20. Derivatives of compositions: the chain rule

At the end of the last lecture we discovered a need for the derivative of a composition. In this lecture we show how to calculate it.

Accordingly, let P have domain [a, b], let Q be such that its domain is the range of P and let S = QP be their composition, i.e.,

$$S(x) = Q(P(x)). \tag{20.1}$$

Then S(x + h) = Q(P(x + h)), implying

$$DQ(S, [x, x+h]) = \frac{S(x+h) - S(x)}{h} = \frac{Q(P(x+h)) - Q(P(x))}{h} \tag{20.2}$$

To find the derivative of S, we must extract the leading term of this expression. Details are in the appendix, where it is shown that

$$\frac{Q(P(x+h)) - Q(P(x))}{h} = P'(x)\,Q'(P(x)) + O[h]. \tag{20.3}$$

Thus

$$S'(x) = P'(x)\,Q'(P(x)) \tag{20.4}$$

or, in mixed notation,

$$\frac{d}{dx}\{Q(P(x))\} = P'(x)\,Q'(P(x)). \tag{20.5}$$

This formula for the derivative of a composition is known as the **chain rule**.

The chain rule is often easiest to work with in differential notation. Define

$$y = P(x), \quad z = Q(y), \tag{20.6}$$

so that z = Q(P(x)) = S(x). Then dy/dx = P′(x), dz/dy = Q′(y) = Q′(P(x)) and dz/dx = S′(x). With these substitutions, (4) reduces to

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}. \tag{20.7}$$

It is important to note that (4), (5) and (7) say exactly the same thing, but in different notations (standard, mixed and differential, respectively). Each has advantages and disadvantages, which is why we use them all.

Our first task for the chain rule is to find the derivative of the exponential function. Recall from Lecture 7 that exp is defined on [0, ∞) by

$$\exp(x) = \lim_{n\to\infty} \phi_n(x) \tag{20.8}$$

where $\{\phi_n \mid n \geq 1\}$ is defined on [0, ∞) by

$$\phi_n(x) = \left(1 + \frac{x}{n}\right)^n. \tag{20.9}$$

That is,

$$\phi_n(x) = (P(x))^n, \tag{20.10}$$

where

$$P(x) = 1 + \frac{x}{n}. \tag{20.11}$$

Because $x \geq 0$, the range of P is $[1, \infty)$. So define Q on $[1, \infty)$ by

$$Q(y) = y^n. \tag{20.12}$$

Then, from (16.20) and Table 18.1, we have both

$$\begin{aligned} P'(x) &= \frac{d}{dx}\left\{1 + \frac{x}{n}\right\} = \frac{d}{dx}\{1\} + \frac{1}{n}\frac{d}{dx}\{x\} \\ &= 0 + \frac{1}{n}\cdot 1 = \frac{1}{n} \end{aligned} \tag{20.13}$$

and

$$Q'(y) = ny^{n-1}. \tag{20.14}$$

Moreover, from (10) and (12),

$$\phi_n(x) = Q(P(x)). \tag{20.15}$$

So the chain rule implies that

$$\begin{aligned} \frac{d}{dx}\{\phi_n(x)\} &= P'(x)\,Q'(P(x)) = \frac{1}{n}\cdot n(P(x))^{n-1} \\ &= (P(x))^{n-1} = \frac{(P(x))^n}{P(x)} = \left\{1+\frac{x}{n}\right\}^{-1}\phi_n(x) \end{aligned} \tag{20.16}$$

on using (10), (13) and (14) with $y = P(x)$.

Now let n become infinitely large. Then x/n becomes arbitrarily close to zero, so that $1 + x/n$ becomes arbitrarily close to 1; and so the right-hand side of (16) approaches the limit of $\phi_n(x)$ as $n \to \infty$, which is $\exp(x)$, by (8). At the same time, the left-hand side of (16) approaches the derivative of the limit of $\phi_n(x)$ as $n \to \infty$, i.e., $d\{\exp(x)\}/dx$. Thus

$$\frac{d}{dx}\{\exp(x)\} = \exp(x). \tag{20.17}$$

Our second task for the chain rule is to calculate the derivative of the function R defined in (19.7) by

$$R(x) = \exp(Ax^m). \tag{20.18}$$

With P, Q defined on $[0, \infty)$ by

$$P(x) = Ax^m \tag{20.19}$$

and

$$Q(y) = \exp(y), \tag{20.20}$$

we have $R(x) = Q(P(x))$. Thus, from the chain rule,

$$\begin{aligned} \frac{d}{dx}\{\exp(Ax^m)\} &= \frac{d}{dx}\{Q(P(x))\} = P'(x)\,Q'(P(x)) \\ &= \frac{d}{dx}\{Ax^m\}Q'(P(x)) \\ &= mAx^{m-1}Q'(P(x)) \end{aligned} \tag{20.27}$$

by (16.20) and Table 18.1. But (17) and (20) imply $Q'(y) = \exp(y)$, and hence $Q'(P(x)) = \exp(P(x))$. Thus (27) yields

$$\frac{d}{dx}\{\exp(Ax^m)\} = mAx^{m-1}\exp(Ax^m). \tag{20.28}$$

A further application of the chain rule yields the probability density function of the distribution fitted in Lecture 19 to data on prairie dog survival, leaf thickness and minnow size. From (19.32), its c.d.f. is defined on $[0, \infty)$ by

$$f(x) \;=\; -\frac{d}{dx}\left\{\frac{1}{R(x)}\right\} \;=\; -\frac{d}{dx}\left\{\frac{1}{\exp(Ax^m)}\right\}. \tag{20.29}$$

Suppose that $y = R(x)$, and that $z = 1/y$. Then, from version (7) of the chain rule,

$$\begin{aligned} f(x) &= -\frac{dz}{dx} \;=\; -\frac{dz}{dy}\frac{dy}{dx} \\ &= -\frac{d}{dy}\left\{\frac{1}{y}\right\}\frac{d}{dx}\{\exp(Ax^m)\} \\ &= -\left(-y^{-2}\right)mAx^{m-1}\exp(Ax^m) \\ &= \frac{mAx^{m-1}\exp(Ax^m)}{\{\exp(Ax^m)\}^2} \;=\; \frac{mAx^{m-1}}{\exp(Ax^m)}, \end{aligned} \tag{20.30}$$

agreeing with (19.34) by virtue of (28).

The product rule now enables us to find where f is increasing or decreasing. From (18) and (30), we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left\{\frac{mAx^{m-1}}{\exp(Ax^m)}\right\} \;=\; \frac{d}{dx}\left\{\frac{1}{R(x)}\cdot mAx^{m-1}\right\} \\ &= \frac{d}{dx}\left\{\frac{1}{R(x)}\right\}mAx^{m-1} \;+\; \frac{1}{R(x)}\frac{d}{dx}\{mAx^{m-1}\} \\ &= -f(x)\cdot mAx^{m-1} \;+\; \frac{mA}{R(x)}\frac{d}{dx}\{x^{m-1}\} \\ &= -f(x)\cdot mAx^{m-1} \;+\; \frac{mA}{R(x)}(m-1)x^{m-2} \\ &= -f(x)\cdot mAx^{m-1} \;+\; \frac{(m-1)}{xR(x)}mAx^{m-1} \\ &= f(x)\left\{-mAx^{m-1} \;+\; \frac{(m-1)}{x}\right\} \\ &= \frac{mAf(x)}{x}\left\{\{x^*\}^m - x^m\right\}, \end{aligned} \tag{20.31}$$

if we define

$$x^* \;=\; \left\{\frac{m-1}{m}\right\}^{\frac{1}{m}}\left\{\frac{1}{A}\right\}^{\frac{1}{m}}. \tag{20.32}$$

Note from (31) that $f'(x) > 0$ if $x < x^*$, $f'(x) < 0$ if $x > x^*$, and $f'(x^*) = 0$. So f has a global maximum at $x = x^*$. In other words, x^* is the mode of the distribution. Because $f = F'$, if $f'(x^*) = 0$ then $F''(x^*) = 0$. So $x = x^*$ is also where the c.d.f. has its inflexion point.

The distribution defined by (30) is a special case of a distribution known as the **Weibull**.[1] It is convenient to set $A = 1/s^m$ and rewrite c.d.f. and p.d.f. as

$$F(x) \;=\; 1 \;-\; \frac{1}{\exp(\{x/s\}^m)} \tag{20.33}$$

and

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[1] In the more general case, m need not be an integer. See Lecture 22.

$$f(x) = \frac{m(x/s)^{m-1}}{s\exp(\{x/s\}^m)}, \tag{20.34}$$

with mode

$$x^* = \left\{1 - \frac{1}{m}\right\}^{\frac{1}{m}} s, \tag{20.35}$$

by (32). Then, if $sf(x)$ is plotted on a vertical axis against x/s on a horizontal axis, the shape of the graph is completely determined by m; accordingly, we call m the **shape** parameter and s the **scale** parameter. The Weibull distribution is very versatile. It was used in Lecture 19 to model variation in prairie dog survival ($m = 1$), leaf thickness ($m = 7$) and minnow size ($m = 5$), and it will be used in Lecture 21 to model variation in rat pupil size ($m = 2$). Figure 1 shows the p.d.f. for $m = 1, \ldots, 6$.

The chain rule yields an exceedingly simple and useful result when P and Q are inverse functions, i.e., when

$$y = P(x) \quad \Leftrightarrow \quad x = Q(y). \tag{20.36}$$

Then $Q(P(x)) = x$, implying

$$P'(x)\,Q'(P(x)) = \frac{d}{dx}\{Q(P(x))\} = \frac{d}{dx}\{x\} = 1. \tag{20.37}$$

In other words, (36) implies $P'(x)\,Q'(y) = 1$, or

$$Q'(y) = \frac{1}{P'(x)}, \tag{20.38}$$

which is often rewritten in differential notation as

$$\frac{dx}{dy} = \left\{\frac{dy}{dx}\right\}^{-1}. \tag{20.39}$$

In particular, if $y = \exp(x)$ then $x = \ln(y)$ from Lecture 7, whereas $P'(x) = d\{\exp(x)\}/dx = \exp(x) = y$ from (17). So (38) yields

$$\frac{d}{dy}\{\ln(y)\} = \frac{1}{y}, \quad y \geq 1 \tag{20.40}$$

or, which is exactly the same thing,

$$\frac{d}{dx}\{\ln(x)\} = \frac{1}{x}, \quad x \geq 1. \tag{20.41}$$

The graphs of the logarithm and its derivative are sketched in Figure 2.

Exercises 20

20.1 Find $S'(x)$ for S defined on $[c + 1, \infty)$ by
(i) $S(x) = (x-c)^{10}$ (ii) $S(x) = (x-c)^{-4}$ (iii) $S(x) = \sqrt{x-c}$ (iv) $S(x) = \ln(x-c)$
In each case, state the domain and range of S.

20.2 If P is an arbitrary smooth function with derivative P' on any subset of $(-\infty, \infty)$, what is $\frac{d}{dx}\{\exp(P(x))\}$?

20.3 If P is an arbitrary smooth function whose range is a subset of $(0, \infty)$, what is $\frac{d}{dx}\{\ln(P(x))\}$?

20.4 Find the mode, m, of the distribution on $[0, \infty)$ with p.d.f. defined by

$$f(x) = \begin{cases} \frac{20}{243}x^3(3-x)^2 & \text{if } 0 \le x < 3 \\ 0 & \text{if } 3 \le x < \infty \end{cases}$$

20.5 Find the mode, m, of the distribution on $[0, \infty)$ with p.d.f. defined by

$$f(x) = \begin{cases} \frac{315}{262144}x^4(4-x)^5 & \text{if } 0 \le x < 4 \\ 0 & \text{if } 4 \le x < \infty \end{cases}$$

20.6 The function R is defined on $[0, \infty)$ by

$$R(t) = \begin{cases} At + Bt^3 & \text{if } 0 \le x < 1 \\ \dfrac{1-t}{1+t} & \text{if } 1 \le x < \infty. \end{cases}$$

What must be the values of A and B if R is smooth on $[0, \infty)$?

20.7 A function F is defined on $[0, \infty)$ by

$$F(t) = 1 - \frac{1}{(t^4 + t^3 + 1)^3}$$

(i) Why must F be the c.d.f. of a random variable, say T, on $[0, \infty)$?
(ii) What is the probability that T exceeds 1?
(iii) Find the p.d.f. of the distribution, and sketch its graph.
(iv) Hence find the mode of the distribution, approximately.

Appendix 20: On the chain rule

The purpose of this appendix is to establish (3). Let P have domain [a, b], let Q be defined on the range of P and let S = QP be their composition. Then

$$S(x) = Q(P(x)), \qquad a \le x \le b. \tag{20.A1}$$

Because P is smooth on [a, b], we have DQ(P, [x, x + h]) = $P'(x) + \varepsilon_P(h,t)$, where $\varepsilon_P(h,t)$ = O[h] and h is a very small positive number. Thus

$$P(x+h) = P(x) + h\{P'(x) + \varepsilon_P(h,t)\} \tag{20.A2a}$$
$$= P(x) + h\{P'(x) + O[h]\}. \tag{20.A2b}$$

Similarly, because S is smooth on [a, b], DQ(S, [x, x + h]) = $S'(x) + O[h]$ or

$$S(x+h) = S(x) + h\{S'(x) + O[h]\}. \tag{20.A3}$$

Moreover, because Q is smooth on the range of P, DQ(Q, [y, y + $\bar{h}$]) = $Q'(y) + O[\bar{h}]$ or

$$Q(y+\bar{h}) = Q(y) + \bar{h}\{Q'(y) + O[\bar{h}]\} \tag{20.A4}$$

where y belongs to the range of P (or domain of Q) and $\bar{h}$ is a very small positive number. *Any* very small positive number will do: If h is one such number, then h times a positive constant is another. So any O[h] is a possible $\bar{h}$. Accordingly, in (A4), we set y = P(x) and take

$$\bar{h} = h\{P'(x) + \varepsilon_P(h,t)\}, \tag{20.A5a}$$
$$= h\{P'(x) + O[h])\}, \tag{20.A5b}$$

which is O[h] by (13.27), (13.29) and Exercise 13.14. Now (A4) implies

$$Q(P(x) + h\{P'(x) + \varepsilon_P(h,t)\}) = Q(P(x)) + h\{P'(x) + O[h]\}\{Q'(P(x)) + O[\bar{h}]\}. \tag{20.A6}$$

Note that we have replaced $\bar{h}$ by expression (A5a) in the left-hand side of (A4) but by expression (A5b) on the right-hand side, because different amounts of precision are required. The extra precision on the left-hand side enables us to exploit (A2a), which reduces (A6) to

$$Q(P(x+h)) = Q(P(x)) + h\{P'(x) + O[h]\}\{Q'(P(x)) + O[\bar{h}]\} \tag{20.A7}$$

$$= Q(P(x)) + h\,P'(x)\,Q'(P(x)) + h\{P'(x)O[\bar{h}] + Q'(P(x))O[h] + O[h]O[\bar{h}]\}, \tag{20.A8}$$

on simplification. But $\bar{h}$ is O[h], and so $P'(x)O[\bar{h}] + Q'(P(x))O[h] + O[h]O[\bar{h}]$ is also O[h], by (13.27)-(13.29). Moreover, Q(P(x)) = S(x), implying Q(P(x+h)) = S(x+h). So (A8) yields

$$S(x+h) = S(x) + h\{P'(x)\,Q'(P(x)) + O[h]\}, \tag{20.A9}$$

implying

$$DQ(S,[x,x+h]) = \frac{S(x+h) - S(x)}{h} = P'(x)\,Q'(P(x)) + O[h], \tag{20.A10}$$

as required.

Answers and Hints for Selected Exercises

20.1 (i) Define P and Q by $P(x) = x - c$ and $Q(y) = y^{10}$, so that $Q(P(x)) = \{P(x)\}^{10}$. Then $S(x) = (x-c)^{10} = \{P(x)\}^{10} = Q(P(x))$, $P'(x) = \frac{d}{dx}\{x-c\} = \frac{d}{dx}\{x\} - \frac{d}{dx}\{c\} = 1 - 0 = 0$ and $Q'(y) = 10y^9$ (by Table 18.1), implying $Q'(P(x)) = 10\{P(x)\}^9 = 10(x-c)^9$. So, by the chain rule, $S'(x) = P'(x)Q'(P(x)) = 1 \cdot 10(x-c)^9 = 10(x-c)^9$. That is,

$$\frac{d}{dx}\left\{(x-c)^{10}\right\} = 10(x-c)^9.$$

Note that P has domain $[c + 1, \infty)$ and range $[1, \infty)$, so that Q has domain $[1, \infty)$ and range $[1, \infty)$. Thus S, like P, has domain $[c + 1, \infty)$ and range $[1, \infty)$.

(ii) As before, $P(x) = x - c \Rightarrow P'(x) = 1$; but now $Q(y) = y^{-4}$, implying $Q'(y) = -4y^{-5}$ (by Appendix 11B). So $S'(x) = P'(x)Q'(P(x)) = 1 \cdot (-4\{P(x)\}^{-5})$, or

$$\frac{d}{dx}\left\{(x-c)^{-4}\right\} = -4(x-c)^{-5} = -\frac{4}{(x-c)^5}.$$

Note that, although Q still has domain $[1, \infty)$, the range of Q is now $(0, 1]$. Thus S has domain $[c + 1, \infty)$ and range $(0, 1]$.

(iii) Again, $P(x) = x - c \Rightarrow P'(x) = 1$; but now $Q(y) = \sqrt{y}$, so that

$$Q'(y) = \frac{1}{2\sqrt{y}} \quad \Rightarrow \quad Q'(P(x)) = \frac{1}{2\sqrt{P(x)}}$$

(by Exercise 17.4). So

$$S'(x) = P'(x)Q'(P(x)) = \frac{P'(x)}{2\sqrt{P(x)}} = \frac{1}{2\sqrt{x-c}}.$$

The domains and ranges of P, Q and S are the same as for (i).

(iv) Now $Q(y) = \ln(y)$, so that $Q'(y) = 1/y$ by (40). Hence

$$S'(x) = P'(x)Q'(P(x)) = P'(x) \cdot \frac{1}{P(x)} = \frac{1}{x-c}.$$

Q still has domain $[1, \infty)$, but its range is now $[0, \infty)$. So S has domain $[c + 1, \infty)$ and range$[0, \infty)$.

20.2 Define Q by $Q(y) = \exp(y)$, so that $Q'(y) = \exp(y) \Rightarrow Q'(P(x)) = \exp(P(x))$. Then, by the chain rule,

$$\frac{d}{dx}\{\exp(P(x))\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)Q'(P(x)) = P'(x)\exp(P(x)).$$

Indeed this result is effectively contained in the lecture.

20.3 Define Q by $Q(y) = \ln(y)$, so that $Q'(y) = 1/y \Rightarrow Q'(P(x)) = 1/P(x)$. Then, by the chain rule,

$$\frac{d}{dx}\{\ln(P(x))\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)Q'(P(x)) = P'(x) \cdot \frac{1}{P(x)} = \frac{P'(x)}{P(x)}.$$

Indeed this result is effectively contained in the solution to Exercise 1(iv).

20.4 Note that $f(x) = g(x)/L$, where

$$g(x) = \begin{cases} x^3(3-x)^2 & \text{if } 0 \le x < 3 \\ 0 & \text{if } 3 \le x < \infty \end{cases}$$

and $L = 243/20$ is a positive constant whose value is irrelevant (because f and g have the same global maximizer). By the chain rule with $P(x) = 3 - x \Rightarrow P'(x) = -1$ and $Q(y) = y^2 \Rightarrow Q'(y) = 2y \Rightarrow Q'(P(x)) = 2P(x) = 2(3-x)$, we have

$$\frac{d}{dx}\{(3-x)^2\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x))$$
$$= (-1)\cdot 2(3-x) = -2(3-x).$$

Thus, on using the product rule, we have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\{x^3(3-x)^2\} \\ &= \frac{d}{dx}\{x^3\}\cdot(3-x)^2 + x^3\cdot\frac{d}{dx}\{(3-x)^2\} \\ &= 3x^2\cdot(3-x)^2 + x^3\cdot\{-2(3-x)\} \\ &= x^2(3-x)\{3(3-x)-2x\} = x^2(3-x)\{9-5x\}. \end{aligned}$$

Because $g'(x) > 0$ if $0 < x < 9/5$ and $g'(x) < 0$ if $9/5 < x < 3$, g has a maximum where $x = 9/5$. Therefore f also has a maximum where $x = 9/5$. So $m = 9/5$.

20.5 As in the previous exercise, $f(x) = g(x)/L$ on $[0, \infty)$, where now

$$g(x) = \begin{cases} x^4(4-x)^5 & \text{if } 0 \le x < 4 \\ 0 & \text{if } 4 \le x < \infty \end{cases}$$

and $L = 262144/315 > 0$. By the chain rule with $P(x) = 4 - x \Rightarrow P'(x) = -1$ and $Q(y) = y^5 \Rightarrow Q'(y) = 5y^4 \Rightarrow Q'(P(x)) = 5\{P(x)\}^4 = 5(4-x)^4$, we have

$$\frac{d}{dx}\{(4-x)^5\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x)) = (-1)\cdot 5(4-x)^4.$$

Thus, on using the product rule, we have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\{x^4(4-x)^5\} \\ &= \frac{d}{dx}\{x^4\}\cdot(4-x)^5 + x^4\cdot\frac{d}{dx}\{(4-x)^5\} \\ &= 4x^3\cdot(4-x)^5 + x^4\cdot\{-5(4-x)^4\} \\ &= x^3(4-x)^4\{4(4-x)-5x\} = x^3(4-x)^4\{16-9x\} \end{aligned}$$

on $[0, 4]$. Because $g'(x) > 0$ if $0 < x < 16/9$ and $g'(x) < 0$ if $16/9 < x < 4$, g has a maximum where $x = 16/9$. Therefore f also has a maximum where $x = 16/9$. So the mode of the distribution is $m = 16/9$.

20.6 First note that, by the chain rule with $P(t) = 1 + t$ and $Q(y) = y^{-1}$ we have

$$\frac{d}{dt}\left\{(1+t)^{-1}\right\} = \frac{d}{dt}\{Q(P(t))\} = P'(t)\cdot Q'(P(t))$$
$$= \{0+1\}\cdot(-\{P(t)\}^{-2}) = -(1+t)^{-2}.$$

Therefore,

$$\frac{d}{dt}\left\{\frac{1-t}{1+t}\right\} = \frac{d}{dt}\left\{\frac{2-(1+t)}{1+t}\right\} = \frac{d}{dt}\left\{\frac{2}{1+t}-1\right\} = 2\frac{d}{dt}\left\{\frac{1}{1+t}\right\}-0 = -2(1+t)^{-2}.$$

(Alternatively, but less simply,

$$\frac{d}{dt}\left\{(1-t)\cdot(1+t)^{-1}\right\} = \frac{d}{dt}\{(1-t)\}\cdot(1+t)^{-1} + (1-t)\cdot\frac{d}{dt}\left\{(1+t)^{-1}\right\}$$
$$= (0-1)\cdot(1+t)^{-1} + (1-t)\cdot\left\{-(1+t)^{-2}\right\}$$
$$= -(1+t)(1+t)^{-2} - (1-t)(1+t)^{-2}$$
$$= -\{1+t+1-t\}(1+t)^{-2},$$

on using the product rule.) So

$$R'(t) = \begin{cases} A+3Bt^2 & \text{if } 0 \le t < 1 \\ -\dfrac{2}{(1+t)^2} & \text{if } 1 \le t < \infty. \end{cases}$$

The continuity condition is $R(1-) = R(1+)$ or $A + B = 0$. The smoothness condition is $R'(1-) = R'(1+)$ or $A + 3B = -1/2$. So $A + B + 2B = -1/2 \Rightarrow 0 + 2B = -1/2 \Rightarrow B = -1/4$. Then $A = -B = 1/4$. Compare your solution to Exercise 15.13.

20.7 (i Because $F(0) = 0$ and $F(\infty) = 1$ and F is a strictly increasing function.

(ii) $\text{Prob}(T > 1) = \text{Int}(f, [1, \infty)) = F(\infty) - F(1) = 1 - 26/27 = 1/27$.

(iii) Define P and Q by $P(t) = t^4 + t^3 + 1$ and $Q(y) = y^{-3}$. Then, by our rule for the derivative of a sum of multiples and Table 18.1, we have $P'(t) = 4t^3 + 3t^2$ and $Q'(y) = -3y^{-4}$, implying $Q'(P(t)) = -3(t^4 + t^3 + 1)^{-4}$. So, from the chain rule,

$$f(t) = F'(t) = \frac{d}{dt}\{1 - Q(P(t))\} = 0 - P'(t)Q'(P(t))$$
$$= -(4t^3+3t^2)\left\{-3(t^4+t^3+1)^{-4}\right\} = \frac{3t^2(4t+3)}{(t^4+t^3+1)^4}.$$

Go to **http://www.math.fsu.edu/~mmg/QuizBank/MAC2311.f97/Answers/assC2.gif** for the graph.

(iv) From the graph, $\text{Max}(f, [0, \infty)) = f(0.52) = 1.9$, approximately. So $m \approx 0.52$.