28. Symmetric distributions

Although previously we have always taken the sample space of a continuous random variable X to be $[0, \infty)$, we have also frequently set $f(x) = 0$ for $x > b$, so that the effective sample space is $[0, b]$. If also $f(x) = 0$ for $x < a$, then the effective sample space becomes $[a, b]$. Accordingly, we now consider distributions on $[a, b]$ instead of on $[0, \infty)$. Our analysis will be more general, because it allows for the possibility that $a < 0$.

The function f is the p.d.f. of a random variable distributed on [a, b] if f is nonnegative and total probability is 1, i.e.

$$f(x) \geq 0, \quad a \leq x \leq b$$
$$\text{Area}(f,[a,b]) = \int_a^b f(x)\,dx = 1 \qquad (28.1a)$$

Correspondingly, F is the c.d.f. of a random variable distributed on [a, b] if

$$F(a) = 0$$
$$F'(x) \geq 0, \quad a \leq x < b \qquad (28.1b)$$
$$F(b) = 1,$$

where f and F are related by the obvious generalization of (19.9), namely,

$$f(x) = F'(x) \quad \Leftrightarrow \quad F(x) = \int_a^x f(t)\,dt. \qquad (28.2)$$

An important special case occurs when the distribution is **symmetric** about the mid-point

$$M = \tfrac{1}{2}(a+b) \qquad (28.3)$$

of the sample space, i.e., when f(x) depends only on $|x - M|$ or, which is the same thing,

$$f(M-t) = f(M+t) \qquad (28.4)$$

for any $t \in [0, (b-a)/2]$. Three such distributions are depicted in Figure 1. The top panel shows the p.d.f. of a **uniform distribution** on [a, b], for which

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b. \qquad (28.5)$$

The middle panel shows the p.d.f. of a (symmetric) **triangular distribution** on [a, b], for which

$$f(x) = \frac{2}{b-a}\left\{1 - \frac{2|x-M|}{b-a}\right\}, \quad a \leq x \leq b. \qquad (28.6)$$

The bottom panel shows the p.d.f. of a bell-shaped distribution on [a, b] with

$$f(x) = \frac{1}{L}e^{-\lambda|x-M|^2}, \quad a \leq x \leq b, \qquad (28.7)$$

where λ is a positive constant, and

$$L = \int_a^b e^{-\lambda|x-M|^2}\,dx \qquad (28.8)$$

is another positive constant that guarantees Int(f, [a, b]) = 1.

The symmetry of such distributions can often be exploited in calculations. For example, because the lighter and darker shaded areas in Figure 1 are equal, i.e.,

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx, \tag{28.9}$$

we have $F(M) = 1 - F(M)$ or $F(M) = 1/2$, implying that the mid-point is also the median, as (3) presumes. Again, because of the symmetry, weight (= probability) is balanced about the mid-point, so that

$$\mu = \frac{a+b}{2} \tag{28.10}$$

as well (see Appendix 28A for a formal proof). Thus symmetry conditions (4) and (9) become

$$f(\mu - t) = f(\mu + t), \qquad 0 \le t \le \tfrac{1}{2}(b-a) \tag{28.11}$$

and

$$\int_a^\mu f(x)\,dx = \int_\mu^b f(x)\,dx. \tag{28.12}$$

Because f is a p.d.f., both of these integrals equal 1/2. More generally, however, any function that satisfies (11) must also satisfy (12), even if it is not a p.d.f. That is, for any function D on [a, b] satisfying

$$D(\mu - t) = D(\mu + t), \qquad 0 \le t \le \tfrac{1}{2}(b-a) \tag{28.13}$$

it must also be true that

$$\int_a^\mu D(x)\,dx = \int_\mu^b D(x)\,dx; \tag{28.14}$$

see Exercise 1. One such function is the dispersion density of a symmetric distribution, i.e., D defined by

$$D(x) = (x-\mu)^2 f(x) \tag{28.15}$$

for f satisfying (11). From (15) and (11) we have

$$D(\mu - t) = (\mu - t - \mu)^2 f(\mu - t) = (-t)^2 f(\mu - t).$$
$$= t^2 f(\mu - t) = t^2 f(\mu + t) = (\mu + t - \mu)^2 f(\mu + t) = D(\mu + t), \tag{28.16}$$

i.e., (13) is satisfied. So Int(D, [a, μ]) = Int(D, [μ, b]) by (14), and (27.2) implies

$$\sigma^2 = \int_a^b D(x)\,dx = \int_a^\mu D(x)\,dx + \int_\mu^b D(x)\,dx = 2\int_a^\mu D(x)\,dx = 2\int_a^\mu (x-\mu)^2 f(x)\,dx \tag{28.17a}$$

$$= 2\int_\mu^b D(x)\,dx = 2\int_\mu^b (x-\mu)^2 f(x)\,dx \tag{28.17b}$$

For a symmetric distribution, variance is often more easily calculated from either (17a) or (17b) than from (27.5).

We illustrate by calculating the variance of the triangular distribution defined by (6). For $x \ge \mu$ we have $|x - \mu| = x - \mu$, so that (6) and (10) imply

$$f(x) = \frac{2}{b-a}\left\{1 - \frac{2(x-\mu)}{b-a}\right\}, \quad \mu \le x \le b. \tag{28.18}$$

Now, from (17)-(18), we have

$$\begin{aligned}\sigma^2 &= 2\int_{\mu}^{b}(x-\mu)^2 f(x)\,dx \\ &= \frac{4}{b-a}\int_{\mu}^{b}(x-\mu)^2\left\{1-\frac{2(x-\mu)}{b-a}\right\}dx \\ &= \frac{4}{b-a}\int_{\mu}^{b}(x-\mu)^2\,dx - \frac{8}{(b-a)^2}\int_{\mu}^{b}(x-\mu)^3\,dx \\ &= \frac{4}{b-a}\frac{(x-\mu)^3}{3}\bigg|_{\mu}^{b} - \frac{8}{(b-a)^2}\frac{(x-\mu)^4}{4}\bigg|_{\mu}^{b} \\ &= \frac{4}{b-a}\frac{(b-\mu)^3}{3} - \frac{8}{(b-a)^2}\frac{(b-\mu)^4}{4} \\ &= \frac{1}{24}(b-a)^2,\end{aligned} \tag{28.19}$$

on using (10). So the triangular distribution has coefficient of variation

$$\kappa = \frac{1}{\sqrt{6}}\frac{b-a}{b+a}. \tag{28.20}$$

One of the most important distributions in applied mathematics, the **Gaussian** or **normal distribution**, can be obtained from an appropriate limit of the bell-shaped distribution in Figure 1. First we set $a = -K$ and $b = K$, so that (10) implies $\mu = 0$. Then (7)-(8) and (12) imply

$$f(x) = \frac{1}{L}e^{-\lambda x^2}, \quad -K \le x \le K \tag{28.21}$$

where

$$L = \int_{-K}^{K} e^{-\lambda x^2}dx = 2\int_{0}^{K} e^{-\lambda x^2}dx. \tag{28.22}$$

Taking the limit as $K \to \infty$, so that the sample space becomes $(-\infty, \infty)$, we have

$$L = 2\int_{0}^{\infty} e^{-\lambda x^2}dx. \tag{28.23}$$

We now make the substitution

$$u = \phi(x) = \lambda x^2. \tag{28.24}$$

Because $u = \lambda x^2$ implies $x = \sqrt{u/\lambda}$, the inverse substitution is defined by

$$x = \zeta(u) = \sqrt{\frac{u}{\lambda}} = \lambda^{-1/2}u^{1/2}, \tag{28.25}$$

implying

$$\zeta'(u) = \frac{d}{du}\left\{\lambda^{-1/2}u^{1/2}\right\} = \lambda^{-1/2}\frac{d}{du}\left\{u^{1/2}\right\} = \lambda^{-1/2}\frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{\lambda}}u^{-1/2}. \tag{28.26}$$

Then, because $\lambda > 0$ implies $\phi(\infty) = \infty$, (21.21) or (22.A1) or (26.17) now reduces (22) to

$$\begin{aligned} L &= 2\int_{\phi(0)}^{\phi(\infty)} e^{-\lambda\{\zeta(u)\}^2}\zeta'(u)\,du = 2\int_0^\infty e^{-u}\frac{u^{-1/2}}{2\sqrt{\lambda}}\,du \\ &= \frac{1}{\sqrt{\lambda}}\int_0^\infty u^{-1/2}e^{-u}\,du \qquad (28.27) \\ &= \frac{1}{\sqrt{\lambda}}\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\frac{\pi}{\lambda}} \end{aligned}$$

from the definition of Γ, and on using (26.31). Thus, letting $K \to \infty$ in (21), we have

$$f(x) = \sqrt{\frac{\lambda}{\pi}}\,e^{-\lambda x^2}, \quad -\infty < x < \infty. \qquad (28.28)$$

From (17), with $\mu = 0$ and letting $b \to \infty$, the variance is

$$\begin{aligned} \sigma^2 &= 2\sqrt{\frac{\lambda}{\pi}}\int_0^\infty x^2e^{-\lambda x^2}dx = 2\sqrt{\frac{\lambda}{\pi}}\int_{\phi(0)}^{\phi(\infty)}\{\zeta(u)\}^2e^{-\lambda\{\zeta(u)\}^2}\zeta'(u)\,du \\ &= 2\sqrt{\frac{\lambda}{\pi}}\int_0^\infty \frac{u}{\lambda}e^{-u}\frac{u^{-1/2}}{2\sqrt{\lambda}}\,du = \frac{1}{\lambda\sqrt{\pi}}\int_0^\infty u^{1/2}e^{-u}\,du \qquad (28.29) \\ &= \frac{1}{\lambda\sqrt{\pi}}\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{\lambda\sqrt{\pi}}\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{\lambda\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} = \frac{1}{2\lambda}, \end{aligned}$$

on using the Gamma function's recursive property (27.12) and the same substitution as before. Thus $\lambda = 1/2\sigma^2$, so that (28) becomes

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}(x/\sigma)^2}, \quad -\infty < x < \infty. \qquad (28.30)$$

This is the p.d.f. of a normal distribution with mean zero and standard deviation σ.

Finally, to obtain the p.d.f. of a normal distribution with arbitrary mean μ, all we have to do is replace x in (30) by $x - \mu$:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty. \qquad (28.31)$$

Integration by substitution readily establishes that this the p.d.f. of a symmetric, bell-shaped distribution on (–∞, ∞) with mean μ and standard deviation σ; see Exercise 7. The c.d.f. is given in Appendix 28B.

In practice, few biologically meaningful random variables have distributions over (–∞, ∞); most, as we have seen, are distributed over [0, ∞), or some subset thereof. Nevertheless, whenever X has sample space [0, ∞), U = ln(X) has sample space (–∞, ∞), so that U can have a normal distribution without violating the constraint that $X \geq 0$. In such cases, the implied distribution of $X = e^U$ on [0, ∞) is said to be **lognormal**. That is, X is lognormal on [0, ∞) if and only if ln(X) is normal on (–∞, ∞).

Reference

Troy, J.B. & J.G.Robson (1992). Steady discharges of X and Y retinal ganglion cells of cat under photopic illuminance. Visual Neuroscience **9**, 535-553.

Exercises 28

28.1 Use a graphical argument to establish that (13) implies (14).

28.2 Find both the variance and coefficient of variation of the distribution defined in Exercise 26.5. Note that this distribution is symmetric on [0, 2].

28.3* Find both the variance and coefficient of variation of the distribution defined in Exercise 26.8. Note that this distribution is symmetric on [0, 2].

28.4 For any positive constant b, a probability distribution is defined on [0, b] by

$$f(x) = \begin{cases} 4x/b^2 & \text{if } 0 \le x \le b/2 \\ 4(b-x)/b^2 & \text{if } b/2 \le x \le b \end{cases}$$

(i) Find the mean, μ

(ii) Find the variance, σ^2

(iii) Deduce that the coefficient of variation is $\kappa = 1/\sqrt{6}$ (regardless of the value of b).

28.5 Find the variance of the uniform distribution defined by (5). Hence verify that the coefficient of variation is

$$\kappa = \frac{1}{\sqrt{3}}\frac{b-a}{b+a}.$$

28.6 Show that the function erf defined by (B6) is the c.d.f. of a probability distribution on $[0, \infty)$. Find its mean, variance and coefficient of variation.
Hint: You will need (26.31) to find the variance.

28.7 Use the substitution $z = x - \mu$ to establish that f defined by (31) is the p.d.f. of a symmetric distribution on $(-\infty, \infty)$ with mean μ and standard deviation σ.

Appendix 28A: The mean of a symmetric distribution is also the median

By the definition of mean, we have

$$\mu \;=\; \int_a^b x\,f(x)dx \;=\; \int_a^b (x-M)f(x)dx \;+\; M\int_a^b f(x)dx. \tag{28.A1}$$

So, because Int(f, [a, b]) = 1, $\mu = M$ is equivalent to I = 0 where

$$I \;=\; \int_a^b (x-M)f(x)dx \;=\; \int_a^M (x-M)f(x)dx \;+\; \int_M^b (x-M)f(x)dx. \tag{28.A2}$$

In the last of these integrals, make the substitution $u = \phi(x) = x - (b-a)/2$ with inverse $x = \zeta(u) = u + (b-a)/2 = M + u - a$. Because $\phi(M) = M - (b-a)/2 = a$, $\phi(b) = b - (b-a)/2 = M$ and $\zeta'(u) = 1$, (26.17) with $g(x) = (x-M)f(x)$ implies

$$\begin{aligned}\int_M^b (x-M)f(x)dx \;&=\; \int_{\phi(M)}^{\phi(b)} (\zeta(u)-M)f(\zeta(u))\zeta'(u)du\\ &=\; \int_a^M (u-a)f(M+u-a)\,du\\ &=\; \int_a^M (u-a)f(M-u+a)\,du\\ &=\; \int_a^M (x-a)f(M-x+a)\,dx,\end{aligned} \tag{28.A3}$$

on using (4). In this integral, make the fresh substitution $u = \phi(x) = M - x + a$, with inverse $x = \zeta(u) = M + a - u$ implying $\zeta'(u) = -1$. Then, because now $\phi(a) = M$ and $\phi(M) = a$, (26.17) with $g(x) = (x-a)f(M-x+a)$ yields

$$\begin{aligned}\int_M^b (x-M)f(x)dx \;&=\; \int_a^M (x-a)f(M-x+a)\,dx\\ &=\; \int_a^M (\zeta(u)-a)f(M-\zeta(u)+a)\zeta'(u)\,du\\ &=\; \int_M^a (M-u)f(u)\{-1\}du\\ &=\; \int_M^a (u-M)f(u)du\\ &=\; -\int_a^M (u-M)f(u)du\\ &=\; -\int_a^M (x-M)f(x)dx\end{aligned} \tag{28.A4}$$

by (22.4). It now follows from (A2) that I = 0.

Appendix 28B: The cumulative distribution function of the normal distribution

In general, it suffices to know the c.d.f. of a symmetric distribution on $[\mu, b]$. For if $x < \mu$ we have

$$\begin{aligned} F(x) &= \mathrm{Int}(f,[a,x]) = 1 - \mathrm{Int}(f,[x,b]) \\ &= 1 - \{\mathrm{Int}(f,[x,\mu]) + \mathrm{Int}(f,[\mu,b])\} \end{aligned} \tag{28.B1}$$

by (2) and (8.25). But symmetry implies that $\mathrm{Int}(f, [\mu, b]) = 1/2$ and

$$\mathrm{Int}(f,[x,\mu] = \mathrm{Int}(f,[\mu,2\mu - x], \tag{28.B2}$$

the two sides of this equation corresponding to equal areas above subdomains of length $\mu - x$. Thus, if $x < \mu$, then

$$\begin{aligned} F(x) &= 1 - \{\mathrm{Int}(f,[\mu,2\mu - x]) + 1/2\} \\ &= 1 - \{1/2 + \mathrm{Int}(f,[\mu,2\mu - x])\} \\ &= 1 - \{\mathrm{Int}(f,[a,\mu]) + \mathrm{Int}(f,[\mu,2\mu - x])\} \\ &= 1 - \mathrm{Int}(f,[a,2\mu - x]) \\ &= 1 - F(2\mu - x) \end{aligned} \tag{28.B3}$$

where $2\mu - x \in [\mu, b]$, on which F is known. Hence assume that $x \geq \mu$. Now, from (31), the c.d.f. for a normal distribution is given by

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt) = \int_{-\infty}^{\mu} f(t)dt + \int_{\mu}^{x} f(t)dt \\ &= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x} e^{-(t-\mu)^2/2\sigma^2} dt. \end{aligned} \tag{28.B4}$$

We now make the substitution $z = \phi(t) = (t - \mu)/\sigma\sqrt{2}$, with inverse $t = \zeta(z) = \mu + \sigma\sqrt{2}\,t$, so that $\zeta'(z) = \sigma\sqrt{2}$. Also, $\phi(\mu) = 0$, and $\phi(x) = (x - \mu)/\sigma\sqrt{2}$. Thus, by (B4) and (26.17),

$$\begin{aligned} F(x) &= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x} e^{-(t-\mu)^2/2\sigma^2} dt \\ &= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{(x-\mu)/\sigma\sqrt{2}} e^{-z^2} \zeta'(z) dz \\ &= \frac{1}{2}\left\{1 + \frac{2}{\sqrt{\pi}} \int_{0}^{(x-\mu)/\sigma\sqrt{2}} e^{-z^2} dz\right\}. \end{aligned} \tag{28.B5}$$

The "error function" erf defined on $[0, \infty)$ by

$$\mathrm{erf}(\theta) = \frac{2}{\sqrt{\pi}} \int_{0}^{\theta} e^{-z^2} dz \tag{28.B6}$$

is one of the "known" functions of applied mathematics, like exp, ln or the Gamma function. Thus, from (B2), (B5) and (B6), the c.d.f. of a normal distribution is given by

$$F(x) = \begin{cases} \dfrac{1}{2}\left\{1 + \mathrm{erf}\left(\dfrac{x-\mu}{\sigma\sqrt{2}}\right)\right\} & \text{if } x \geq \mu \\ \dfrac{1}{2}\left\{1 - \mathrm{erf}\left(\dfrac{\mu - x}{\sigma\sqrt{2}}\right)\right\} & \text{if } x < \mu \end{cases} \tag{28.B7}$$

The graph of erf is sketched in Figure 2.

Appendix 28C: The mean and variance of the exponential distribution

In this appendix, we illustrate the usefulness of partial integrals by calculating the mean and variance of the exponential distribution. From (26.29),

$$f(x) = \frac{1}{s}e^{-x/s} \tag{28.C1}$$

and Int(f, [0, ∞)) = 1 implies

$$\int_0^\infty e^{-x/s}\,dx = s. \tag{28.C2}$$

From (26.12) and (C1), the mean is defined by

$$\mu = \int_0^\infty \frac{x}{s}e^{-x/s}\,dx. \tag{28.C3}$$

This integral is easiest to evaluate if we regard $e^{-x/s}$, not as an ordinary function of x with parameter s, but instead as a bivariate function of x and s, say,

$$P(x,s) = e^{-x/s} \tag{28.C4}$$

(where $s > 0$). Now, from Exercise 20.2, if Ω is an ordinary function of s then

$$\frac{d}{ds}\left\{e^{\Omega(s)}\right\} = \frac{d\Omega}{ds}e^{\Omega(s)}. \tag{28.C5}$$

So, if instead Ω is a bivariate function of x and s, because partial differentiation with respect to s is equivalent to ordinary differentiation with x held constant, we instead have

$$\frac{\partial}{\partial s}\left\{e^{\Omega(x,s)}\right\} = \frac{\partial\Omega}{\partial s}e^{\Omega(x,s)}. \tag{28.C6}$$

Set

$$\Omega(x,s) = -\frac{x}{s}. \tag{28.C7}$$

Then, holding x constant, we have

$$\frac{\partial\Omega}{\partial s} = \frac{\partial}{\partial s}\left(-\frac{x}{s}\right) = -x\frac{d}{ds}\left(\frac{1}{s}\right) = -x\frac{d}{ds}\left(s^{-1}\right) = -x\left(-s^{-2}\right) = \frac{x}{s^2}. \tag{28.C8}$$

Thus, from (C4) and (C6)-(C8), we have

$$\frac{\partial}{\partial s}\{P(x,s)\} = \frac{\partial}{\partial s}\left\{e^{\Omega(x,s)}\right\} = \frac{\partial\Omega}{\partial s}e^{\Omega(x,s)} = \frac{x}{s^2}e^{-x/s} \tag{28.C9}$$

We can now evaluate μ very easily, with the help of Lecture 25. From (C3) and (C9),

$$\mu = \int_0^\infty \frac{x}{s}e^{-x/s}\,dx = s\int_0^\infty \frac{x}{s^2}e^{-x/s}\,dx = s\int_0^\infty \frac{\partial}{\partial s}\{P(x,s)\}\,dx. \tag{28.C10}$$

But Lecture 25 implies that

$$\int_0^\infty \frac{\partial}{\partial s}\{P(x,s)\}\,dx = \frac{d}{ds}\left\{\int_0^\infty P(x,s)\,dx\right\}dx. \tag{28.C11}$$

So, from (C2) and (C11),

$$\begin{aligned}\mu \;&=\; s\,\frac{d}{ds}\left\{\int_0^\infty P(x,s)\,dx\right\} \;=\; s\,\frac{d}{ds}\left\{\int_0^\infty e^{-x/s}\,dx\right\} \\ &=\; s\,\frac{d}{ds}\{s\} \;=\; s\cdot 1 \;=\; s\end{aligned} \tag{28.C12}$$

(agreeing with (26.27) when c = 1).

From (27.5) and (C1),

$$\sigma^2+\mu^2 \;=\; \int_0^\infty x^2 f(x)\,dx \;=\; \frac{1}{s}\int_0^\infty x^2 e^{-x/s}\,dx. \tag{28.C13}$$

From (C.3) and (C.12),

$$s \;=\; \int_0^\infty \frac{x}{s}e^{-x/s}\,dx, \tag{28.C14}$$

implying

$$\int_0^\infty x e^{-x/s}\,dx \;=\; s^2. \tag{28.C15}$$

So, from (C.11) with $P(x,s) = xe^{-x/s}$ in place of (C4), we have

$$\int_0^\infty \frac{\partial}{\partial s}\left(xe^{-x/s}\right)dx \;=\; \frac{d}{ds}\left\{\int_0^\infty xe^{-x/s}\,dx\right\} \;=\; \frac{d}{ds}\left\{s^2\right\} \;=\; 2s, \tag{28.C16}$$

on using (C15). But from (C9) we have

$$\frac{\partial}{\partial s}\left\{e^{-x/s}\right\} \;=\; \frac{x}{s^2}e^{-x/s}, \tag{28.C17}$$

so that (because x is held constant when differentiating with respect to s)

$$\frac{\partial}{\partial s}\left\{xe^{-x/s}\right\} \;=\; x\frac{\partial}{\partial s}\left\{e^{-x/s}\right\} \;=\; \frac{x^2}{s^2}e^{-x/s}. \tag{28.C18}$$

Thus (C16) implies

$$\int_0^\infty \frac{x^2}{s^2}e^{-x/s}\,dx \;=\; 2s. \tag{28.C19}$$

Now, from (C12)-(C13) and (C19), we obtain

$$\sigma^2+s^2 \;=\; \frac{1}{s}\int_0^\infty x^2 e^{-x/s}\,dx \;=\; s\int_0^\infty \frac{x^2}{s^2}e^{-x/s}\,dx \;=\; 2s^2 \tag{28.C20}$$

(because s is held constant when integrating with respect to x). Thus $\sigma^2 = 2s^2 - s^2 = s^2$ and

$$\kappa \;=\; \frac{\sigma}{\mu} \;=\; \frac{s}{s} \;=\; 1. \tag{28.C21}$$

Answers and Hints for Selected Exercises

28.2 $\sigma^2 = \frac{1}{5}$, $\kappa = \frac{1}{\sqrt{5}}$.

28.3 $\sigma^2 = \frac{1}{7}$, $\kappa = \frac{1}{\sqrt{7}}$.

28.4 This distribution is symmetric about the midpoint $m = b/2$ of the interval $[0, b]$ because for any t such that $0 \le t \le b/2$ we have $f(m + t) = 4(b - \{m + t\})/b^2 = 4(b - m - t)/b^2 = 4(b - b/2 - t)/b^2 = 4(b/2 - t)/b^2 = 4(m - t)/b^2 = f(m-t)$. So

(i) $\mu = m = b/2$

(ii)

$$\begin{aligned}
\sigma^2 &= 2\int_0^\mu (t-\mu)^2 f(t)\,dt = 2\int_0^\mu (t-\mu)^2 \frac{4t}{b^2}\,dt = \frac{8}{b^2}\int_0^\mu (t-\mu)^2 t\,dt \\
&= \frac{8}{b^2}\int_0^\mu (t^2 - 2\mu t + \mu^2)\cdot t\,dt = \frac{8}{b^2}\int_0^\mu (t^3 - 2\mu t^2 + \mu^2 t)\,dt \\
&= \frac{8}{b^2}\int_0^\mu \frac{d}{dt}\left\{\frac{t^4}{4} - 2\mu\frac{t^3}{3} + \mu^2\frac{t^2}{2}\right\}dt = \frac{8}{b^2}\left(\frac{t^4}{4} - 2\mu\frac{t^3}{3} + \mu^2\frac{t^2}{2}\right)\Bigg|_0^\mu \\
&= \frac{8}{b^2}\left(\frac{\mu^4}{4} - 2\mu\frac{\mu^3}{3} + \mu^2\frac{\mu^2}{2}\right) = \frac{8}{b^2}\cdot\frac{\mu^4}{12} = \frac{1}{24}b^2
\end{aligned}$$

(because $\mu = b/2$). So $\sigma = b/\sqrt{24} = b/(2\sqrt{6})$ and

(iii) $\kappa = \sigma/\mu = 1/\sqrt{6}$.