The mean and median of a distribution.

What does it mean to be above average in some respect? A possible answer is above the middle, or median, of a relevant distribution, defined as the value exceeded with probability 0.5. That is, if the relevant random variable, say \( X \), is distributed on \([0, \infty)\) with p.d.f. \( f \) and c.d.f. \( F \) and \( M \) denotes the median, then

\[
\Pr(X \leq M) = F(M) = \int_0^M f(x) \, dx = \frac{1}{2} \tag{26.1a}
\]

or, equivalently,

\[
\Pr(X \geq M) = 1 - F(M) = \int_M^\infty f(x) \, dx = \frac{1}{2}. \tag{26.1b}
\]

For example, from (22.37)-(22.38), the c.d.f. and p.d.f. of a Weibull distribution with shape parameter \( c \geq 1 \) and scale parameter \( s > 0 \) are defined by

\[
F(x) = 1 - e^{-(x/s)^c} \tag{26.2a}
\]

and

\[
f(x) = c s (x/s)^{c-1} e^{-(x/s)^c} \tag{26.2b}
\]

where, in general, \( s \) and \( c \) may be any positive numbers, although in this lecture we assume that \( c \geq 1 \). If \( c = 1 \), then \( f(x) \) is equal to \( (s/x) e^{-x/s} \), which is

\[
\int_{s/x} f(x) \, dx = 1. \tag{26.3}
\]

The mean is defined as the value at which a cardboard lamina of area \( \frac{1}{2} \), cut to the shape of the region between the horizontal axis and the graph of the p.d.f., would balance if no other weight is applied to either side. The mean is illustrated in Figure 1, where shaded area equals 0.5. Note that \( M \) lies above the mode for \( c = 2 \) but below the mode for \( c = 5 \) in Figure 1. If \( c > 1 \), then the distribution is unimodal with mode \( \frac{1}{1-1/c} s > 0 \), by (20.35). Either way, (1)-(2) imply

\[
\exp\left(-\frac{M/s}{c}\right) = \frac{1}{2} \text{ or } M = s \ln\left(\frac{1}{2}\right)^{1/c} \tag{26.4}
\]

(Exercise 1). The median is illustrated for \( c = 2 \) and \( c = 5 \) by Figure 1, where shaded area equals 0.5. Note that \( M \) lies above the mode for \( c = 2 \) but below the mode for \( c = 5 \) in Figure 1. If \( c > 1 \), then the distribution is unimodal with mode \( \frac{1}{1-1/c} s > 0 \), by (20.35). Either way, (1)-(2) imply

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\]
or clockwise, for \( x > \mu \) and negative, or anticlockwise, for \( x < \mu \). But \( f(x) \) is not a weight; rather, it is a weight density, i.e., a weight per unit length. Therefore

\[
T(x) = (x - \mu)f(x)
\]

is not a turning effect; rather, it is a turning effect density, or turning effect per unit length. Accordingly, just as \( \int f(x) \, dx \) is the weight associated with the interval \([a, b]\), so \( \int T(x) \, dx \) is the turning effect associated with the interval \([a, b]\). Hence the total clockwise or anticlockwise moment about the balance point is

\[
\int_{-\infty}^{\infty} x f(x) \, dx = \int_{\mu}^{\infty} T(x) \, dx - \int_{-\infty}^{\mu} T(x) \, dx
\]

(26.6)

From (8.25) and using (16.20) in conjunction with Table 18.1, we have

\[
\begin{align*}
\int_{-\infty}^{\infty} T(x) \, dx &= \int \frac{e^{\sqrt{V-1}y}}{V + 1} \, dy \quad \text{for } V = 0.768, \text{ Clearly,}
\end{align*}
\]

(19.2)

The p.d.f. is described by (19.2), the p.d.f. is defined as the probability density function for a random variable. To illustrate, consider mean survival time for Lecture 15’s melanoma patients. From the definition of the median, it is the unique number \( \mu \) which is not a finite quantity. So an alternative and potentially useful distribution function for which \( \mu \) is not a finite quantity is

\[
\mu = \int_{0}^{\infty} x f(x) \, dx
\]

(26.9)

Using elementary properties of integrals, we can rewrite (8) as

\[
\int_{0}^{\infty} x f(x) \, dx = \mu.
\]

(26.10)

So from (10), and on using (16.20) in conjunction with Table 18.1, we have

\[
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\int_{0}^{\infty} x f(x) \, dx &= \mu
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\int_{0}^{\infty} x f(x) \, dx &= \mu
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We can simplify this integral by using the substitution
\[
\frac{\mu}{2} \int_0^{\infty} e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x} = \int_0^{\infty} e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x}
\]

implying
\[
\frac{s}{x} = \frac{(n)}{x} = n
\]

because \(n\) implies \(x = ns\), the inverse substitution is defined by
\[
\frac{s}{x} = \left(\frac{x}{\phi}\right) = n
\]

\(26.20\)

\[
\int_0^{\infty} e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x} = \int_0^{\infty} e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x}
\]

\(26.19\)

\[
\int_0^{\infty} \left(\frac{s}{x}\right) e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x} = \int_0^{\infty} \left(\frac{s}{x}\right) e^{-\left(\frac{s}{x}\right) t} \frac{dx}{x}
\]

\(26.18\)

\[
\int_0^{\infty} \frac{\mu}{2} - \frac{x}{2} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\}
\]

\(26.17\)

\[
\int_0^{\infty} \frac{\mu}{2} - \frac{x}{2} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\}
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\(26.16\)

\[
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\(26.14\)

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\(26.12\)

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\]

\(26.09\)

\[
\int_0^{\infty} \frac{\mu}{2} - \frac{x}{2} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\} = \int_0^{\infty} \left\{\frac{n}{2} - \frac{x}{2}\right\}
\]
\[ \int_0^\infty s \phi(0) = 0 \] and, because \( s > 0 \),
\[ \phi(\infty) = \infty. \]  (Note, however, that \( s < 0 \) would imply \( \phi(\infty) = -\infty. \))

From (21), we have
\[ \int_a^b g(x) \, dx = \int_a^b g(\phi(u)) \phi'(u) \, du = \phi(a) - \phi(b) \]
for arbitrary \( g \).  With \( g \) defined by
\[ g(x) = \frac{c(x/s)}{c \cdot e^{-x/s}}, \]
(20) reduces to
\[ \mu = \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

Because the right-hand side of (26) depends only on \( s \) and \( c \), we can

on using (21)-(23), because the right-hand side of (26) (23)
\[ \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

\[ \mu = \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

Now, \( I + c + I = I/c \), and in mathematics one always prefers the simplest form
\[ \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

Of an expression. So why do we not write

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Because \( c \) is positive, if \( x \to \infty \) then \( \phi(x) \to \infty \) also. So \( (21)-(23) \)

(20) reduces to
\[ \mu = \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

imply

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\[ u = \phi(x) = x \]
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replacing \( x \) by \( u \): because the right-hand side of (26) (23)
\[ \int_0^\infty \frac{c \phi(u)/s}{c \cdot e^{-\phi(u)/s}} = \int_0^\infty \frac{c \phi(u)}{c \cdot e^{-\phi(u)/s}}. \]

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The Gamma function is a "known" function of mathematics, just like exp or ln. In terms of $\Gamma$, the mean of the Weibull is simply
\[ \mu = s \Gamma \left( 1 + \frac{1}{c} \right), \tag{26.34} \]

The domain of the Gamma function is the largest interval on which the integral in (26) corresponds to a finite area, or "converges," which turns out to be $(0, \infty)$. On this domain, $\Gamma$ is concave up with global minimum $0.8856$ and range $[0.8856, \infty)$. See Figure 3. Because $1 \leq c < \infty$, however, we have $1 + 1/c \leq 2$. Thus, as far as the mean of Figure 4 is concerned, it suffices to know $\Gamma$ only on $[1, 2]$. The restriction of $\Gamma$ to this subdomain is graphed in Figure 4. Note that $\Gamma(1) = 1 = \Gamma(2)$ (26.35)

For example, rat pupil radius in Lecture 22 has a Weibull distribution with shape parameter $c = 2$ (as in Figure 2) and scale parameter $s = 0.713$. So, by (34) and Figure 4, mean rat pupil radius is $s \Gamma(3/2) = 0.886s = 0.886 \times 0.713 = 0.63$ mm. Similarly, for the Weibull in Figure 19.3, implying that mean leaf thickness in Dicerandra linearifolia is $s \Gamma(8/7) = 0.152s \times 0.935 = 0.14$ mm. Again, $c = 5$ and $s = 17.84$ for the Weibull in Figure 19.5, so that mean size (above base length) in D'Arcy Thompson's minnows is $s \Gamma(6/5) = 17.8s \times 0.9182 = 16.4$ mm. Finally, $c = 1$ and $s = 1.286$ in Figure 19.1, so mean life expectancy among prairie dogs is $s \Gamma(2) = s = 1.286$ years.

A Gamma function has a recursive property, namely,
\[ \Gamma(r + 1) = r \Gamma(r) \tag{26.36} \]
for any $r > 0$ (see Exercise 4 and Appendix 26). If, for example, we require both $\Gamma(0.5)$ and $\Gamma(3.7)$, we can use (36) to obtain reasonably accurate answers from Figure 4, even though neither 0.5 nor 3.7 belongs to $[1, 2]$. For example, setting $r = 0.5$ in (36) yields $\Gamma(1.5) = 0.5 \Gamma(0.5)$, so that $\Gamma(0.5) = \frac{2}{5} \times 0.886 = 1.772$. Similarly, setting $r = 2.7$ in (36) yields $\Gamma(3.7) = 2.7 \Gamma(2.7)$, and setting $r = 1.7$ yields $\Gamma(2.7) = 1.7 \Gamma(1.7)$, so that $\Gamma(3.7) = 2.7 \times 1.7 \times \Gamma(1.7) = 2.7 \times 1.7 \times 0.909 = 4.17$.

The quantity $\Gamma(0.5)$ will surface again in Lecture 28, in an important context. So we conclude by noting for later reference that $\Gamma(0.5) = 1.772$ is merely a numerical approximation to a precise relationship, namely,

\[ \frac{1}{\sqrt{\pi}} = \int_0^{\infty} e^{-x^2} dx \]
where $\pi$ is the ratio between circumference and diameter of a circle.

\[ \pi = \frac{\text{circumference}}{\text{diameter}} \]
Exercises 26

26.1 (i) Verify that \( M = \frac{1}{s/\ln(2)} \) for the Weibull distribution defined by (2).

(ii) Verify that \( M \) lies above or below the mode according to whether \( c < c^* \) or \( c > c^* \), where \( c^* = \frac{1}{\ln(2) - 1} \approx 3.26 \).

26.2 Find the median of the distribution defined on \([0, \infty)\) by

\[
f(x) = \begin{cases} 2/3 & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x < 3 \\ 0 & \text{otherwise} \end{cases}
\]

26.3 Show that \( \frac{d}{du} \left( x \right) e^{-u} \) and \( \frac{d}{du} \left( x + 1 \right) e^{-u} \) are \( x e^{-u} \).

Hence establish (35).

26.4 Use mathematical induction (Appendix 17B) to show that if \( r \) is an integer, then \( \Gamma(r+1) = r! \), where \( r! \) (or \( r \) factorial) is the product of the first \( r \) positive integers, i.e., \( r! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (r-1) \cdot r \).

26.5 The p.d.f. of a distribution on \([0, \infty)\) is defined by

\[
f(x) = \begin{cases} x(2-x)/L & \text{if } 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}
\]

where \( L \) is a constant. Find (i) \( L \) (ii) \( m \) (iii) \( \mu \) (iv) the c.d.f.

26.6 The p.d.f. of a distribution on \([0, \infty)\) is defined by

\[
f(x) = \begin{cases} x^2(2-x)/L & \text{if } 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}
\]

where \( L \) is a constant. Find (i) \( L \) (ii) \( m \) (iii) \( \mu \) (iv) the c.d.f.

26.7* The p.d.f. of a distribution on \([0, \infty)\) is defined by

\[
f(x) = \begin{cases} x^2(2-x)^2/L & \text{if } 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}
\]

where \( L \) is a constant. Find (i) \( L \) (ii) \( m \) (iii) \( \mu \) (iv) the c.d.f.

26.8 The exponential distribution defined by (36) and the distributions defined in Exercises 22.9-22.11 are all special cases of the "Gamma" distribution. The p.d.f. of the Gamma with shape parameter \( c \) and scale parameter \( s \) is defined by

\[
f(x) = \frac{x^{c-1}e^{-x/s}}{L}.
\]

The expected distribution defined by (36) and the distributions defined in Exercises 22.9-22.11 are all special cases of the "Gamma" distribution. The p.d.f. of the Gamma with shape parameter \( c \) and scale parameter \( s \) is defined by

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where \( \lambda \), \( b \) are parameters and \( L \) is a constant chosen to ensure that \( \int_{0}^{\infty} f(x) \, dx = 1 \).

\[
\begin{align*}
\text{f(x)} &= \begin{cases} 
\frac{1}{L} e^{-\lambda x} & \text{if } 0 \leq x \leq b \\
0 & \text{if } b \leq x \leq \infty
\end{cases}
\end{align*}
\]

\( \int_{0}^{\infty} \text{f(x)} \, dx = 1 \)

Hint: For (i)-(iv), use (21)-(24) and (33).
The purpose of this appendix is to establish that $\Gamma(r + 1) = r\Gamma(r)$ for any $r > 0$. The key observation is that the Gamma function $\Gamma(z)$ approaches zero as $z \to \infty$. No matter how large the value of $r$, $\Gamma(r + 1)$ is the product of something that gets bigger and bigger as $u \to \infty$ and something that gets smaller and smaller. At the same time, however, the larger the value of $r$, the more rapidly $\Gamma(z)$ increases. Hence $\Gamma(r + 1) = r\Gamma(r)$.

Exercise 1. The easiest way to see this result is to compare the graph of $z = u/r$ with that of $z = \ln(u)$. The first is a straight line with positive slope through the origin of coordinates; the second is a concave down curve (Figure 22.1). Because $\Gamma(z)$ keeps increasing as $z \to \infty$, there must come a point beyond which $\Gamma(z) > \ln(u)$ for all $u > 0$. Therefore, $\Gamma(r)$ must also approach zero as $r \to \infty$. Hence $u - \Gamma(r)$ must approach zero as $u \to \infty$.

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26.5 (i) $\frac{1}{I} = (I) \neq (\lambda, i) = I \neq I = \pi \neq \pi (\lambda, i) = I \neq \pi \neq \frac{3}{4} = 3 (i)$

Because $e^x$ approaches infinity much more rapidly than $x$ as $x \to \infty$, from

$$I = \left\{ \frac{x^\alpha}{x+1} - \frac{1}{I} \right\}_{x^\alpha}^{\infty} = \left\{ \frac{1}{x} \cdot \vartheta(I+x) - 1 \right\}_{x^\alpha}^{\infty} =$$

$$\left\{ 0 \cdot \vartheta(I+0) + \vartheta(I+x+) \right\}_{x^\alpha}^{\infty} = \int_{x^\alpha}^{\infty} \vartheta(I+n) - \vartheta(I) = 0 \int_{x^\alpha}^{\infty} \vartheta(I+n) =$$

$$\text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} = \text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} = \text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} =$$

and the fundamental theorem

$\mathbb{Z} = x$ with

Similarly, from (26), because $e^x$ becomes arbitrarily large as $x \to \infty$, with $x \to \infty$.

$$I = \left\{ \frac{x^\alpha}{1} - 1 \right\}_{x^\alpha}^{\infty} = \left\{ \frac{1}{x} \cdot \vartheta(I+x) - 1 \right\}_{x^\alpha}^{\infty} =$$

$$\left\{ 0 \cdot \vartheta(I+0) - \vartheta(I-x) \right\}_{x^\alpha}^{\infty} = \int_{x^\alpha}^{\infty} \vartheta(I+n) - \vartheta(I) = 0 \int_{x^\alpha}^{\infty} \vartheta(I+n) =$$

$$\text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} = \text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} = \text{np} \left\{ \vartheta(I+n) - \frac{1}{p} \right\}_{0}^{\infty} =$$

Now, from (26) with $I = x$ and the fundamental theorem

$\vartheta(I+n) = \vartheta(I+n) + (\vartheta(I+n) - 0) = 0$.

The product rule yields

$$26.3$$

Answers and Hints for Selected Exercises
26.6 (i) \[ L = \frac{4}{3} \]
(ii) \[ m = \frac{4}{3} \]
(iii) \[ \mu = \frac{6}{5} \]

(iv) \[ F(t) = \frac{1}{16} t^3 (8 - 3t) \]

26.7 (i) Define \( g \) by
\[ g(x) = \begin{cases} 
  x(2 - x) & \text{if } 0 \leq x < 2 \\
  0 & \text{if } 2 \leq x < \infty
\end{cases} \]

Then \( f(x) = \frac{g(x)}{L} \). So
\[ \int_0^\infty f(x) \, dx = 1 \implies \int_0^\infty \frac{g(x)}{L} \, dx = 1 \implies \frac{1}{L} \int_0^\infty g(x) \, dx = 1, \]
implying \( L = \int_0^\infty g(x) \, dx \).

26.7 (i) Clearly, \( 0 < m < 2 \). For \( x < 2 \), we have
\[ f(x) = \frac{3x}{4} (2 - x)^2 \]
and
\[ f'(x) = \frac{3}{4} \frac{d}{dx} x(2 - x)^2 \]

\[ = \frac{3}{4} \frac{d}{dx} (2 - x) \cdot x^2 + x(2 - x) \cdot 2x \]

\[ = 2x^2 - \frac{4}{3} x^3 + \frac{1}{4} x^4 \]

\[ \int_0^2 \left( 2x^2 - \frac{4}{3} x^3 + \frac{1}{4} x^4 \right) \, dx = \frac{2}{3} \cdot \frac{2}{3} - \frac{4}{3} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{4}{3}. \]

That is, \( m = \frac{2}{3} \).
(iii) From (12),
\[
\mu = \int_0^\infty x^3 \frac{x - 2}{x^2 + 1} \, dx
\]

From (12), (3) implies
\[
\int_0^\infty x^3 \frac{x - 2}{x^2 + 1} \, dx = 1
\]

Then \( F(x) = (x)^{\alpha} \). So, as in the previous exercise, \( L/\Gamma(x)^{\alpha} = 1 \).

(iv) If \( t > 2 \) then \( F(t) = 1 \). If \( 0 < t < 2 \), then
\[
F(t) = \int_0^t f(x) \, dx
\]

Define \( \beta \) by
\[
\int_0^\infty \left( \frac{1}{x^\alpha} + \frac{x}{x^\alpha} - \frac{x}{x^\alpha} \right) \, dx = \int_0^\infty \frac{x^\alpha}{x^\alpha} \, dx
\]

Note that \( F(2) = 1 \).

26.8 (i) Define \( g \) by
\[
g(x) = \begin{cases} x^2 (2 - x)^2 & \text{if } 0 \leq x < 2 \\ 0 & \text{if } 2 \leq x < \infty \end{cases}
\]

Then \( f(x) = \frac{g(x)}{L} \). So, as in the previous exercise, \( L = \int_0^\infty g(x) \, dx \), implying
\[
L = \int_0^\infty x^2 (2 - x)^2 \, dx
\]

Note that \( F(2) = 1 \). If \( 0 < t < 2 \) then
\[
\int_0^t f(x) \, dx = \int_0^t \frac{x^\alpha}{x^\alpha} \, dx
\]

From (12) (ii)
A simpler way to show that \( m = 1 \) in this case will emerge in Lecture 27.

\[
\begin{align*}
1 &= \left( 0 - \frac{4}{3}x^3 + \frac{1}{5}x^5 \right) \frac{9I}{\Sigma I} = \\
&= \left( \int_0^1 x^3 + \frac{4}{3}x^5 - x \right) \frac{9I}{\Sigma I} \\
&= \int_0^1 x \left( x^3 + \frac{4}{3}x^5 - x \right) \frac{9I}{\Sigma I} = \\
&= \int_0^1 x \left( x + \frac{4}{3}x^4 - \frac{1}{5}x^5 \right) \frac{9I}{\Sigma I} = \\
&= \int \left( x - \frac{4}{3}x^5 + \frac{1}{5}x^6 \right) \frac{9I}{\Sigma I} = \\
&= \int x^4 - \frac{4}{3}x^5 + \frac{1}{5}x^6 \frac{9I}{\Sigma I} = \\
&= \left( \frac{16}{15} \right).
\end{align*}
\]

(ii) Clearly, \( 0 < m < 2 \). For \( x < 2 \), we have

\[
\begin{align*}
\int_0^1 x \left( x^3 + \frac{4}{3}x^5 - x \right) \frac{9I}{\Sigma I} = \\
&= \int_0^1 x \left( x + \frac{4}{3}x^4 - \frac{1}{5}x^5 \right) \frac{9I}{\Sigma I} = \\
&= \int \left( x - \frac{4}{3}x^5 + \frac{1}{5}x^6 \right) \frac{9I}{\Sigma I} = \\
&= \int x^4 - \frac{4}{3}x^5 + \frac{1}{5}x^6 \frac{9I}{\Sigma I} = \\
&= \left( \frac{16}{15} \right).
\end{align*}
\]

So \( \mu = \frac{16}{15} \) when \( 0 < x < 1 \) but \( \mu = \frac{15}{16} \) when \( 1 < x < 2 \), implying that \( f \) has a maximum \( \frac{15}{16} \) where \( x = 1 \). This maximizer is the mode, \( \mu \). That is, \( m = 1 \).

(iii) From (12),

\[
\begin{align*}
\mu &= \int_0^1 x \int_0^1 \left( x^3 + \frac{4}{3}x^5 - x \right) \frac{9I}{\Sigma I} = \\
&= \int_0^1 x \left( x + \frac{4}{3}x^4 - \frac{1}{5}x^5 \right) \frac{9I}{\Sigma I} = \\
&= \int \left( x - \frac{4}{3}x^5 + \frac{1}{5}x^6 \right) \frac{9I}{\Sigma I} = \\
&= \int x^4 - \frac{4}{3}x^5 + \frac{1}{5}x^6 \frac{9I}{\Sigma I} = \\
&= \left( \frac{16}{15} \right).
\end{align*}
\]

A simpler way to show that \( m = 1 = \mu \) in this case will emerge in Lecture 27.
(iv) If $t > 2$ then $F(t) = 1$. If $0 \leq t \leq 2$, then 
\[
F(t) = \int_0^t f(x) \, dx
\]
\[
= \left[ \frac{15}{16} x^2 \right]_0^t = \frac{15}{16} t^2.
\]

Note that $F(2) = 1$.

On using the product rule,

\[
\frac{d}{dt} \left\{ \frac{s}{x} - I - e \right\} = \frac{1}{s/x} \frac{d}{dx} x^c - \frac{1}{e} e^{-x/s}
\]
\[
= \frac{1}{s/x} \frac{d}{dx} x^c - \frac{1}{e} e^{-x/s} + x^c - 1 \frac{d}{dx} e^{-x/s}
\]
\[
= x^c - 2 e^{-x/s}.
\]

So \( \frac{d}{dt} F(t) > 0 \) when \( 0 < x < (c-1)s \) but \( \frac{d}{dt} F(t) < 0 \) when \( (c-1)s < x < \infty \), implying that \( F \) has a maximum where \( x = (c-1)s \). That is, \( m = (c-1)s \).