30 Trigonometric function properties

In Lecture 23, we studied properties of the trigonometric functions C and S, or cos and sin, from a graphical perspective. In this lecture, we redefine them geometrically to study their properties further.

In calculus, it is usual to measure angles in terms of radians, as opposed to degrees. A radian is the unit of arc along a circle of radius 1, or unit circle for short. Because 2π is this circle's circumference, 360 degrees correspond to 2π. So a degree corresponds to $\pi/180$ radians, and a radian to $180/\pi$ degrees. Figure 1(a) shows an arc of a unit circle. The triangle OMP is right-angled. So

$$\frac{MO}{OP} = \cos(\angle MOP), \qquad \frac{MP}{OP} = \sin(\angle MOP). \tag{30.1}$$

But OP = 1, because OP is a radius, and

$$\begin{aligned}\angle MOP = \angle LOP &= \text{arc}(PL)\\ &= x,\end{aligned} \tag{30.2}$$

say, where x is measured in radians. Thus

$$MO = \cos(x), \qquad MP = \sin(x). \tag{30.3}$$

These equations define a pair of functions C and S according to

$$C(x) = \cos(x), \qquad S(x) = \sin(x). \tag{30.4}$$

For the sake of simplicity, we assume for the time being that C and S have common domain $[0, \pi/2]$, i.e., P may vary between L and U, or along a quarter of the circle in Figure 1(a). Then the graphs of C and S are as shown in Figure 2. Of course, C and S are the very same functions as those we met in Lecture 23.

On $[0, \pi/2]$, C is decreasing and S is increasing, so both functions are invertible. Because x is an arc of circumference, possible other names for C and S are cosarc and sinarc, respectively, suggesting arccos and arcsin as names for their inverses. In practice, cosarc and sinarc are never used (although they would have the advantage of making absolutely clear that angles are measured in radians, not degrees); however, arccos and arcsin are the standard names for their inverses. So

$$y = \cos(x),\ 0 \le x \le \tfrac{\pi}{2} \quad \Leftrightarrow \quad x = \arccos(y),\ 0 \le y \le 1 \tag{30.5}$$

and

$$y = \sin(x),\ 0 \le x \le \tfrac{\pi}{2} \quad \Leftrightarrow \quad x = \arcsin(y),\ 0 \le y \le 1. \tag{30.6}$$

The inverse functions arccos and arcsin are graphed in Figure 2 below the graphs of cos and sin. Note that both cos and sin are concave down on $[0, \pi/2]$, and that arccos is concave down on [0, 1]. In contrast, arcsin is concave up on [0, 1].

All four functions are perfectly smooth. To calculate the derivatives of cos and sin, we may proceed as follows. In Figure 1(c), arc(LP) = x and arc(PQ) = h, implying arc(LQ) = x + h; moreover, the triangle ONQ is right-angled. Thus

$$\frac{ON}{OQ} = \cos(\angle NOQ), \qquad \frac{QN}{OQ} = \sin(\angle NOQ). \tag{30.7}$$

But OQ = 1 because OQ is a radius. Thus

$$\angle NOQ = \angle LOQ = x + h, \tag{30.8}$$

implying

$$NO = \cos(x+h), \qquad QN = \sin(x+h). \tag{30.9}$$

So

$$\begin{aligned} QR &= QN - RN \\ &= QN - MP \\ &= \sin(x+h) - \sin(x) \\ &= \mathrm{Diff}(S,[x,x+h]) \end{aligned} \tag{30.10}$$

and

$$\begin{aligned} PR &= MN \\ &= OM - ON \\ &= \cos(x) - \cos(x+h) \\ &= -\mathrm{Diff}(C,[x,x+h]). \end{aligned} \tag{30.11}$$

But h = arc(PQ). So

$$\begin{aligned} DQ(S,[x,x+h]) &= \frac{QR}{\mathrm{arc}(PQ)} = \frac{QR}{PQ}\frac{PQ}{\mathrm{arc}(PQ)} \\ &= \cos(\angle RQP)\frac{PQ}{\mathrm{arc}(PQ)} \end{aligned} \tag{30.12a}$$

and

$$\begin{aligned} DQ(C,[x,x+h]) &= -\frac{PR}{\mathrm{arc}(PQ)} = -\frac{PR}{PQ}\frac{PQ}{\mathrm{arc}(PQ)}. \\ &= -\sin(\angle RQP)\frac{PQ}{\mathrm{arc}(PQ)}. \end{aligned} \tag{30.12b}$$

Now look at Figure 1(b). Here

$$\angle NOP + \angle ONQ = \angle PQN + \angle QPO, \tag{30.13}$$

or

$$x + \frac{\pi}{2} = \angle PQR + \angle QPO. \tag{30.14}$$

As h approaches 0, Q approaches P, and so $\angle OPQ$ becomes a right angle, i.e.,

$$\lim_{h\to 0}\angle OPQ = \frac{\pi}{2}. \tag{30.15}$$

From (14), therefore, $\angle RQP$ approaches x as h approaches zero, implying that ($\cos(\angle RQP)$ approaches cos(x) and $\sin(\angle RQP)$ approaches sin(x). From Figure 1(d), however, we see that arc(PQ) becomes coincident with PQ as h approaches zero, so that PQ/arc(PQ) approaches 1 in (12). So (12a) implies that

$$S'(x) = \lim_{h\to 0} DQ(S,[x,x+h]) = \cos(x), \tag{30.16a}$$

whereas (12b) implies

$$C'(x) = \lim_{h\to 0} DQ(C,[x,x+h]) = -\sin(x). \tag{30.16a}$$

In other words,

$$\frac{d}{dx}\{\cos(x)\} = -\sin(x), \qquad \frac{d}{dx}\{\sin(x)\} = \cos(x), \tag{30.17}$$

confirming the results we obtained in Lecture 23; and the fundamental theorem at once yields the first two rows of Table 1.

Restrictions	DERIVATIVE on $[a,b]$, $b>a$	ANTIDERIVATIVE on $[a,b]$, $b>a$
$0 \le a < b \le \frac{\pi}{2}$	$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$	$\int_a^x \sin(t)\,dt = -\cos(x) + \cos(a)$
$0 \le a < b \le \frac{\pi}{2}$	$\frac{d}{dx}\{\sin(x)\} = \cos(x)$	$\int_a^x \cos(t)\,dt = \sin(x) - \sin(a)$
$0 \le a < b < 1$	$\frac{d}{dx}\{\arccos(x)\} = -\frac{1}{\sqrt{1-x^2}}$	$\int_a^x \frac{1}{\sqrt{1-t^2}}\,dt = -\arccos(x) + \arccos(a)$
$0 \le a < b < 1$	$\frac{d}{dx}\{\arcsin(x)\} = \frac{1}{\sqrt{1-x^2}}$	$\int_a^x \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin(x) - \arcsin(a)$

Table 30.1 Some derivatives and integrals of trigonometric functions

Once we know the derivatives of sin and cos, the derivatives of their inverses arcsin and arccos follow almost immediately from (20.39), i.e., from

$$\frac{dx}{dy} = \left\{\frac{dy}{dx}\right\}^{-1}. \tag{30.18}$$

With $y = \sin(x)$ or $x = \arcsin(y)$, (18) implies

$$\frac{d}{dy}\{\arcsin(y)\} = \left\{\frac{d}{dx}\{\sin(x)\}\right\}^{-1} = \frac{1}{\cos(x)} = \frac{1}{\cos(\arcsin(y))}. \tag{30.19}$$

With $y = \cos(x)$ or $x = \arccos(y)$, (18) implies

$$\frac{d}{dy}\{\arccos(y)\} = \left\{\frac{d}{dx}\{\cos(x)\}\right\}^{-1} = \frac{1}{\sin(x)} = -\frac{1}{\sin(\arccos(y))}. \tag{30.20}$$

From Pythagoras' theorem (applied to triangle OPM in Figure 1) or (23.29), however,

$$\cos^2(x) + \sin^2(x) = 1. \tag{30.21}$$

If $y = \sin(x)$ then (21) implies $\cos(x) = \sqrt{1-y^2}$, whereas if $y = \cos(x)$ then (21) implies $\sin(x) = \sqrt{1-y^2}$. Thus (20) and (19) yield

$$\frac{d}{dy}\{\arccos(y)\} = -\frac{1}{\sqrt{1-y^2}}, \qquad \frac{d}{dy}\{\arcsin(y)\} = \frac{1}{\sqrt{1-y^2}} \tag{30.22}$$

on [0, 1], and the fundamental theorem now yields the third and fourth rows of Table 1. Note that the domain of the derivatives of arccos and arcsin cannot be extended from [0, 1) to [0, 1] because the right-hand sides of (20) become infinite as $y \to 1$. Note also that rows three and four of Table 1 imply

$$\arccos(x) + \arcsin(x) = \arccos(a) + \arcsin(a) \tag{30.23}$$

for any value of x or a, which is just another way of saying that arccos(x) + arcsin(x) is a constant. The value of this constant is $\arccos(0) + \arcsin(0) = \pi/2 + 0 = \pi/2$. Thus

$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}. \tag{30.24}$$

You should check that this result is consistent with (22). Its effect is illustrated in Figure 2 by the vertical dots and dashes.

Arcs of circumference are counted as positive if measured anticlockwise but negative if measured anticlockwise. So the domain of cos or sin is readily extended to include $[-\pi/2, 0)$ by allowing P in Figure 1(a) to slide clockwise below L; see Figure 3(a), where P can vary between W and L. We still have OM = cos(x) and MP = sin(x), as in (3), but MP counts as negative now because it falls below the horizontal axis. Thus $\sin(x) < 0$ when $-\pi/2 \le x < 0$. On the other hand, OM still counts as positive, because it still falls to the right of the vertical axis, and so $\cos(x) > 0$ when $-\pi/2 < x \le 0$. See Figure 3(b), where the graph of sin on $[-\pi/2, \pi/2]$ is shown solid, and that of cos shown dashed. Note that sin is invertible, i.e., $y = \sin(x) \Leftrightarrow x = \arcsin(y)$, whereas cos is not invertible. Figure 3(b) extends the domain of arcsin from [0, 1] to [-1,1].

Similarly, the domain of cos or sin is extended to include $(\pi/2, \pi]$ by allowing P in Figure 1(a) to slide anticlockwise past U; see Figure 3(c), where P can vary between U and V. As before, OM = cos(x) and MP = sin(x), but OM counts as negative whereas MP counts positive. Thus $\sin(x) \ge 0$, $\cos(x) \le 0$ when $\pi/2 < x \le \pi/2$. See Figure 3(d), where the graph of cos on $[0, \pi]$ is shown solid, and that of sin shown dashed. Note that cos is invertible, i.e., $y = \cos(x) \Leftrightarrow x = \arccos(y)$, whereas sin is not invertible. Figure 3(d) extends the domain of arccos from [0, 1] to [-1,1].

Three quarters of the circle have now been incorporated into the domain of cos or sin. There are two equivalent ways to include the fourth quarter; we can allow P to vary between V and W by letting it slide either clockwise past W in Figure 3(a) or anticlockwise past V in Figure 3(c). In the first case, we add $(-\pi/2, -\pi)$ to the domain; in the second case, we add $(\pi, 3\pi/2)$ to the domain; but either way, OM = cos(x) and MP = sin(x) both count as negative, and for any P between V and W both $\cos(\angle MOP)$ and $\sin(\angle MOP)$ are uniquely specified, although both can be written in two different ways. To be precise, if x lies between π and $3\pi/2$ and corresponds to point P in the extension of Figure 3(a), then $x - 2\pi$ lies between $-\pi/2$ and π in the extension of Figure 3(c) and corresponds to exactly the same point, so that

$$\cos(x) \quad = \quad \cos(\angle MOP) \quad = \quad \cos(x - 2\pi) \tag{30.25}$$

and

$$\sin(x) \quad = \quad \sin(\angle MOP) \quad = \quad \sin(x - 2\pi). \tag{30.26}$$

At this juncture, the domain of cos or sin has been extended only to $[-\pi, 3\pi/2]$. But the above procedure can be carried out indefinitely, with each clockwise revolution of P adding a negative subdomain and each clockwise revolution adding a positive one. Thus we can extend the domain of cos or sin to $(-\infty, \infty)$. Moreover, a clockwise or anticlockwise revolution of P must always return it to the same point on the circle, implying $\cos(x) = \cos(x \pm 2\pi)$ and $\sin(x) = \sin(x \pm 2\pi)$, for any x. So the functions cos and sin are periodic with period 2π, as we know already from Lecture 23; that is, (23.2) is satisfied with both g = C = cos and g = S = sin. See Figure 4, where a horizontal shift of $\pm 2\pi$ units has no effect on the graph.

As far as Table 1 is concerned, the only significant effect of extending the domain of cos or sin is that arccos and arcsin are uninvertible on $(-\infty, \infty)$. From Figure 4, π is the length of the largest subdomain on which either function is invertible. As implied by Figure 3, it is customary to choose $[0, \pi]$ as the domain of arccos and $[-\pi/2, \pi/2]$ as the domain of arcsin, because both include the subdomain $[0, \pi/2]$ which most commonly arises in applications. Except for this restriction on the domains of arccos and arcsin, the argument that yielded (17)-(24) is still valid. So we replace Table 1 by Table 2. Moreover, by using the chain rule and product rule, this table is readily extended to include derivatives of other trigonometric functions defined as compositions of cos or sin. See Appendix 30 and Exercises 1-7.

Restrictions	DERIVATIVE on $[a, b]$, $b > a$	ANTIDERIVATIVE on $[a, b]$, $b > a$
none	$\frac{d}{dx}\{\cos(x)\} = -\sin(x)$	$\int_a^x \sin(t)\,dt = -\cos(x) + \cos(a)$
none	$\frac{d}{dx}\{\sin(x)\} = \cos(x)$	$\int_a^x \cos(t)\,dt = \sin(x) - \sin(a)$
$0 \le a < b < \pi$	$\frac{d}{dx}\{\arccos(x)\} = -\frac{1}{\sqrt{1-x^2}}$	$\int_a^x \frac{1}{\sqrt{1-t^2}}\,dt = -\arccos(x) + \arccos(a)$
$-\frac{\pi}{2} \le a < b < \frac{\pi}{2}$	$\frac{d}{dx}\{\arcsin(x)\} = \frac{1}{\sqrt{1-x^2}}$	$\int_a^x \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin(x) - \arcsin(a)$

Table 30.2 Some derivatives and integrals of trigonometric functions: a generalization of Table 1

Exercises 30

30.1 The function sec is defined on $(-\pi/2, \pi/2)$ by

$$\sec(x) = \frac{1}{\cos(x)}.$$

Show that

$$\frac{d}{dx}\{\sec(x)\} = \sin(x)\sec^2(x).$$

30.2 The function csc is defined on $(0, \pi/2)$ by $\csc(x) = \frac{1}{\sin(x)}$. Find $\frac{d}{dx}\{\csc(x)\}$.

30.3 For sec defined in Exercise 1, what is the domain of arcsec? Find $\frac{d}{dx}\{\mathrm{arcsec}(x)\}$.

30.4 For csc defined in Exercise 2, what is the domain of arccsc? Find $\frac{d}{dx}\{\mathrm{arccsc}(x)\}$.

30.5 For tan and arctan defined by (A1) and (A3), find $\frac{d}{dx}\{\tan(x)\}$ and $\frac{d}{dx}\{\arctan(x)\}$.

30.6 With tan defined by (A1), cot is defined by cot(x) = tan(x). What is the domain of cot? Find $\frac{d}{dx}\{\cot(x)\}$.

30.7 For cot defined in Exercise 6, what is the domain of arccot? Find $\frac{d}{dx}\{\mathrm{arccot}(x)\}$.

Appendix 30: The relationship between elevation and gradient

Apart from sin and cos, the most important trigonometric function is the quotient tan, defined on $(-\pi/2, \pi/2)$ by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}. \tag{30.A1}$$

Note that

$$\tan(\angle MOP) = \frac{MP}{OM} = \frac{\sin(\angle MOP)}{\cos(\angle MOP)} \tag{30.A2}$$

both in Figure 1(a), where MP, $\sin(\angle MOP)$ and $\tan(\angle MOP)$ are positive, and in Figure 3(a), where MP, $\sin(\angle MOP)$ and $\tan(\angle MOP)$ are negative. The function tan is increasing and therefore invertible, with inverse denoted by arctan, i.e.,

$$y = \tan(x), \; -\tfrac{\pi}{2} < x < \tfrac{\pi}{2} \quad \Leftrightarrow \quad x = \arctan(y), \; -\infty < y < \infty. \tag{30.A3}$$

Both tan and arctan are graphed in Figure 5.

Now, in Figure 6, θ is the elevation at P on the graph of the smooth function F, i.e., θ degrees is the angle you see between line of sight OP and horizontal OM in the diagram itself. If $y = F(x)$ were drawn to a scale of 1 unit per millimeter on each axis, then the gradient at P would satisfy

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{Q \to P} \frac{RQ}{PR} = \frac{MP}{OM} = \tan\left(\frac{\pi\theta}{180}\right), \tag{30.A4}$$

and so elevation would be

$$\theta = \frac{180}{\pi}\arctan\left(\frac{dy}{dx}\right) \tag{30.A5}$$

degrees. In practice, however, graphs are drawn to a scale of, say, K units per mm on the horizontal axis and L units per mm on the vertical axis, and usually $K \neq L$. Thus, in Figure 6, actual vertical distance RQ represents $RQ \cdot L$ units of dependent variable, whereas actual horizontal distance PR represents $PR \cdot K$ units of independent variable; that is, $\delta y = RQ \cdot L$ and $\delta x = PR \cdot K$. So, in practice, (A4) must be replaced by

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{Q \to P} \frac{L \cdot RQ}{K \cdot PR} = \frac{L}{K} \lim_{Q \to P} \frac{RQ}{PR} = \frac{L}{K}\frac{MP}{OM} = \frac{1}{r}\tan\left(\frac{\pi\theta}{180}\right), \tag{30.A6}$$

where

$$r = \frac{K}{L} \tag{30.A7}$$

is the scale ratio. Thus, in practice, elevation is given not by (A5), but rather by

$$\theta = \frac{180}{\pi}\arctan\left(r\,\frac{dy}{dx}\right) \tag{30.A8}$$

or, equivalently,

$$\theta(x) = \frac{180}{\pi}\arctan\left(r\,F'(x)\right). \tag{30.A9}$$

Because arctan is a strictly increasing and continuous function (Figure 5), (A9) implies that any zero, extremum or discontinuity of F' must correspond to a zero, extremum or discontinuity of θ. In other words, (A9) establishes (14.10).

For example, in Figure 1.3, volume of blood V in a human left ventricle is defined on [0.05, 0.35] by

$$V(t) = \frac{5}{432}\left\{8779 + 70560t - 924000t^2 + 3136000t^3 - 3360000t^4\right\} \qquad (30.A10)$$

(see Appendix 2B). The plot rectangle represents 0.9 units on the horizontal axis and 125 units on the vertical axis, so that the scale ratio would be 0.9/125 if the plot rectangle were a square, i.e., a rectangle with (horizontal to vertical) aspect ratio 1. But the actual plot rectangle has aspect ratio ϕ_∞, where ϕ_∞ is the golden ratio defined by (5.19), because the figure was drawn using Mathematica's default option. So, in fact,

$$r = \frac{K}{L} = \frac{0.9}{125\phi_\infty} = \frac{9}{625(1+\sqrt{5})} = 0.00445. \qquad (30.A11)$$

From (A10),

$$\begin{aligned} V'(t) &= \frac{5}{432}\left\{70560 - 1848000t + 9408000t^2 - 13440000t^3\right\} \\ &= \frac{350}{9}(1-20t)(10t-3)(20t-7), \end{aligned} \qquad (30.A12)$$

after simplification. So, from (A9), elevation in Figure 1.6 is defined by

$$\begin{aligned} \theta(t) &= \frac{180}{\pi}\arctan\left(r\,V'(t)\right) = \frac{180}{\pi}\arctan\left\{\frac{14(1-20t)(10t-3)(20t-7)}{25(1+\sqrt{5})}\right\} \\ &= \frac{180}{\pi}\arctan\left\{0.173(1-20t)(10t-3)(20t-7)\right\} \end{aligned} \qquad (30.A13)$$

degrees (at least on subdomain [0.05, 0.35]). For example, $\theta(0.08) = -51$ degrees.

Answers and Hints for Selected Exercises

30.2 By the chain rule with $P(x) = \sin(x)$ and $Q(y) = y^{-1}$ we have

$$\frac{d}{dx}\{\csc(x)\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x))$$
$$= \{\cos(x)\}\cdot(-\{P(x)\}^{-2}) = -\cos(x)\cdot\{\sin(x)\}^{-2} = -\{\csc(x)\}^2\cdot\cos(x)$$

You can rewrite this result as

$$\frac{d}{dx}\{\csc(x)\} = -\cot(x)\cdot\csc(x).$$

30.4 As x increases from 0 to $\pi/2$, $\sin(x)$ increases from 0 to 1, so $1/\sin(x)$ decreases from ∞ to 1. The range of csc is therefore $[1, \infty)$. So the domain of arccsc is $[1, \infty)$. By the definition of inverse, $x = \text{arccsc}(y) \Leftrightarrow y = \csc(x)$. So

$$\frac{dx}{dy} = \left\{\frac{dy}{dx}\right\}^{-1}$$

implies

$$\frac{d}{dy}\{\text{arccsc}(y)\} = \left\{\frac{d}{dx}\{\csc(x)\}\right\}^{-1} = \frac{-1}{\csc^2(x)\cdot\cos(x)}$$
$$= \frac{-1}{y^2\cdot\sqrt{1-\sin^2(x)}} = \frac{-1}{y^2\cdot\sqrt{1-(1/y)^2}} = \frac{-1}{y\sqrt{y^2-1}}$$

or, which is exactly the same thing,

$$\frac{d}{dx}\{\text{arccsc}(x)\} = \frac{-1}{x\sqrt{x^2-1}}.$$