Although previously we have always taken the sample space of a continuous random variable $X$ to be $[0, \infty)$, we have also set $f(x) = 0$ for $x > b$, so that the effective sample space is $[0, b]$. If also $f(x) = 0$ for $x < a$, then the effective sample space becomes $[a, b]$. Accordingly, we now consider distributions on $[a, b]$ instead of on $[0, \infty)$. Our analysis will be more general, because it allows for the possibility that $a < 0$. The function $f$ is the p.d.f. of a random variable distributed on $[a, b]$ if $f$ is nonnegative and total probability is 1, i.e.

$$\int_a^b f(x) \, dx = 1$$

where $\lambda$ is a positive constant, and

$$f(a) = f(b) = 0,$$

for any $a \in [0, b]$. These distributions are depicted in Figure 1. The top panel shows the p.d.f. of a symmetric distribution on $[a, b]$ for which

$$f(x) = \begin{cases} \frac{a-x}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.5)

for the mid-point

$$M = \frac{a + b}{2}$$

(28.3)

An important special case occurs when the distribution is symmetric about the mid-point

$$f(M - t) = f(M + t)$$

(28.4)

for any $t \in [0, (b-a)/2]$. Three such distributions are depicted in Figure 1. The top panel shows the p.d.f. of a uniform distribution on $[a, b]$, for which

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.6)

The middle panel shows the p.d.f. of a symmetric triangular distribution on $[a, b]$, for which

$$f(x) = \begin{cases} \frac{2(b-a)}{b-a - 2x - M(b-a)} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.7)

The bottom panel shows the p.d.f. of a bell-shaped distribution on $[a, b]$ with

$$f(x) = \begin{cases} \frac{1}{L} e^{\lambda(x-M)^2/(2)} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.8)

where $\lambda$ is a positive constant, and $L$ is another positive constant that guarantees

$$\int_a^b f(x) \, dx = 1.$$

The symmetry of such distributions can often be exploited in calculations. For example, because the lighter and darker shaded areas in Figure 1 are equal, i.e.,

$$\int_a^b f(x) \, dx = 1$$

(28.8)

The function $F$ is the c.d.f. of a random variable distributed on $[a, b]$ for which

$$F(a) = 0, \quad F(x) \geq 0, \quad a \leq x < b,$$

(28.2)

and $\int_a^b f(x) \, dx = 1$.

$$F(x) = \int_a^x f(x) \, dx$$

(28.1a)

For any $t \in [0, (b-a)/2]$, the c.d.f. of a symmetric distribution is

$$F(M-t) = F(M+t)$$

(28.2a)

The density $f$ and c.d.f. $F$ are related by the obvious generalization of (19.9), namely,

$$f(x) = \frac{d}{dx} F(x)$$

(28.3)

where $f$ and $F$ are related by the obvious generalization of (19.9), namely,

$$F(a) = 0, \quad F(x) \geq 0, \quad a \leq x < b,$$

(28.1a)

and $\int_a^b f(x) \, dx = 1$.

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

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The middle panel shows the p.d.f. of a symmetric triangular distribution on $[a, b]$, for which

$$f(x) = \begin{cases} \frac{2(b-a)}{b-a - 2x - M(b-a)} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.7)

The bottom panel shows the p.d.f. of a bell-shaped distribution on $[a, b]$ with

$$f(x) = \begin{cases} \frac{1}{L} e^{\lambda(x-M)^2/(2)} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(28.8)
We illustrate by calculating the variance of the triangular distribution defined by (27.2). For a symmetric distribution, variance is often more easily calculated from either (17a).

Now, from (17)-(18), we have

\[
\int_a^b \frac{a - q}{(q - x)^2} \cdot \frac{a - q}{x} \, dx = (x)_{\int_a}^{b} = (x)_{\int_a}^{b}.
\]

For \( x \neq a, b \), so that (6) and (10) imply (6). Hence, (6) implies (13) is satisfied. So if function D satisfies (11), then

\[
D(x) = D(x)_{\int_a}^{b}, \quad \text{and}\quad D(x) = D(x)_{\int_a}^{b}.
\]

and

\[
D(x)_{\int_a}^{b} = D(x)_{\int_a}^{b}.
\]

For a symmetric distribution, variance is often more easily calculated from either (17a).

One such function is the dispersion density of a symmetric distribution, see Exercise 1. The dispersion density of a symmetric distribution, (28.15)

\[
D(x) = \frac{(x - \mu)^2}{f(x)}
\]

for \( f \) satisfying (11). From (15) and (11) we have

\[
D(x)_{\int_a}^{b} = D(x)_{\int_a}^{b} \cdot (28.16)
\]

i.e., (13) is satisfied. So integrating (11), we have

\[
D(x) = D(x)_{\int_a}^{b} = D(x)_{\int_a}^{b}.
\]

The dispersion density of a symmetric distribution, (28.15)

\[
D(x) = \frac{(x - \mu)^2}{f(x)}
\]

for \( f \) satisfying (11). From (15) and (11), we have

\[
D(x) = D(x)_{\int_a}^{b}.
\]

(28.16)
Then, because \( \int_{-\infty}^{\infty} \phi = 1 \), \( \phi \) implies \( 0 < \gamma < x \gamma = x \).

By the inverse substitution is defined by
\[
\gamma = (\phi) = n
\]

We now make the substitution
\[
\int_{0}^{L} x \gamma \phi = \int_{0}^{L} = 1
\]

Taking the limit as \( K \to \infty \), so that the sample space becomes \( (0, \infty) \), we have
\[
\int_{0}^{\infty} = (x) f = n
\]

where
\[
K \leq x \leq K - \int_{0}^{\infty} = (x) f
\]

(28.26)

One of the most important distributions in applied mathematics, the **Gaussian** distribution, can be obtained from an appropriate limit of the bell-shaped distribution in Figure 1. First, we set \( a = -K \) and \( b = K \), so that \( 0 \leq x \leq 1 \). Then
\[
\phi = \frac{e^{-\frac{1}{2}}}{\lambda^{1/2}}
\]

(28.17)

By using (10), the triangular distribution has coefficient of variation
\[
\frac{\sigma}{\mu} = \frac{1}{3}
\]

(28.19)

Then, because \( 0 < \gamma < x \gamma = x \gamma = x \).

Because \( u = \lambda x \), the inverse substitution is defined by
\[
\gamma = (\phi) = n
\]

(28.20)

We now make the substitution
\[
\int_{0}^{L} x \gamma \phi = \int_{0}^{L} = 1
\]

Taking the limit as \( K \to \infty \), so that the sample space becomes \( (0, \infty) \), we have
\[
\int_{0}^{\infty} = (x) f = n
\]

where
\[
K \leq x \leq K - \int_{0}^{\infty} = (x) f
\]

(28.26)

One of the most important distributions in applied mathematics, the **Gaussian** distribution, can be obtained from an appropriate limit of the bell-shaped distribution in Figure 1. First, we set \( a = -K \) and \( b = K \), so that \( 0 \leq x \leq 1 \). Then
\[
\phi = \frac{e^{-\frac{1}{2}}}{\lambda^{1/2}}
\]

(28.17)

By using (10), the triangular distribution has coefficient of variation
\[
\frac{\sigma}{\mu} = \frac{1}{3}
\]

(28.19)
\[ L = 2e^{-\lambda \zeta(u)} \frac{\phi(0)}{\phi(\infty)} \int du = 2e^{-u} - \frac{1}{2} \frac{2}{\lambda} \int_{0}^{\infty} du = 1 \]

From the definition of \( \Gamma \), and on using (26.31). Thus, letting \( K \to \infty \) in (21), we have

\[ f(x) = \lambda e^{-\lambda x^2}, \quad -\infty < x < \infty. \] (28.28)

From (17), with \( \mu = 0 \) and letting \( b \to \infty \), the variance is

\[ \sigma^2 = 2 \lambda \pi x^2 e^{-\lambda x^2} dx = 0 \int_{0}^{\infty} \frac{2 \lambda \pi \zeta(u)}{I} du. \]

This is the p.d.f. of a normal distribution with mean zero and standard deviation \( \sigma \). Finally, to obtain the p.d.f. of a normal distribution with arbitrary mean \( \mu \) and standard deviation \( \sigma \), we have to do is replace \( x \) by \( x - \mu \):

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty. \] (28.30)

Integration by substitution readily establishes that this the p.d.f. of a symmetric, bell-shaped distribution on \( (-\infty, \infty) \) with mean \( \mu \) and standard deviation \( \sigma \); see Exercise 7.

The c.d.f. is given in Appendix 28B.

In practice, few biologically meaningful random variables have distributions that

\[ \text{lognormal}. \]

That is, \( X \) is lognormal on \([0, \infty)\) if and only if \( \ln(X) \) is normal on \((-\infty, \infty)\).

In such cases, the implied distribution of \( X \) is \( \ln(X) = \zeta \), with sample space \((-\infty, \infty)\) and on some subset thereof.

Nevertheless, whenever \( X \) has sample space \([0, \infty)\), the implied distribution over \([0, \infty)\) of \( \ln(X) \) is normal over \((-\infty, \infty)\).

That is, \( X \) is lognormal if and only if \( \ln(X) \) is normal.

\[ \mu \]

\[ \sigma \]

Reference

Exercises 28

28.1 Use a graphical argument to establish that (13) implies (14).

28.2 Find both the variance and coefficient of variation of the distribution defined in Exercise 26.5. Note that this distribution is symmetric on [0, 2].

28.3* Find both the variance and coefficient of variation of the distribution defined in Exercise 26.8. Note that this distribution is symmetric on [0, 2].

28.4 For any positive constant b, a probability distribution is defined on [0, b] by

\[ f(x) = \begin{cases} \frac{4}{b^3 - x^3} & \text{if } 0 \leq x \leq b/2 \\ \frac{4}{b^3 - x^3} & \text{if } b/2 \leq x \leq b \end{cases} \]

(i) Find the mean, \( \mu \)
(ii) Find the variance, \( \sigma^2 \)
(iii) Deduce that the coefficient of variation is \( \kappa = 1/6 \) (regardless of the value of b).

28.5 Find the variance of the uniform distribution defined by (5). Hence verify that the coefficient of variation is

\[ \kappa = \frac{1}{3(b - a)} \]
Appendix 28A: The mean of a symmetric distribution is also the median

By the definition of mean, we have

\[ \mu = \int_a^b x f(x) \, dx \]

and

\[ \mu = M \]

where

\[ M = \frac{1}{b-a} \int_a^b x \, dx \]

So, because \( \int_a^b f(x) \, dx = 1 \),

\[ \mu = M \]

is equivalent to \( I = 0 \) where

\[ I = \int_a^b \frac{1}{b-a} (x - M) f(x) \, dx \]

In the last of these integrals, make the substitution \( u = \phi(x) = x - \frac{b-a}{2} \) with inverse \( x = \zeta(u) = u + \frac{b-a}{2} = M + u - a \). Because \( \phi(M) = M - \frac{b-a}{2} = a \),

\[ \phi(b) = b - \frac{b-a}{2} = M \]

and \( \phi(q) = q \)

In the last of these integrals, make the substitution \( u = \phi(x) = \frac{1}{b-a}(x - M) \) with inverse \( x = \zeta(u) = u + \frac{b-a}{2} = M + u - a \).

\[ I = \int_a^b \frac{1}{b-a} (x - M) f(x) \, dx \]

By the definition of mean, we have

**Theorem 28A.** The mean of a symmetric distribution is also the median.
In general, it suffices to know the c.d.f. of a symmetric distribution on $[\mu, \nu]$. For $\mu < x < \nu$,

We have

The "error function" erf defined on $[0, \infty)$ by

where $2 \pi - x \in [\mu, \nu]$, on which $F$ is known. Hence assume that $x \geq \mu$. Now, from (31),

the c.d.f. for a normal distribution is given by

We now make the substitution $z = \frac{t - \mu}{\sigma}$, with inverse $t = \mu + \sigma z$,

so that $2 \pi / (\pi - x) = (z) \phi$ and $\phi(\mu) = 0$. Also, $\phi(z) = \frac{\phi(z)}{\phi(\mu)}$ with inverse $z = \frac{t - \mu}{\sigma}$.

Thus, from (2B2), (B5) and (26.17), the c.d.f. of a normal distribution is given by

The graph of erf is sketched in Figure 2.
In this appendix, we illustrate the usefulness of partial integrals by calculating the mean and variance of the exponential distribution.

\( f(x) = \frac{1}{s} e^{-x/s} \) (28.C1) 

and \( \int_0^\infty f(x) \, dx = 1 \) implies 

\[ e^{-x/s} \int_0^\infty \, dx = s. \] (28.C2) 

From (26.12) and (C1), the mean is defined by 

\[ \mu = \int_0^\infty x \frac{1}{s} e^{-x/s} \, dx. \] (28.C3) 

This integral is easiest to evaluate if we regard \( e^{-x/s} \), not as an ordinary function of \( x \) with parameter \( s \), but instead as a bivariate function of \( x \) and \( s \), say, 

\[ P(x,s) = e^{-x/s}. \] (28.C4) 

Then, holding \( x \) constant, we have 

\[ \frac{\partial}{\partial s} e^{\Omega(x,s)} = \frac{\partial}{\partial s} e^{\Omega(x,s)} \] (28.C5) 

So, from (C2) and (C11), 

\[ s \cdot \frac{\partial}{\partial s} e^{\Omega(x,s)} = \frac{\partial}{\partial s} e^{\Omega(x,s)} \] (28.C6) 

Thus, from (C4) and (C6)-(C8) we have 

\[ \frac{\partial}{\partial s} e^{\Omega(x,s)} = e^{\Omega(x,s)} \] (28.C7) 

\[ \frac{s}{x} = (e^{-s})x - \left( e^{-s} \right) \frac{sp}{p} x = \left( e^{-s} \right) \frac{sp}{p} x - \left( e^{-s} \right) \frac{sp}{p} x = \frac{sp}{p} \] (28.C8) 

So, from (C2) and (C11), 

\[ \frac{\partial}{\partial s} e^{\Omega(x,s)} = \frac{\partial}{\partial s} e^{\Omega(x,s)} \] (28.C9) 

Thus, from Exercise 20.2, if \( \Omega \) is an ordinary function of \( s \) then 

\[ \frac{d}{ds} e^{\Omega(s)} = \frac{d}{ds} e^{\Omega(s)} \] (28.C10) 

Set 

\[ \frac{d}{ds} e^{\Omega(x,s)} = \frac{d}{ds} e^{\Omega(x,s)} \] (28.C11)
\[ (28.\text{C}1) \quad \frac{s}{s} = \frac{\eta}{\sigma} = x \]

and

\[ s = z - s - z = \varphi \]

Thus \( \varphi = \sigma + s + \varphi \)

because \( s \) is held constant when integrating with respect to \( x \).

Now, from \((C12)-(C13)\) and \((C19)\), we obtain

\[ (28.\text{C}19) \quad \frac{s}{s} = \int \left[ \frac{\eta}{\sigma} \right] \frac{s}{s} \]
Because \( \mu = \frac{b}{2} \). So \( \sigma = \frac{b}{24} = \frac{b}{2\pi^2} \) and

\[ \mu = \frac{\mu}{\mu} = \frac{\mu}{\mu} \]

This distribution is symmetric about the midpoint in the interval \( [0, b] \).