14. Smoothness and concavity: an algebraic perspective

We have already established, in Lecture 13, that a function \( V \) is growing or decaying at time \( t \) according to whether \( \frac{dV}{dt} > 0 \) or \( \frac{dV}{dt} < 0 \). What about times when \( \frac{dV}{dt} = 0 \)? Then \( V \) is stationary, i.e., neither growing nor decaying. Typically, a function becomes stationary not only on \([0,0.35]\), where \( \frac{dV}{dt} = 0 \), but also at \( t = 0.3 \), where \( \frac{dV}{dt} = 0 \).

Note, however, that \( V \) can become stationary where there is no local extremum. For example, in Figure 1.3, \( V \) is stationary not only on \([0.35, 0.4]\), where \( V = 50 \) implies \( \frac{dV}{dt} = 0 \) (Exercise 2), but also at \( t = 0.75 \), where \( \frac{dV}{dt}(0.75) = 0 \) even though \( t = 0.75 \) is not a local extremizer. What happens here is that \( V \) starts to increase again as soon as \( t \) has stopped increasing.

\[
\frac{dV}{dt} = \left(1 + \frac{\partial}{\partial t}\right) V
\]

We have

\[
\left(1 + \frac{\partial}{\partial t}\right) V = \frac{dV}{dt}
\]

This is a smooth function, see Figure 1(a). It therefore makes local minimum in Figure 2(a)-(b), while \( \frac{dV}{dt} \) changes sign from negative to positive at \( t = t^* \), i.e., because \( V \) stops decreasing and starts increasing at \( t = t^* \), i.e., because \( \frac{dV}{dt} \) changes sign from negative to positive at \( t = t^* \).

We have already established, in Lecture 13, that \( V \) is growing or decaying at time \( t \) according to whether \( \frac{dV}{dt} > 0 \) or \( \frac{dV}{dt} < 0 \). Which about times when \( \frac{dV}{dt} = 0 \)?
The graph of $v'$ is plotted in Figure 1(b), directly below the graph of $v$. Observe from the dashed lines that, e.g., inflow is increasing at 2520 ml/s$^2$ after 0.29 seconds, even though blood is still being discharged through the aorta at over 22 ml/s, because $v'(0.29) = 2520$ when $v(0.29) = -22.4$. Similarly, $v'(0.32) = 513.3$ ml/s$^2$ when $v(0.32) = 25.2$ ml/s and $v'(0.34) = -1291$ ml/s$^2$ when $v(0.34) = 18$ ml/s.

Because $v'$ is the derivative of $v = V'$, we refer to $v'$ as the second derivative of $V$ and denote it by $V''$ (as well as by $\theta''$). We will establish these results in Appendix 28. Taking them on faith until then, we observe that $\theta$ has a discontinuity whenever $\theta$ is increasing or decreasing (i.e., when $\theta$ is not relatively monotone), and vice versa. In cases where $\theta$ is relatively monotone, $\theta$ has a local extremum of elevation wherever $\theta'$ is zero.

Furthermore, if $\theta$ is decreasing or increasing, it is increasing or decreasing, or down according to whether $\theta$ is increasing or decreasing. From Lecture 1, $\theta''$ is concave up or down according to whether $\theta'$ is increasing or decreasing. From Lecture 1, $\theta'$ is concave up or down according to whether $\theta''$ is increasing or decreasing.

Thus, comparing Figure 2(b) with Figure 2(d), $V$ has an inflection point at $t = s$, where $\theta$ has a local maximum. Comparing Figure 2(d) with Figure 2(b), we see that $t = s$ is also where $\theta$ has a local maximum. Further comparing $\theta$ with Figure 2(d), we see that $\theta$ has a local extremum of elevation when $V$ has an inflection point at $t = s$, and vice versa. In cases where $\theta$ is relatively monotone, $\theta$ has a local extremum of elevation whenever $\theta'$ is zero.

On the other hand, second derivatives are often encountered in practice. For example, in Figure 1 we have $v'(t) > 0$ if $t < s$ but $v'(t) < 0$ if $t > s$, where $s$ is the maximizer of $v$ defined by $v''(s) = 0$ or, from (7), $77 - 784s + 1680s^2 = 0$. The second method is invariably easier. For example, in Figure 1 we have $v'(t) > 0$ if $t < s$ but $v'(t) < 0$ if $t > s$, where $s$ is the maximizer of $v$ defined by $v''(s) = 0$ or, from (7), $77 - 784s + 1680s^2 = 0$. The second method is invariably easier. For example, in Figure 1 we have $v'(t) > 0$ if $t < s$ but $v'(t) < 0$ if $t > s$, where $s$ is the maximizer of $v$ defined by $v''(s) = 0$ or, from (7), $77 - 784s + 1680s^2 = 0$. The second method is invariably easier. For example, in Figure 1 we have $v'(t) > 0$ if $t < s$ but $v'(t) < 0$ if $t > s$, where $s$ is the maximizer of $v$ defined by $v''(s) = 0$ or, from (7), $77 - 784s + 1680s^2 = 0$. The second method is invariably easier.
This equation is a quadratic equation, whose roots are 

\[ s = \frac{14 - 31}{60} = 0.140537 \quad \text{and} \quad s = \frac{14 + 31}{60} = 0.326129. \]

Only the second root belongs to \([0.28, 0.35]\). Thus inflow \( v \) increases on \([0.28, 0.326]\) but decreases on \([0.326, 0.35]\), with maximum \( v(s) = 26.8 \text{ ml/s} \).

Correspondingly, \( V \) has an inflection point where \( t = s \).

In effect, relationship (10b) between elevation and derivative enables us to redefine inflection points as local extremizers of the derivative (as opposed to local extremizers of elevation). Correspondingly, a function \( V \) is concave up or concave down according to whether \( V'' \) is positive or negative; see Figure 3. These newer definitions are more useful in practice because \( V'' \) is invariably easier to calculate than \( V' \).

We conclude this lecture by emphasizing that (13.5), (13.18) and (7a) yield explicit expressions for the first and second derivatives of an arbitrary polynomial of order up to 4. That is,

\[
\begin{align*}
V(t) & = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 \\
V'(t) & = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 \\
V''(t) & = 2c_2 + 6c_3 t + 12c_4 t^2.
\end{align*}
\]

Exercises

Exercise 14

14.1 A function \( f \) is defined on \([0, 5]\) by \( f(t) = 17 - 18t + 8t^2 - t^3 \).

(i) Find all local extrema

(ii) Where is \( f \) concave upward? Concave downward?

(iii) Find both Max(\( f, [0, 5]\)) and Min(\( f, [0, 5]\)).

(iv) Using Mathematica or otherwise, sketch the graphs of \( f \) and \( f' \), one above the other.

14.2 A function \( f \) is defined on \([0, 4]\) by \( f(t) = 3t^3 - 14t^2 + 9t + 8 \).

(i) Find all local extrema

(ii) Where is \( f \) concave upward? Concave downward?

(iii) Find both Max(\( f, [0, 4]\)) and Min(\( f, [0, 4]\)).

(iv) Sketch the graphs of \( f \) and \( f' \), one above the other.

14.3 A function \( f \) is defined on \([0, 3]\) by \( f(t) = t(9t - 2t^2 - 12)/3 \).

(i) Find an expression for \( f'(t) \)

(ii) Hence find all local extrema of \( f \)

(iii) Where is \( f \) concave upward? Where is \( f \) concave downward?

(iv) Find both Max(\( f, [0, 3]\)) and Min(\( f, [0, 3]\))

(v) Sketch the graphs of \( f \) and \( f' \), one above the other.
14.4 The function $f$ is defined on $[0, 7]$ by 
$$f(t) = \frac{(2t^2 - 19t + 41)(t-1)}{6}.$$ 
(i) Find an expression for $f'(t)$.
(ii) Hence find all local extrema of $f$.
(iii) Where is $f$ concave upward? Where is $f$ concave downward?
(iv) Find both Max$(f, [0, 7])$ and Min$(f, [0, 7])$.
(v) Sketch the graphs of $f$ and $f'$, one above the other.

14.5 A function $f$ is defined on $[2, 8]$ by 
$$f(t) = t(27t - 2t^2 - 108).$$ Where is it concave up? Concave down? Find its global maximum.

14.6* A smooth function $f$ with domain $[0, 10]$ and range $[-1, 5]$ has global minimizer $t = 0$, global maximizer $t = 3$, local minimizer $t = 8$, an inflection point at $t = 5$ and

\[ f(10) = 4, \text{ Find a possible formula for } f. \]

A smooth function $f$ with domain $[0, 10]$ and range $[-1, 5]$ has global minimizer $t = 0$, global maximizer $t = 3$, local minimum $t = 8$, an inflection point at $t = 5$ and

\[ f(10) = 4, \text{ Find a possible formula for } f. \]

\[ f(10) = 4, \text{ Find a possible formula for } f. \]

\[ f(10) = 4, \text{ Find a possible formula for } f. \]

\[ f(10) = 4, \text{ Find a possible formula for } f. \]
14.1 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Assignment B, #3)

14.5 Because \( f(t) = -\frac{108}{t} + \frac{27}{t^2} - \frac{2}{t^3} \), (13) with \( c_0 = 0 = c_4 \), and \( \frac{c_1}{t} = \frac{c_2}{t^2} \) yields \( f(t) = -108 + 27t - 2t^2 \) when \( 9/2 < t \leq 8 \), implying that \( f \) is convex up on \([2, 9/2]\) and concave down on \([9/2, 8]\] with an inflection point where \( t = 9/2 \).

So \( f'(t) \) is decreasing on \([2, 3]\), increasing on \([3, 6]\) and decreasing again on \([6, 8]\].

So the global maximizer is either \( t = 2 \) or \( t = 6 \). But \( f(2) = -124 \) and \( f(6) = -108 \), which is bigger. So the global maximum is -108.

The function must satisfy (at least) six constraints, namely, \( f(0) = -1 \), \( f(3) = 5 \), \( f'(3) = 0 \), \( f''(5) = 0 \), \( f'(8) = 0 \) and \( f(10) = 4 \). The simplest such function is probably a fifth-order polynomial.

\[
\begin{align*}
\text{The simplest such function is probably a fifth-order polynomial.} \\
\text{The function must satisfy (at least) six constraints, namely, } f(0) = 0, f'(0) = 0, f''(0) = 0, f'(3) = 0, f''(5) = 0, f'(8) = 0, f(10) = 5. \\
\end{align*}
\]

The simplest fifth-order polynomial is probably a fifth-order polynomial.