which is Area(g+h, [a, b]), is in principle painted by a third brush tracking the graph of g + h, but in practice the same effect is achieved by painting from left to right with the second brush held above the first. In other words, the dark and light shaded regions are equal in area, which establishes (14). Similarly, the area at bottom right is Area(h+g, [a, b]); it requires no new brush to track h + g because holding the first brush above the second achieves the same effect. Again, the dark and light shaded regions are equal in area to those in the other panels, which establishes that

$$\text{Area}(h+g,[a,b]) \;=\; \text{Area}(h,[a,b]) \;+\; \text{Area}(g,[a,b]). \qquad (9.14b)$$

Of course, (14a) and (14b) are equivalent, because g + h is the same function as h + g. A similar paintbrush argument (see Figure 4, where the light and dark shaded regions all have the same area) reveals the second result, namely, that

$$\text{Area}(kg,[a,b]) \;=\; k\,\text{Area}(g,[a,b]), \qquad (9.15a)$$

for any nonnegative constant k, i.e., that the area enclosed by k times a function is k times the area that the function encloses. In this equation, kg is a shorthand for a function that labels things in g's domain by k times as much as g labels them. In other words, if a function z is defined by $z(t) = k \cdot g(t)$, then kg is a shorthand for z. We call z a **multiple** of g.[1]

We can combine (14) and (15) into a single result as follows. If q is another nonnegative constant, then (15) implies

$$\text{Area}(qh,[a,b]) \;=\; q\,\text{Area}(h,[a,b]), \qquad (9.15b)$$

whereas (14) yields

$$\text{Area}(kg+qh,[a,b]) \;=\; \text{Area}(kg,[a,b]) \;+\; \text{Area}(qh,[a,b]). \qquad (9.16)$$

Combining (15) and (16), we have

$$\text{Area}(kg+qh,[a,b]) \;=\; k\cdot\text{Area}(g,[a,b]) \;+\; q\cdot\text{Area}(h,[a,b]), \qquad (9.17)$$

which is the result we sought: (14) is a special case of (17) with k = 1 and q = 1, whereas (15) is a special case of (17) with k = 0 or q = 0.

Now, to obtain (17), we implicitly assumed that k and q are both nonnegative. We will show in Lecture 12, however, that (17) holds for any k or q provided g, h and kg + qh are all nonnegative; in other words, (17) holds whenever it is well defined. So we can use it to obtain expression (13) for F(t). We first define functions g and h by

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[1] A sum of multiples is called a linear combination. In particular, a polynomial is a linear combination of power functions.

$$\begin{aligned} g(x) &= 1 \\ h(x) &= x. \end{aligned} \tag{9.18}$$

Their graphs, $y = g(x) = 1$ and $y = h(x) = x$, are sketched in Figure 5. By definition, the shaded area in Figure 5(a) is Area(g, [a, t]). But it is also that of a rectangle, whose width is $t - a$ and whose height is 1. Thus

$$\text{Area}(g,[a,t]) \quad = \quad (t-a)\cdot 1 \quad = \quad t-a. \tag{9.19}$$

Similarly, the shaded area in Figure 5(b) is Area(h, [a, t]), by definition. But it is also the area of a trapezium, of width $t - a$, minimum height a and maximum height $a + t$. As indicated in the diagram, two such trapeziums make a rectangle of width $t - a$ and height $t + a$, and the area of each trapezium is half that of the rectangle. Thus

$$\text{Area}(h,[a,t]) \quad = \quad \frac{1}{2}(t-a)\cdot(t+a) \quad = \quad \frac{1}{2}t^2 - \frac{1}{2}a^2 \tag{9.20}$$

(indicating that the area can also be calculated as a difference in areas of triangles). We assume, of course, that $t \geq a$. Now recall from (3) that

$$f(x) \quad = \quad \begin{cases} 0 & \text{if } 0 \leq x \leq 0.05 \\ 9300x - 465 & \text{if } 0.05 \leq x \leq 0.1 \\ 465 & \text{if } 0.1 \leq x \leq 0.15 \\ 930 - 3100x & \text{if } 0.15 \leq x \leq 0.3, \end{cases} \tag{9.21}$$

from which (17) yields

$$f \quad = \quad \begin{cases} 0 & \text{on } [0, 0.05] \\ 9300h - 465g & \text{on } [0.05, 0.1] \\ 465g & \text{on } [0.1, 0.15] \\ 930g - 3100h & \text{on } [0.15, 0.3] \end{cases} \tag{9.22}$$

Then, e.g., for $t \in [0.05, 0.1]$ we have

$$\begin{aligned} \text{Area}(f,[0.05,t]) &= \text{Area}(9300h - 465g,[0.05,t]) \\ &= 9300\,\text{Area}(h,[0.05,t]) \quad - \quad 465\,\text{Area}(g,[0.05,t]) \\ &= 9300\left\{\frac{1}{2}t^2 - \frac{1}{2}0.05^2\right\} \quad - \quad 465\{t - 0.05\} \\ &= 4650t^2 - 465t + 11.625, \end{aligned} \tag{9.23}$$

in agreement with (6) above. Similarly, for $t \in [0.15, 0.3]$ we obtain

$$\begin{aligned} \text{Area}(f,[0.15,t]) &= \text{Area}(930g - 3100h,[0.15,t]) \\ &= 930\,\text{Area}(g,[0.15,t]) \quad - \quad 3100\,\text{Area}(h,[0.15,t]) \\ &= 930\{t - 0.15\} \quad - \quad 3100\left\{\frac{1}{2}t^2 - \frac{1}{2}0.15^2\right\} \\ &= 930t - 1550t^2 - 104.625, \end{aligned} \tag{9.24}$$

in agreement with (11)-(12) above.

Exercises 9

9.1 A piecewise-linear function f is defined on [0, 6] by

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ 2x & \text{if } 1 \le x \le 2 \\ 8 - 2x & \text{if } 2 \le x \le 4 \\ 0 & \text{if } 4 \le x \le 6 \end{cases}$$

The functions F, L and U are defined on the same domain by

$$\begin{aligned} F(t) &= \text{Area}(f,[0,t]) \\ L(t) &= \text{Min}(f,[0,t]) \\ U(t) &= \text{Max}(f,[0,t]) \end{aligned}$$

(i) Sketch the graphs of f, L and U. Distinguish them clearly.

(ii) Use two different methods to obtain an explicit formula for F(t). Verify that your results agree.
<u>Hint</u>: You need to consider each of the four subdomains separately, i.e., you need separate expressions for $0 \le t \le 1$, for $1 \le t \le 2$, for $2 \le t \le 4$ and for $4 \le t \le 6$.

(iii) Use your results to verify that Area(f, [0, 3]) = 8.

(iv) Find both Area(L, [0, 6]) and Area(U, [0, 6]).

9.2* A piecewise-linear function f is defined on [0, 7] by

$$f(x) = \begin{cases} 3 + 2x & \text{if } 0 \le x \le 2 \\ 13 - 3x & \text{if } 2 \le x \le 4 \\ x - 3 & \text{if } 4 \le x \le 5 \\ 2 & \text{if } 5 \le x \le 7 \end{cases}$$

The functions F, L and U are defined on the same domain by

$$\begin{aligned} F(t) &= \text{Area}(f,[0,t]) \\ L(t) &= \text{Min}(f,[0,t]) \\ U(t) &= \text{Max}(f,[0,t]) \end{aligned}$$

(i) Sketch the graphs of f, L and U. Distinguish them clearly.

(ii) Use two different methods to obtain an explicit formula for F(t). Verify that your results agree.

(iii) What is Area(f, [0, 7])?

(iv) What is Area(L, [0, 7])?

9.3 For a cardiac cycle, $v(t)$ is ventricular inflow at time t and $R(t) = \text{Area}(v, [0.4, t])$ is ventricular recharge during the interval $[0.4, t]$. If v is defined on $[0.4, 0.9]$ by

$$v(x) = \begin{cases} 600(5x-2) & \text{if } 0.4 \le x \le 0.5 \\ 300 & \text{if } 0.5 \le x \le 0.55 \\ 375(3-4x) & \text{if } 0.55 \le x \le 0.75 \\ 400(4x-3) & \text{if } 0.75 \le x \le 0.825 \\ 160(9-10x) & \text{if } 0.825 \le x \le 0.9, \end{cases}$$

show that R is defined on $[0.4, 0.9]$ by

$$R(t) = \begin{cases} 240-1200t+1500t^2 & \text{if } 0.4 \le t \le 0.5 \\ -135+300t & \text{if } 0.5 \le t \le 0.55 \\ -\frac{2895}{8}+1125t-750t^2 & \text{if } 0.55 \le t \le 0.75 \\ 510-1200t+800t^2 & \text{if } 0.75 \le t \le 0.825 \\ -579+1440t-800t^2 & \text{if } 0.825 \le t \le 0.9. \end{cases}$$

In other words, show that the recharge trace in the upper half of Figure 6 corresponds to the inflow trace in the lower half. What is the stroke volume?
Hint: Repeatedly apply $\text{Int}(v, [a, t]) = \text{Int}(v, [a, c]) + \text{Int}(v, [c, t])$, for appropriate a ($\le c$) and t ($\ge c$).

Answers and Hints for Selected Exercises

9.1 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (First Test, ##2-3)

9.2 (ii) For $0 \le t \le 2$, Area(f, [0, t]) is the area of a trapezium of width t, minimum height 3 and maximum height $3 + 2t$. So

$$F(t) = \text{Area}(f,[0,t]) = \frac{1}{2}t\cdot\{3 + 3 + 2t\} = t(t+3).$$

In particular, $F(2) = 10$.

For $2 \le t \le 4$, Area(f, [2, t]) is the area of a trapezium of width $t-2$, maximum height 7 and maximum height $13 - 3t$. So

$$\text{Area}(f,[2,t]) = \frac{1}{2}(t-2)\cdot\{7 + 13 - 3t\} = 13t - \tfrac{3}{2}t^2 - 20$$

and

$$\begin{aligned} F(t) = \text{Area}(f,[0,t]) &= \text{Area}(f,[0,2]) + \text{Area}(f,[2,t]) \\ &= F(2) + \text{Area}(f,[2,t]) \\ &= 13t - \tfrac{3}{2}t^2 - 10 \end{aligned}$$

In particular, $F(4) = 18$.

For $4 \le t \le 5$, Area(f, [4, t]) is the area of a trapezium of width $t-4$, minimum height 1 and maximum height $t - 3$. So

$$\text{Area}(f,[4,t]) = \frac{1}{2}(t-4)\cdot\{1 + t - 3\} = \tfrac{1}{2}t^2 - 3t + 4$$

and

$$\begin{aligned} F(t) = \text{Area}(f,[0,t]) &= \text{Area}(f,[0,4]) + \text{Area}(f,[4,t]) \\ &= F(4) + \text{Area}(f,[4,t]) \\ &= \tfrac{1}{2}t^2 - 3t + 22 \end{aligned}$$

In particular, $F(5) = 39/2$.

Finally, for $5 \le t \le 7$, Area(f, [5, t]) is the area of a rectangle of width $t-5$ and height 2. So Area(f, [5, t]) = $2(t-5)$ and $F(t) = F(5) + \text{Area}(f, [5, t]) = 2t + 19/2$. Gathering our results together, we find that F is the join defined on [0, 7] by

$$f(t) = \begin{cases} t^2 + 3t & \text{if } 0 \le t \le 2 \\ 13t - \tfrac{3}{2}t^2 - 10 & \text{if } 2 \le t \le 4 \\ \tfrac{1}{2}t^2 - 3t + 22 & \text{if } 4 \le t \le 5 \\ 2t + \tfrac{19}{2} & \text{if } 5 \le t \le 7 \end{cases}$$

(iii) Area(f, [0, 7]) = F(7) = 47/2.

(iv) Note that Area(L, [0, 4]) is less than Area(f, [0, 4]) by the area of a triangle Δ with vertices at (0, 3), (2, 7) and P, where P is the point where the line $y = 13 - 3x$ meets the horizontal line $y = 3$. Because $13 - 3x = 3$ implies $x = 10/3$, P = (10/3, 3). So Δ has base 10/3 and height $7 - 3 = 4$, hence area 20/3. Therefore Area(L, [0, 4]) = Area(f, [0, 4]) – 20/3 = F(4) – 20/3 = 18 – 20/3 = 34/3, implying Area(L, [0, 7]) = Area(L, [0, 4]) + Area(L, [4, 7]) = 34/3 + 3 = 43/3.