From index function to ordinary function. Ventricular recharging

An essential difference between ordinary and index functions is that an ordinary function, say $F$, yields local properties whereas an index function, say $\text{Index}$, yields global properties. For example, if $F(t)$ denotes the volume of blood discharged into the aorta during the first $t$ seconds of a cardiac cycle, then an ordinary function $V_{min}(t)$ is generated by $\text{Index} = \text{Min}$ according to

\[ V_{min}(t) = \text{Min}V, [0, t] \]  \hspace{1cm} (9.1)

The graph of $V_{min}$ is sketched in Figure 1 as the solid curve, with that of $V$ shown dashed for comparison.

Again, we can generate an ordinary function $F$ from the function $f$ graphed in the first three panels of Figure 2 by defining

\[ F(t) = \text{Area}(f, [0, t]) \]  \hspace{1cm} (9.2)

Here $x$ denotes a generic thing in the domain of $f$, whereas $t$ denotes a generic thing in the domain of $F$; we must use different letters, because the right-hand boundary of the shaded region in Figure 2 is at $x = t$. The physiological interpretation of $F$, as we will discover in Lecture 12, is that $F(t)$ is the volume of blood discharged into the aorta during the first $t$ seconds of a cardiac cycle with ventricular outflow defined by Figure 2. Here we focus merely on how to calculate $F$ from $f$, whose algebraic definition is

\[ f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.05 \\ 465(20x - 1) & \text{if } 0.05 \leq x \leq 0.1 \\ 465 & \text{if } 0.1 \leq x \leq 0.15 \\ 310(3 - 10x) & \text{if } 0.15 \leq x \leq 0.3. \end{cases} \]  \hspace{1cm} (9.3)

From (3), there are four cases, according to which subdomain of $[0, 0.3]$ contains $t$. First, the easiest case is when $t \in [0, 0.05]$: because $f(x) = 0$ for $0 \leq x \leq 0.05$ and $t \leq 0.05$, we have $f(x) = 0$ for $0 \leq x \leq t$, and so $F(t) = 0$ for $t \in [0, 0.05]$. Second, the next case is when $t \in [0.05, 0.1]$. Then, from (8.17) with $a = 0$, $c = 0.05$, and $b = t$, we have

\[ \text{Area}(f, [0, t]) = \text{Area}(f, [0, 0.05]) + \text{Area}(f, [0.05, t]) \]  \hspace{1cm} (9.4)

We have $\text{Area}(f, [0, 0.05]) = 0$, so in particular $F(0.05) = 0$. Third, the second case is when $t \in [0, 0.05]$. Finally, the fourth case is when $t \in [0.05, 0.1]$. Thus the result holds for all $t \in [0, 0.3]$.
\[ F(t) = \begin{cases} \frac{465t}{2} & \text{if } 0 \leq t \leq 0.05 \\ \frac{465t - 465}{11.625} & \text{if } 0.05 \leq t \leq 0.1 \\ 0.5t - 0.5t^2 & \text{if } 0.1 \leq t \leq 0.15 \\ 0 & \text{if } 0.15 \leq t \leq 0.3 \end{cases} \]

In particular, \( F(0.1) = 11.625 \).

The third case to consider is when \( t \in [0.1, 0.15] \). Now, from (8.17) with \( a = 0 \), \( c = 0.1 \) and \( b = t \), we have

\[ \text{Area}(f, [0, 0.1]) + \text{Area}(f, [0.1, t]) = F(0.1) + \text{Area}(f, [0.1, t]) \]

so, from (2) and (10), for \( 0.1 \leq t \leq 0.15 \) we have

\[ F(t) = 11.625 + (t - 0.1)f(t) = 11.625 + 465(t - 0.1) \]

\[ = 465t - 34.875 \]

From Figure 2(b), however, \( \text{Area}(f, [0.15, t]) \) is the area of a trapezium of width \( t - 0.15 \), maximum height \( f(0.15) = 465 \) and minimum height \( f(t) = 310(3 - 10t) \). We can place two such trapeziums together to form a rectangle of area \( f(0.15) + f(t) \). The area of each trapezium is half that of the rectangle. So for \( 0.15 \leq t \leq 0.3 \) we have

\[ \text{Area}(f, [0.15, t]) = \frac{1}{2} \left( f(0.15) + f(t) \right) \cdot (t - 0.15) \]

\[ \begin{align*}
\text{Area}(f, [0.15, t]) & = \frac{1}{2} (465 + 310(3 - 10t)) (t - 0.15) \\
& = 234.875 + 0.5(t - 0.15)(465 + 310(3 - 10t)) \\
& = 234.875 + 0.5(t - 0.15)(930 - 1550t^2 - 69.75) \\
\end{align*} \]

In particular, \( F(0.3) = 69.75 \) is the stroke volume. Gathering together (6), (9) and (12), we find that \( F \) is the join defined on \([0, 0.3]\) by

\[ F(t) = \begin{cases} \frac{465t}{2} & \text{if } 0 \leq t \leq 0.05 \\ \frac{465t - 465}{11.625} & \text{if } 0.05 \leq t \leq 0.1 \\ 0.5t - 0.5t^2 & \text{if } 0.1 \leq t \leq 0.15 \\ 0 & \text{if } 0.15 \leq t \leq 0.3 \end{cases} \]
The above calculation can arguably be simplified with the help of two general results about Area. The first result is that the area enclosed by a sum of nonnegative functions, say \( g \) and \( h \), equals the sum of the areas enclosed by each in the sense that

\[
\text{Area}(g + h, [a,b]) = \text{Area}(g, [a,b]) + \text{Area}(h, [a,b]).
\]

(9.14a)

The easiest way to obtain this result is from Figure 3. Imagine that \( \text{Area}(g, [a,b]) \) at top left in Figure 3 has been painted from left to right with a magic brush that tracks the graph of \( g \), so that the width of the brush at \( x \) is always \( g(x) \) and no paint leaks outside the shaded area. Similarly imagine that \( \text{Area}(h, [a,b]) \) at top right in Figure 3 has been painted with a brush that tracks the graph of \( h \). The area at bottom left, which is \( \text{Area}(g+h, [a,b]) \), is in principle painted by a third brush tracking the graph of \( g + h \), but in practice the same effect is achieved by painting from left to right with the second brush held above the first. In other words, the dark and light shaded regions are equal in area, which establishes (14). Similarly, the area at bottom right is \( \text{Area}(h+g, [a,b]) \); it requires no new brush to track \( h + g \) because holding the first brush above the second achieves the same effect. Again, the dark and light shaded regions are equal in area to those in the other panels, which establishes that for any nonnegative constant \( k \), the area enclosed by \( k \cdot g \) is equal to \( k \times \text{Area}(g, [a,b]) \). Hence the area of the shaded region in Figure 4, where the light and dark shaded regions all have the same area (reveals the second result, namely, that a similar paintbrush argument (see Figure 4) reveals the second result, namely, that for any nonnegative constant \( k \), the area enclosed by \( k \cdot g \) is equal to \( k \times \text{Area}(g, [a,b]) \). Of course, (14) and (15) are equivalent, because holding the first brush above the second brush achieves the same effect.

\[
\text{Area}(k \cdot g + q \cdot h, [a,b]) = \text{Area}(k \cdot g, [a,b]) + \text{Area}(q \cdot h, [a,b]).
\]

(9.16)

Combining (15) and (16), we have

\[
\text{Area}(q \cdot h, [a,b]) = q \cdot \text{Area}(h, [a,b]),
\]

(9.15b)

whereas (14) yields

\[
\text{Area}(k \cdot g + q \cdot h, [a,b]) = k \cdot \text{Area}(g, [a,b]) + q \cdot \text{Area}(h, [a,b]),
\]

(9.17)

which is the result we sought: (14) is a special case of (17) with \( k = 1 \) and \( q = 1 \), whereas (15) is a special case of (17) with \( k = 0 \) or \( q = 0 \).

Now, to obtain (17), we implicitly assumed that \( k \) and \( q \) are both nonnegative, whereas (17) is a special case of (17) with \( k = q = 0 \). We will show in Lecture 12, however, that (17) holds for any \( k \) or \( q \) provided that \( k \) and \( q \) are both nonnegative. The above calculation can arguably be simplified with the help of two general results about Area. The first result is that the area enclosed by a sum of nonnegative functions, say \( g \) and \( h \), equals the sum of the areas enclosed by each in the sense that

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\text{Area}(k \cdot g + q \cdot h, [a,b]) = \text{Area}(k \cdot g, [a,b]) + \text{Area}(q \cdot h, [a,b]).
\]

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Combining (15) and (16), we have

\[
\text{Area}(q \cdot h, [a,b]) = q \cdot \text{Area}(h, [a,b]),
\]

(9.15b)

whereas (14) yields

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\text{Area}(k \cdot g + q \cdot h, [a,b]) = k \cdot \text{Area}(g, [a,b]) + q \cdot \text{Area}(h, [a,b]),
\]

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\[
\text{Area}(k \cdot g + q \cdot h, [a,b]) = k \cdot \text{Area}(g, [a,b]) + q \cdot \text{Area}(h, [a,b]).
\]

(9.17)
\[ g(x) = 1 \]
\[ h(x) = x. \]

Their graphs, \( y = g(x) = 1 \) and \( y = h(x) = x \), are sketched in Figure 5. By definition, the shaded area in Figure 5(a) is \( \text{Area}(g, [a, t]) \). But it is also that of a rectangle, whose width is \( t - a \) and whose height is 1. Thus

\[ \text{Area}(g, [a, t]) = (t - a) \cdot 1 = t - a. \]

Similarly, the shaded area in Figure 5(b) is \( \text{Area}(h, [a, t]) \), by definition. But it is also the area of a trapezium, of width \( t - a \), minimum height \( a \) and maximum height \( a + t \). As indicated in the diagram, two such trapeziums make a rectangle of width \( t - a \) and height \( t + a \), and the area of each trapezium is half that of the rectangle. Thus the shaded area in Figure 5 is \( \text{Area}(h, [a, t]) \). By definition, the area of a trapezium of width \( w \), height \( h_1 \) and \( h_2 \), is \( \frac{1}{2}(h_1 + h_2)w \). Thus

\[ \text{Area}(h, [a, t]) = \frac{1}{2}((a + t) + t)(t - a) = \frac{1}{2}(2a + 2t)(t - a) = (a + t)(t - a). \]

Thus \( a + t \) and \( t - a \) are the lengths of the bases of the trapezium, and \( t \) is its height. Thus

\[ \text{Area}(h, [a, t]) = \frac{1}{2}((a + t) + (a + t))(t - a) = \frac{1}{2}(2a + 2t)(t - a) = (a + t)(t - a). \]

Therefore, the shaded area in Figure 5 is

\[ (a + t)(t - a). \]

We assume, of course, that \( t \geq a \). Now recall from (3) that

\[ f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.05 \\ 9300x - 465 & \text{if } 0.05 \leq x \leq 0.1 \\ 465 & \text{if } 0.1 \leq x \leq 0.15 \\ 930 - 3100x & \text{if } 0.15 \leq x \leq 0.3 \end{cases} \]

from which, from (17) yields

\[ f = 0 \text{ on } [0, 0.05] \]
\[ 9300h - 465g, [0.05, t] \]
\[ 465g, [0.1, 0.15] \]
\[ 9300h - 3100g, [0.15, 0.3] \]

Then, e.g., for \( t \in [0.05, 0.1] \) we have

\[ \text{Area}_f([0.05, t]) = \text{Area}_{9300h}([0.05, t]) - \text{Area}_{465g}([0.05, t]) = \frac{1}{2}t^2 - \frac{1}{2}0.05^2 \cdot (4650t - 9300) \]

\[ = 4650t^2 - 465t + 11.625. \]

Similarly, for \( t \in [0.15, 0.3] \) we obtain

\[ \text{Area}_f([0.15, t]) = \text{Area}_{9300g}([0.15, t]) - \text{Area}_{3100h}([0.15, t]) = 930t^2 - 1550t^2 - 104.625, \]
Exercises 9.1

A piecewise-linear function $f$ is defined on $[0, 6]$ by

\[
f(x) = \begin{cases} 
2 & \text{if } 0 \leq x \leq 1 \\
2x - 8 & \text{if } 1 \leq x \leq 2 \\
-2x + 8 & \text{if } 2 \leq x \leq 4 \\
4 & \text{if } 4 \leq x \leq 6
\end{cases}
\]

The functions $F$, $L$, and $U$ are defined on the same domain by

\[
F(t) = \text{Area}(f, [0, t]) \\
L(t) = \text{Min}(f, [0, t]) \\
U(t) = \text{Max}(f, [0, t])
\]

(i) Sketch the graphs of $f$, $L$, and $U$. Distinguish them clearly.

(ii) Use two different methods to obtain an explicit formula for $F(t)$. Verify that your results agree. Hint: You need to consider each of the four subdomains separately, i.e., you need separate expressions for $0 \leq t \leq 1$, for $1 \leq t \leq 2$, for $2 \leq t \leq 4$, and for $4 \leq t \leq 6$.

(iii) Use your results to verify that $\text{Area}(f, [0, 3]) = 8$.

(iv) Find both $\text{Area}(L, [0, 6])$ and $\text{Area}(U, [0, 6])$.

Exercises 9.2

A piecewise-linear function $f$ is defined on $[0, 7]$ by

\[
f(x) = \begin{cases} 
3 & \text{if } 0 \leq x \leq 2 \\
3x - 3 & \text{if } 2 \leq x \leq 4 \\
2x + 3 & \text{if } 4 \leq x \leq 5 \\
7 & \text{if } 5 \leq x \leq 7
\end{cases}
\]

The functions $F$, $L$, and $U$ are defined on the same domain by

\[
F(t) = \text{Area}(f, [0, t]) \\
L(t) = \text{Min}(f, [0, t]) \\
U(t) = \text{Max}(f, [0, t])
\]

(i) Sketch the graphs of $f$, $L$, and $U$. Distinguish them clearly.

(ii) Use two different methods to obtain an explicit formula for $F(t)$. Verify that your results agree.

(iii) What is $\text{Area}(f, [0, 7])$?

(iv) What is $\text{Area}(L, [0, 7])$?
For a cardiac cycle, $v(t)$ is ventricular inflow at time $t$ and $R(t) = \text{Area}(v, [0.4, t])$ is ventricular recharge during the interval $[0.4, t]$. What is the stroke volume?

In other words, show that the recharge trace in Figure 6 corresponds to the inflow trace in the lower half.

Show that $R$ is defined on $[0.4, 0.9]$ by

$$R(t) = \begin{cases} 0.825 \leq t \leq 0.9, & -579 + 1440t - 800t^2 \\ 0.75 \leq t \leq 0.825, & 210 - 1200t + 800t^2 \\ 0.5 \leq t \leq 0.75, & -379 + 1125t - 750t^2 \\ 0.05 \leq t \leq 0.5, & 300 \\ 0.4 \leq t \leq 0.5, & 600 \end{cases}$$

For a cardiac cycle, $v(t)$ is ventricular inflow at time $t$ and $R(t) = \text{Area}(v, [0.4, t])$.
Answers and Hints for Selected Exercises

9.1 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (First Test, ##2-3)

9.2 (ii) For $0 \leq t \leq 2$, area of a trapezium of width $t$, minimum height 3 and maximum height 3 + 2t. So

$$F(t) = \text{Area}(f, [0, t]) = \frac{1}{2} t \cdot (3 + 3 + 2t) = t(t + 3).$$

In particular, $F(2) = 10$.

For $2 \leq t \leq 4$, area of a trapezium of width $t-2$, maximum height 7 and maximum height 13 – 3t. So

$$\text{Area}(f, [2, t]) = \frac{1}{2} (t - 2) \cdot (7 + 13 - 3t) = 13t - \frac{3}{2}t^2 - 20.$$ And

$$F(t) = \text{Area}(f, [0, t]) = \text{Area}(f, [0, 2]) + \text{Area}(f, [2, t]) = F(2) + 13t - \frac{3}{2}t^2 - 20.$$ In particular, $F(4) = 18$.

For $4 \leq t \leq 5$, area of a trapezium of width $t-4$, minimum height 1 and maximum height $t - 3$. So

$$\text{Area}(f, [4, t]) = \frac{1}{2} (t - 4) \cdot (1 + t - 3) = \frac{1}{2} t^2 - 3t + 22.$$ And

$$F(t) = \text{Area}(f, [0, t]) = \text{Area}(f, [0, 4]) + \text{Area}(f, [4, t]) = F(4) + \frac{1}{2} t^2 - 3t + 22.$$ In particular, $F(5) = 39/2$.

Finally, for $5 \leq t \leq 7$, area of a rectangle of width $t-5$ and height 2. So

$$\text{Area}(f, [5, t]) = 2(t-5)$$ And

$$F(t) = \text{Area}(f, [0, t]) = \text{Area}(f, [5, t]) + \text{Area}(f, [5, t]) = 2t + 19/2.$$ Gathering our results together, we find that $F$ is the join defined on $[0, 7]$ by

\[
\begin{align*}
\text{Area}(f, [0, t]) &= t^2 + 3t & \text{if } 0 \leq t \leq 2 \\
&= 13t - \frac{3}{2}t^2 - 10 & \text{if } 2 \leq t \leq 4 \\
&= \frac{1}{2} t^2 - 3t + 22 & \text{if } 4 \leq t \leq 5 \\
&= 2t + 19/2 & \text{if } 5 \leq t \leq 7 \\
\end{align*}
\]

9.2 (iii) $\text{Area}(f, [0, 7]) = F(7) = 47/2$.

9.2 (iv) Note that the area of a trapezium of width $t$ and minimum height 3 is less than the area of a triangle with vertices at $(0, 3), (2, 7)$ and $P$, where $P$ is the point where the line $y = 13 - 3x$ meets the horizontal line $y = 3$. Because $13 - 3x = 3$ implies $x = 10/3$, P = (10/3, 3).

So $\Delta$ has base $10/3$ and height 4, hence area $\Delta = 10/3$. Therefore, $\text{Area}(L, [0, 4]) = \text{Area}(f, [0, 4]) - \Delta = F(4) - 10/3 = 18 - 10/3 = 34/3$, implying $\text{Area}(L, [0, 7]) = \text{Area}(L, [0, 4]) + \text{Area}(L, [4, 7]) = 34/3 + 3 = 43/3$. (First Test, #2-3)