23. Periodic functions: models of rhythms in nature

Many things in nature cycle, i.e., their variation with time repeats itself after a specified lapse of time, or **period**, which we denote by p. The period of such cycles, or rhythms, is often about a day – in which case, the rhythms are called **circadian**. For example, Figure 1, which is based on data from Exercises 1.4-1.7, shows heights above the zero of a tide staff at Morro in California during a 75-hour period stretching from midnight on February 12, 1919 to 3 a.m. on the morning of February 16, 1919. A pattern of minor high and low between major high and low appears to repeat itself approximately three times, or about once every 25 hours.

Such natural rhythms can often be modelled by **periodic functions**, i.e., functions f on $(-\infty, \infty)$ satisfying

$$f(t \pm p) = f(t). \tag{23.1}$$

Shifting the graph of such a function to the right or left by precisely p units has no effect, as indicated by the dots in Figure 2. An important class of periodic functions are those with period $2\pi = 6.283$, the circumference of a circle with radius 1. Such functions are called **trigonometric**.

At first, trigonometric functions may appear unsuitable for modelling biological rhythms, because a day isn't anywhere close to 2π in units we commonly use. If g is any such function, however, i.e., if

$$g(x \pm 2\pi) = g(x) \tag{23.2}$$

for any x, then f defined on $(-\infty, \infty)$ by

$$f(t) = g\left(\frac{2\pi t}{p}\right) \tag{23.3}$$

is periodic with period p because (3) and (2) with $x = 2\pi t/p$ imply

$$f(t \pm p) = g\left(\frac{2\pi(t \pm p)}{p}\right) = g\left(\frac{2\pi t}{p} \pm 2\pi\right) = g\left(\frac{2\pi t}{p}\right) = f(t). \tag{23.4}$$

In other words, compositions of trigonometric functions can have any period. Thus, despite first appearances, trigonometric functions can be used to model biological rhythms.

The simplest way to define a trigonometric function (or, for that matter, any function) is by using a graph. So let us define a trigonometric function by using the graph in Figure 3(a). Although the function has domain $(-\infty, \infty)$, it suffices to define its graph on subdomain $[0, 2\pi]$, because (2) implies that its graph on any other subdomain can be obtained by shifting its graph on $[0, 2\pi]$ sufficiently far to the right or left; see Figure 4(a). We will call this function S. Note that

$$S(0) = 0, \quad S(\pi) = 0, \quad S(2\pi) = 0 \tag{23.5a}$$

and

$$S(\pi/2) = 1, \quad S(3\pi/2) = -1. \tag{23.5b}$$

Now, in principle, no matter how we define a function – even graphically – if that function is smooth then it has a derivative. In particular, the function S defined on $[0, 2\pi]$ by Figure 3(a) and extended to $(-\infty, \infty)$ by its periodicity, i.e., by (3), is clearly smooth, hence it has a derivative. Let us call it C. Then

$$C(t) = S'(t). \tag{23.6}$$

defines a new periodic function on $(-\infty, \infty)$. What is its graph like? We can deduce its general shape from Figure 3(a). Because S has local extrema at $t = \pi/2$ and $t = 3\pi/2$, we know that $S'(\pi/2) = 0 = S'(3\pi/2)$. Because S is increasing on $[0, \pi/2)$ and $(3\pi/2, 2\pi)$ but decreasing on $(\pi/2, 3\pi/2)$, we know that $S'(t) > 0$ on $[0, \pi/2)$ and $(3\pi/2, 2\pi)$ but that $S'(t) < 0$ on $(\pi/2, 3\pi/2)$. So, from (6), we know that $C(t) > 0$ on $[0, \pi/2)$ and $(3\pi/2, 2\pi)$ but that $C(t) < 0$ on $(\pi/2, 3\pi/2)$, and that

$$C(\pi/2) = 0, \quad C(3\pi/2) = 0. \qquad (23.7a)$$

Many functions have all of these properties, but only one them is C. Its graph was worked out by our clever worm, who crawled along the graph of S and, for every value of t between 0 and 2π, calculated not elevation, as in Lecture 3, but rather the leading term of the difference quotient DQ(S, [t, t+h]). The worm's results are shown in Figure 3(b). You can see at a glance that all of the above properties are satisfied, and that in addition

$$C(0) = 1, \quad C(\pi) = -1, \quad C(2\pi) = 1. \qquad (23.7b)$$

Moreover, our worm has extended its definition of C from $[0, 2\pi)$ to $(-\infty, \infty)$ by sliding the graph on $[0, 2\pi)$ to the right or left as far as is necessary; see Figure 4(b).

From Figure 4 we see, with hindsight, that if our worm had been extremely clever, it could have found the graph of C without ever calculating even a single difference quotient. The graph of C can be obtained from the graph of S merely by sliding it $\pi/2$ units to the left. That is

$$C(\theta) = S(\theta + \pi/2) \qquad (23.8a)$$

for any value of θ, as is clear from comparing the solid disk in Figure 4(a) with that on the left of Figure 4(b). Equivalently, the graph of S can be obtained from the graph of C merely by sliding it $\pi/2$ units to the right. That is

$$S(\theta) = C(\theta - \pi/2) \qquad (23.8b)$$

for any value of θ, as is clear from comparing the open disks. If θ denotes time, we say that S **lags** C with **delay** $\pi/2$ because each sucessive extremum of S is later – i.e., further to the right – than the corresponding extremum of C. You should check that (8) is consistent with (5) and (7).[1]

The dashed curve in Figure 4(b) is the graph of the negative of C. Observe that if we slide the solid curve π units to the right, then it coincides with the dashed curve. That is,

$$C(\theta) = -C(\theta + \pi) \qquad (23.9)$$

for any value of θ, as is clear from comparing the solid disks in Figure 4(b). The effect of sliding S's graph π units to the right is similar: it coincides with its negative, as indicated by the open disks in Figure 4(a). Thus

$$S(\theta) = -S(\theta + \pi) \qquad (23.10)$$

for any θ as well.

The period of S or C can be lengthened or shortened by stretching or shrinking its graph along the horizontal axis. In this way, functions with arbitrary period p are obtained. If $p > 2\pi$, then stretching the graphs of S and C by factor

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[1] More generally, that if the graph of a (not necessarily periodic) function f on $(-\infty, \infty)$ is obtained from the graph of a function g on $(-\infty, \infty)$ by sliding it α units to the right, then $f(t) = g(t-\alpha)$ for any value of t. Similarly, sliding the graph of g to the left by β units would yield the graph of f defined by $f(t) = g(t+\beta)$; β units to the left is the same as $-\alpha$ units to the right.

$$A = \frac{p}{2\pi} \qquad (23.11)$$

(> 1) produces new functions $\overline{S}$ and $\overline{C}$, respectively, with period p on $(-\infty, \infty)$. Suppose, for example, that p = L, where

$$L = \frac{621}{25} = 24.84 \qquad (23.12)$$

hours (or 24 hours and 50 minutes) is approximately the length of a lunar day, or the period of a circadian rhythm. Then the stretch factor is $A = L/2\pi = 621/50\pi = 3.95$, or almost 4. Figure 5 shows the effect of the stretch: subdomain $[0, 2\pi] = [0, L/A] = [0, 6.28]$ for S becomes subdomain $[0, p] = [0, L] = [0, 24.84]$ for $\overline{S}$. Moreover, a point with coordinates (x, S(x) on the graph of S becomes the point with coordinates $(t, \overline{S}(t))$ on the graph of $\overline{S}$. The height of the point is unchanged; thus

$$S(x) = \overline{S}(t) \qquad (23.13)$$

But the horizontal displacement of the point is increased by factor A. That is,

$$t = Ax \qquad (23.14)$$

or $x = t/A$. Substituting in (13), we have $S(t/A) = \overline{S}(t)$. In other words, the function $\overline{S}$ is defined by

$$\begin{aligned}\overline{S}(t) &= S(t/A) = S(2\pi t/p) = S(2\pi t/L)\\ &= S(50\pi t/621) = S(0.253t). \qquad (23.15a)\end{aligned}$$

Similarly (Figure 6), the function $\overline{C}$ is defined by

$$\begin{aligned}\overline{C}(t) &= C(t/A) = C(2\pi t/p) = C(2\pi t/L)\\ &= C(50\pi t/621) = C(0.253t). \qquad (23.15b)\end{aligned}$$

Note that if, on the other hand, $p < 2\pi$, then the graphs of S and C must be shrunk by factor $p/2\pi = 1/s$ (<1) to produce the graphs of $\overline{S}$ and $\overline{C}$, respectively. Nevertheless, (15) still holds, because shrinking by factor 1/s is exactly the same thing as stretching by factor s.

A simple modification of the above analysis yields compositions $\tilde{S}$ and $\tilde{C}$ whose period is approximately half a lunar day. We set $p = L/2 = 621/50 = 12.42$ hours. Then, from (11), the stretch factor becomes $A = p/2\pi = L/4\pi = 1.977$, or almost 2. Figures 7-8 show its effect. In place of (15), we have

$$\begin{aligned}\tilde{S}(t) &= S(t/A) = S(2\pi t/p) = S(4\pi t/L)\\ &= S(100\pi t/621) = S(0.506t). \qquad (23.16a)\end{aligned}$$

and

$$\begin{aligned}\tilde{C}(t) &= C(t/A) = C(2\pi t/p) = C(4\pi t/L)\\ &= C(100\pi t/621) = C(0.506t). \qquad (23.16b)\end{aligned}$$

Now, from Exercise 2, if each of two functions has period p then so does their sum or product. Furthermore, neither adding a constant nor multiplying by a constant affects its periodicity; see Exercise 3. Thus, from $\overline{S}$ or $\overline{C}$, each of which has period L, we can generate countless other functions with period L. For example, Figure 9(a) shows the graph of g defined by

$$g(t) = 3.62367 - 0.0425951\,\overline{C}(t) + 0.919763\,\overline{S}(t)$$

$$= 3.62367 - 0.0425951\,C(0.253t) + 0.919763\,S(0.253t) \qquad (23.17)$$

Similarly, from $\tilde{S}$ or $\tilde{C}$, each of which has period $L/2$, we can generate countless other functions with period $L/2$. For example, Figure 9(b) shows the graph of h defined by

$$\begin{aligned} h(t) &= 0.1692\,\tilde{C}(t) - 1.49049\,\tilde{S}(t) \\ &= 0.1692\,C(0.506t) - 1.49049\,S(0.506t) \qquad (23.18) \end{aligned}$$

Notice that, because h has period $L/2$, the pattern on $[0, L/2]$ repeats itself on $[L/2, L]$; but then the combined pattern repeats itself on $[L, 2L]$, on $[2L, 3L]$, and so on. Thus h has period L (in addition to period $L/2$). But g has period L as well. So f defined by

$$f(t) = g(t) + h(t) \qquad (23.19)$$

must likewise have period L, as Figure 9(c) confirms. Also shown in Figure 9(c) are the tidal data from Figure 9.1. It appears that f is an excellent model of the data. So, on using (19), we can interpret the Morro tide as a sum, or **superposition**, of oscillations with different periods, namely, L and $L/2$.

Having period p means completing a cycle every p units of time. So the number of cycles completed per unit time is $1/p$. We call this number the **frequency** of the oscillation, and denote it by ν. That is,

$$\nu = \frac{1}{p}. \qquad (23.20)$$

Increasing the frequency decreases the period, and vice versa. For example, because g has frequency $1/L$ and h has frequency $1/(L/2) = 2/L$, halving the period doubles the frequency. We can use (19)-(20) to reinterpret the Morro tide as a sum of oscillations with different frequencies, namely, $1/L$ and $2/L$; specifically, because f has period L, we interpret the tide as a sum of oscillations with frequencies ν and 2ν. We refer to ν as the **fundamental** frequency, because it is the lowest of the superposition.

Now, from Figure 3, the function C is clearly smooth. To find its derivative, we recall from (8a) that

$$C(\theta) = S(\theta + \pi/2) \qquad (23.21)$$

for any θ. Defining $P(\theta) = \theta + \pi/2$, we have $P'(\theta) = 1$ and $C(\theta) = S(P(\theta))$. Hence, by (21) and the chain rule, $C'(\theta) = P'(\theta)S'(P(\theta)) = 1 \cdot S'(P(\theta)) = S'(P(\theta))$. But $S'(t) = C(t)$, by definition, and so $S'(P(\theta)) = C(P(\theta)) = C(\theta + \pi/2)$. From (8), however, we have $C(t) = S(t + \pi/2)$ for any value of t; so, with $t = \theta + \pi/2$, we obtain $C(\theta + \pi/2) = S(\theta + \pi/2 + \pi/2) = S(\theta + \pi) = -S(\theta)$, by (10). The upshot is that

$$C'(\theta) = -S(\theta) \qquad (23.22)$$

for any value of θ.

The functions we have introduced as C and S are better known as cos and sin, for reasons to be discussed in Lecture 30. That is, $C(t) = \cos(t)$ and $S(t) = \sin(t)$, for all $t \in (-\infty, \infty)$. So, from (6) and (22), we have

$$\frac{d}{dt}\{\sin(t)\} = \cos(t) \qquad (23.23a)$$

$$\frac{d}{dt}\{\cos(t)\} = -\sin(t) \qquad (23.23b)$$

for every value of t. These results enable us to expand our list of known integrals and derivatives as in Table 1; see Exercise 5.

DERIVATIVE on $(-\infty, \infty)$	ANTIDERIVATIVE on $(-\infty, \infty)$	SOURCE
$\frac{d}{dx}\{\cos(cx)\} = -c\sin(cx)$	$\int_a^x \sin(ct)\,dt = -\frac{1}{c}\cos(cx) + \text{const}$	Exercise 5
$\frac{d}{dx}\{\sin(cx)\} = c\cos(cx)$	$\int_a^x \cos(ct)\,dt = \frac{1}{c}\sin(cx) + \text{const}$	Exercise 5

Table 23.1 Some derivatives and integrals considered known by the end of this lecture

Finally, from (23) and the fundamental theorem we can readily deduce an important trigonometric identity, namely, that $\{\cos(t)\}^2 + \{\sin(t)\}^2 = 1$, for any value of t. To obtain this result, we first define a function W on $(-\infty, \infty)$ by

$$W(t) = \{\cos(t)\}^2 + \{\sin(t)\}^2 = \{C(t)\}^2 + \{S(t)\}^2. \tag{23.24}$$

Then

$$W'(t) = \frac{d}{dt}\left(\{C(t)\}^2 + \{S(t)\}^2\right) = \frac{d}{dt}\left(\{C(t)\}^2\right) + \frac{d}{dt}\left(\{S(t)\}^2\right). \tag{23.25}$$

If we define Q on $[-1, 1]$ by $Q(y) = y^2$, so that $Q'(y) = 2y$, then $\{C(t)\}^2 = Q(C(t))$; and so, by the chain rule,

$$\begin{aligned}\frac{d}{dt}\left(\{C(t)\}^2\right) &= \frac{d}{dt}\{Q(C(t))\}\\ &= C'(t)\cdot Q'(C(t)) = C'(t)\cdot 2C(t) = -2S(t)C(t),\end{aligned} \tag{23.26}$$

on using (22). Similarly,

$$\begin{aligned}\frac{d}{dt}\left(\{S(t)\}^2\right) &= \frac{d}{dt}\{Q(S(t))\}\\ &= S'(t)\cdot Q'(S(t)) = S'(t)\cdot 2S(t) = 2C(t)S(t),\end{aligned} \tag{23.27}$$

on using (6). Substituting from (26)-(27) into (25) yields $W'(t) = 0$. So, by the fundamental theorem, we have

$$W(x) = W(0) + \int_0^x W'(t)\,dt = W(0) + \int_0^x 0\,dt = W(0) + 0 = W(0) \tag{23.28}$$

for any value of x. On using (5), (7) and (24), however, we have $W(0) = \{C(0)\}^2 + \{S(0)\}^2 = 1^2 + 0^2 = 1$. So $W(x) = 1$, for any value of x. In other words, by (24),

$$\cos^2(x) + \sin^2(x) = 1, \tag{23.29}$$

where we have introduced a standard shorthand for the products cos•cos and sin•sin, namely, $\cos^2$ and $\sin^2$, respectively.

Reference

Schureman, Paul (1994) Manual of Harmonic Analysis and Prediction of Tides (U.S. Coast and Geodetic Survey Special Publication No. 98). U.S. Government Printing Office, Washington

Exercises 23

23.1 Figures 7 and 8 show graphically that $\tilde{S}$ and $\tilde{C}$ have period L/2. Verify this result algebraically, i.e., use (14) to show that $\tilde{S}(t+L/2)$ = $\tilde{S}(t)$ and $\tilde{C}(t+L/2)$ = $\tilde{C}(t)$.

23.2 If each of two functions has period p on (-∞, ∞), show that their sum and product also have period p on (-∞, ∞).

23.3 The function f is defined on (-∞, ∞) by f(t) = α + βg(t), where the function g has period p. Show that f also has period p.

23.4* (i) Use the data from Exercises 1.4-1.7 to verify Figure 9.
(ii) The table below gives heights y in feet above the zero of the tide staff at Morro in California at hourly intervals from 1 a.m. on February 19, 1919 (t = 145) to 11 p.m. on the same day (t = 167). Use these data, in conjunction with those from Exercises 1.15-1.17, to model the tide at Morro during the 75-hour period between 8 p.m. on February 16, 1919 (t = 92) and 11 p.m. on February 19 (t = 167) as a sum of oscillations with frequencies 1/L and 2/L (where L is given by (9)). Find an explicit approximation for the height of the tide in terms of C and S.

t	y	t	y	t	y	t	y	t	y
145	4.8	150	2.7	155	3.2	160	3.2	165	2.9
146	4.9	151	2.3	156	3.6	161	2.8	166	3.7
147	4.5	152	2.2	157	3.8	162	2.5	167	4.2
148	3.8	153	2.4	158	3.8	163	2.3		
149	3.2	154	2.8	159	3.6	164	2.4		

23.5 Verify Table 1.

23.6 Show that (i) Area(C, [0, π/2]) = 1 and (ii) Area(S, [0, π]) = 2.

23.7 In each of the following cases, find $F'(x)$ and $F''(x)$:

(i) $F(x) = x^3\cos(4x) + (3x^2+2)\sin(x)$

(ii) $F(x) = \cos(4x^2)$

(iii) $F(x) = \sin(4x^2)$

(iv) $F(x) = \cos(4x^2) + (3x+2)\sin(4x^2)$

(v) $F(x) = \cos(4x^2) + e^{3x}\sin(4x^2)$

23.8 (i) Find $\frac{d}{dt}\{\sin(t)\cos(t)\}$. Write your answer solely in terms of cos(t).

(ii) Use the substitution $u = \arcsin(t/2)$ to evaluate

$$\int_0^2 \sqrt{4-t^2}\, dt.$$

Interpret your answer geometrically.

23.9 A smooth function g is defined on $[0, \infty)$ by

$$g(t) \;=\; \begin{cases} A\cos(\pi t) + t^2 & \text{if } 0 \le t < 2 \\ Be^{-2t} & \text{if } 2 \le t < \infty \end{cases}$$

(i) Find the values of A and B.

(ii) Calculate Int(g, [0,4]).

(iii) A function G is defined on $[0, \infty)$ by G(x) = Int(g, [0, x]). Find an explicit formula for G(x) on $[0, \infty)$.

23.10 If P is an arbitrary smooth function with derivative P' on any subset of $(-\infty, \infty)$, what is $\frac{d}{dx}\{\sin(P(x))\}$?

23.11 If P is an arbitrary smooth function with derivative P' on any subset of $(-\infty, \infty)$, what is $\frac{d}{dx}\{\cos(P(x))\}$?

23.12 Using the chain rule in conjunction with the product rule and properties of exponentials and logarithms, find $F'(x)$ in each of the following cases:

(i) $F(x) = \left(e^{\cos(x)}\right)^2$ (ii) $F(x) = x^5\cos(x)$

(iii) $F(x) = \ln(1+\cos(x))$ (iv) $F(x) = \ln\left(x^5 e^{\cos(x)}\right)$

23.13 Using the chain rule in conjunction with the product rule and properties of exponentials and logarithms, find $F'(x)$ in each of the following cases:

(i) $F(x) = \left(e^{\sin(x)}\right)^3$ (ii) $F(x) = x^3\cos(2x)$

(iii) $F(x) = \ln(1+\sin(x))$ (iv) $F(x) = \ln\left(x^7 e^{\sin(x)}\right)$

23.14 Using the chain rule in conjunction with the product rule and properties of exponentials and logarithms, find $F'(x)$ in each of the following cases:

(i) $F(x) = \left(e^{x^3+x^2+x+1}\right)^3$ (ii) $F(x) = x^4\cos(x^2+2)$

(iii) $F(x) = \ln\left(1+4x^6\right)$ (iv) $F(x) = \ln\left(x^7 e^{x^3+x^2+x+1}\right)$

23.15 Using the chain rule in conjunction with the product rule and properties of exponentials and logarithms, find $F'(x)$ in each of the following cases:

(i) $F(x) = \left(e^{x^4+x^3}\right)^5$ (ii) $F(x) = x^5\sin(x^3+2)$

(iii) $F(x) = \ln\left(1+6x^{10}\right)$ (iv) $F(x) = \ln\left(x^6 e^{x^4+x^3}\right)$

23.16 The function G defined on $[0, \infty)$ by $G(t) = \ln\left(1+\sin^2(t)\right)$ is known to satisfy

$$\frac{G(t+h)-G(t)}{h} = \frac{\sin(2t)}{1+\sin^2(t)} + O[h]$$

as $h \to 0+$. What must be the value of

$$\int_0^{\pi/2} \frac{\sin(2x)}{1+\sin^2(x)}\,dx?$$

Write your answer as simply as possible.

23.17 (i) Find $\frac{d}{dt}\left\{\cos^3(t)\right\}$.

(ii) Use the substitution $u = \arcsin(x)$ to evaluate

$$\int_0^1 x\sqrt{1-x^2}\,dx.$$

Answers and Hints for Selected Exercises

23.1 From (14), $\tilde{S}(t) = S(4\pi t/L)$. So $\tilde{S}(t + L/2) = S(4\pi\{t+L/2\}/L) = S(4\pi t/L + 2\pi) = S(4\pi t/L) = \tilde{S}(t)$ because S, by virtue of having period 2π, satisfies $S(x + 2\pi) = S(x)$ for any x. Similarly for $\tilde{C}$.

23.2 Suppose that f and g both have period p, and that h is their sum. Then $f(t\pm p) = f(t)$ and $g(t\pm p) = g(t)$ for any $t \in (-\infty, \infty)$, and $h(x) = f(x) + g(x)$ for any $x \in (-\infty, \infty)$. In particular, both $h(t) = f(t) + g(t)$ and $h(t\pm p) = f(t\pm p) + g(t\pm p)$. But $f(t\pm p) = f(t)$ and $g(t\pm p) = g(t)$, because f and g both have period p. Therefore $h(t\pm p) = f(t) + g(t) = h(t)$, implying that h has period p. Similarly for product.

23.3 Because g is periodic, $g(t+p) = g(t)$. But $f(t+p) = \alpha + \beta g(t+p)$. Therefore $f(t+p) = \alpha + \beta g(t) = f(t)$, implying that f has period p.

23.6 (i) From (5), (23a) and the fundamental theorem,

$$\text{Area}(C,[0,\pi/2]) = \int_0^{\pi/2} \cos(t)\,dt = \int_0^{\pi/2} \frac{d}{dt}\{\sin(t)\}\,dt = \sin(t)\Big|_0^{\pi/2}$$
$$= \sin(\pi/2) - \sin(0) = S(\pi/2) - S(0) = 1 - 0 = 1$$

23.8 (i) By the product rule,

$$\frac{d}{dt}\{\sin(t)\cos(t)\} = \frac{d}{dt}\{\sin(t)\}\cdot\cos(t) + \sin(t)\cdot\frac{d}{dt}\{\cos(t)\}$$
$$= \cos(t)\cdot\cos(t) + \sin(t)\cdot\{-\sin(t)\} = 2\cos^2(t) - 1$$

(ii) We have $\phi(t) = \arcsin(t/2)$. The substitution $u = \phi(t)$ and the inverse substitution $t = \zeta(u)$ imply one another. But $u = \arcsin(t/2)$ implies $\sin(u) = t/2$ $\Rightarrow t = 2\sin(u)$. Therefore ζ is defined by $\zeta(u) = 2\sin(u)$, and so $\zeta'(u) = 2\cos(u)$. Also, $\phi(0) = \arcsin(0/2) = \arcsin(0) = 0$ and $\phi(2) = \arcsin(2/2) = \arcsin(1) = \pi/2$. So, with $f(t) = \sqrt{4-t^2}$, we have

$$\int_0^2 \sqrt{4-t^2}\,dt = \int_0^2 f(t)\,dt = \int_{\phi(0)}^{\phi(2)} f(\zeta(u))\zeta'(u)\,du$$
$$= \int_0^{\pi/2} \sqrt{4-\{\zeta(u)\}^2}\,\zeta'(u)\,du = \int_0^{\pi/2} \sqrt{4-4\sin^2(u)}\,2\cos(u)\,du$$

$$= \int_0^{\pi/2} 2\cos(u)\, 2\cos(u)\,du \;=\; 2\int_0^{\pi/2} 2\cos^2(u)\,du$$

$$= 2\int_0^{\pi/2}\left\{1+\frac{d}{du}\{\sin(u)\cos(u)\}\right\}du \;=\; 2\{u+\sin(u)\cos(u)\}\Big|_0^{\pi/2}$$

$$= 2\{\pi/2+\sin(\pi/2)\cos(\pi/2)-\sin(0)\cos(0)\} \;=\; \pi,$$

on using (i).

Because $y = \sqrt{4-t^2}$, $0 \le t \le 2$ is the equation of a quarter of circle with center at (t, y) = (0, 0) and radius 2, Area(f, [0, 2]) is a quarter of the area of a circle of radius 2, i.e., a quarter of $\pi \cdot 2^2$

23.9 (i) Applying the smoothness conditions $g(2-) = g(2+)$ and $g'(2-) = g'(2+)$ yields $A = -6$ and $B = -2e^4$.

(ii) So

$$\text{Int}(g,[0,4]) = \int_0^2\{-6\cos(\pi t)+t^2\}dt + \int_2^4\{-2e^{-2(t-2)}\}dt$$

$$= \int_0^2 \frac{d}{dt}\left\{-\frac{6}{\pi}\sin(\pi t)+\frac{t^3}{3}\right\}dt + \int_2^4 \frac{d}{dt}\left\{e^{-2(t-2)}\right\}dt$$

$$= \left\{-\frac{6}{\pi}\sin(\pi t)+\frac{t^3}{3}\right\}\Big|_0^2 + e^{-2(t-2)}\Big|_2^4$$

$$= \frac{8}{3} + e^{-4} - 1 = \frac{5}{3} + e^{-4} = 1.685.$$

23.10 Define Q by $Q(y) = \sin(y)$, so that $Q'(y) = \cos(y) \Rightarrow Q'(P(x)) = \cos(P(x))$. Then, by the chain rule,

$$\frac{d}{dx}\{\sin(P(x))\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)Q'(P(x)) = P'(x)\cos(P(x)).$$

23.11 Define Q by $Q(y) = \cos(y)$, so that $Q'(y) = -\sin(y) \Rightarrow Q'(P(x)) = -\sin(P(x))$. Then, by the chain rule,

$$\frac{d}{dx}\{\cos(P(x))\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)Q'(P(x)) = -P'(x)\sin(P(x)).$$

23.12 (i) $F(x) = e^{2\cos(x)}$. So, by Exercise 20.2,

$$F'(x) = \frac{d}{dx}\{2\cos(x)\}\cdot e^{2\cos(x)} = 2\{-\sin(x)\}\cdot e^{2\cos(x)} = -2\sin(x)\cdot\left(e^{\cos(x)}\right)^2$$

(ii) By the product rule,

$$F'(x) = \frac{d}{dx}\{x^5\}\cdot\cos(x) + x^5\cdot\frac{d}{dx}\{\cos(x)\}$$
$$= 5x^4\cdot\cos(x) + x^5\cdot\{-\sin(x)\} = x^4\{5\cos(x) - x\sin(x)\}.$$

(iii) By Exercise 20.3 with $P(x) = 1 + \cos(x) \Rightarrow P'(x) = 0 - \sin(x) = -\sin(x)$,

$$F'(x) = \frac{d}{dx}\{\ln(P(x))\} = \frac{P'(x)}{P(x)} = -\frac{\sin(x)}{1+\cos(x)}.$$

(iv) Here $F(x) = \ln(x^5) + \ln(e^{\cos(x)}) = 5\ln(x) + \cos(x)$ by properties of the logarithm. So $F'(x) = 5/x - \sin(x)$.

23.13 (i) By Exercise 20.2,

$$F'(x) = \frac{d}{dx}\left(e^{3\sin(x)}\right) = \frac{d}{dx}\left(3\sin(x)\right)\cdot e^{3\sin(x)}$$
$$= 3\cos(x)e^{3\sin(x)} = 3\cos(x)\left(e^{\sin(x)}\right)^3$$

(ii) By the product rule and Exercise 11 with $P(x) = 2x \Rightarrow P'(x) = 2$,

$$F'(x) = \frac{d}{dx}\left\{x^3\right\}\cdot\cos(2x) + x^3\cdot\frac{d}{dx}\{\cos(2x)\}$$
$$= 3x^2\cos(2x) + x^3\cdot\{-2\sin(2x)\} = x^2\{3\cos(2x) - 2x\sin(2x)\}$$

(iii) By Exercise 20.3,

$$F'(x) = \frac{d}{dx}\{1+\sin(x)\}\cdot\frac{1}{1+\sin(x)} = \{0+\cos(x)\}\frac{1}{1+\sin(x)} = \frac{\cos(x)}{1+\sin(x)}$$

(iv) By properties of the logarithm,

$$F(x) = \ln(x^7) + \ln\left(e^{\sin(x)}\right) = 7\ln(x) + \sin(x).$$

So $F'(x) = 7/x + \cos(x)$.

23.14 (i) $F'(x) = 3(3x^2+2x+1)\cdot\left(e^{x^3+x^2+x+1}\right)^3$

(ii) $F'(x) = 4x^3\cos(x^2+2) - 2x^5\sin(x^2+2)$

(iii) $F'(x) = 24x^5/(1+4x^6)$

(iv) $F(x) = 7\ln(x) + x^3 + x^2 + x + 1$, implying $F'(x) = 7/x + 3x^2 + 2x + 1$

23.15 (i) $F'(x) = 5x^2(4x+3)\cdot\left(e^{x^4+x^3}\right)^5$

(ii) $F'(x) = 5x^4\sin(x^3+2) + 3x^7\cos(x^3+2)$

(iii) $F'(x) = 60x^9/(1+6x^{10})$

(iv) $F(x) = 6\ln(x) + x^4 + x^3$, implying $F'(x) = 6/x + 4x^3 + 3x^2$

23.16 Extracting the leading term of the difference quotient, we have

$$G'(t) \;=\; \frac{\sin(2t)}{1+\sin^2(t)}.$$

Hence, by the fundamental theorem,

$$\int_0^{\pi/2} \frac{\sin(2x)}{1+\sin^2(x)}\,dx \;=\; \int_0^{\pi/2} G'(x)\,dx \;=\; G(\pi/2)-G(0)$$
$$= \ln\big(1+\sin^2(\pi/2)\big) \;-\; \ln\big(1+\sin^2(0)\big)$$
$$= \ln\big(1+1^2\big) \;-\; \ln\big(1+0^2\big)$$
$$= \ln(2) \;-\; \ln(1) \;=\; \ln(2).$$

23.17 (i) By the chain rule with $P(t) = \cos(t)$ and $Q(y) = y^3$, hence $P'(t) = -\sin(t)$ and $Q'(y) = 3y^2$, we have

$$\frac{d}{dt}\left\{\cos^3(t)\right\} \;=\; \frac{d}{dt}\{Q(P(t))\} \;=\; P'(t)\cdot Q'(P(t))$$
$$= -\sin(t)\cdot 3\{P(t)\}^2 \;=\; -3\sin(t)\cdot\cos^2(t)$$

(ii) We substitute $u = \phi(x) = \arcsin(x)$. So the inverse substitution is $x = \zeta(u) = \sin(u)$, implying $\zeta'(u) = \cos(u)$. Also, $\phi(0) = \arcsin(0) = 0$ and $\phi(1) = \arcsin(1) = \pi/2$. So, with $f(x) = x\sqrt{1-x^2}$, we have

$$\int_0^1 x\sqrt{1-x^2}\,dx \;=\; \int_0^1 f(x)\,dx \;=\; \int_{\phi(0)}^{\phi(1)} f(\zeta(u))\zeta'(u)\,du$$
$$= \int_0^{\pi/2} \zeta(u)\sqrt{1-\{\zeta(u)\}^2}\,\zeta'(u)\,du \;=\; \int_0^{\pi/2} \sin(u)\sqrt{1-\sin^2(u)}\,\cos(u)\,du$$
$$= \int_0^{\pi/2} \sin(u)\cdot\cos(u)\cdot\cos(u)\,du \;=\; -\frac{1}{3}\int_0^{\pi/2}\{-3\sin(u)\cos^2(u)\}\,du$$
$$= -\frac{1}{3}\int_0^{\pi/2}\left\{\frac{d}{du}\{\cos^3(u)\}\right\}du \;=\; -\frac{1}{3}\cos^3(u)\Big|_0^{\pi/2}$$
$$= -\frac{1}{3}\{\cos^3(\pi/2)-\cos^3(0)\} \;=\; -\frac{1}{3}\{0^3-1^3\} \;=\; \frac{1}{3}$$