2 Lanchestrian Models of Human Combat

Although verbal models of war and military strategy have been around for centuries,[1] it is conventionally agreed that truly mathematical models of war did not appear until about 100 years ago.[2] The models first independently developed by Lanchester (1916) and Osipov (1995) around that time have since been embellished and extended by numerous others, and it will help to introduce them here in a relatively recent guise. Accordingly, we adopt the notation of Adams and Mesterton-Gibbons (2003) and Johnson and MacKay (2015), and refer to these models collectively as Lanchestrian models.[3]

In Lanchester's models, of which there are two, each army is assumed to consist of individuals with equivalent fighting abilities. Because these models do not include recruitment or reinforcement, group sizes decline from their original values. The first model yields what Lanchester called his "square law." The second yields his "linear law." We deal with each model in turn.

Lanchester's square law arises when two armies fight in such a way that individuals on either side can concentrate their attacks on their opponents. Let $m(t)$ and $n(t)$ be the number of surviving individuals in Group 1 and Group 2, respectively, at time t. Group sizes at the start of the fight are $m_0 = m(0)$ and $n_0 = n(0)$. Let α_m be the fighting ability of each individual in Group 1, and α_n be the fighting ability of each individual in Group 2. The fighting ability expresses the rate at which an individual can kill opponents within a particular context. Under Lanchester's first model, rates of mortality are described by the following pair of ordinary differential equations or ODEs:

$$\frac{dm}{dt} = -\alpha_n \, n \tag{2.1a}$$

$$\frac{dn}{dt} = -\alpha_m \, m \tag{2.1b}$$

Absent any recruitment, both $\frac{dm}{dt}$ and $\frac{dn}{dt}$ are negative, so that m and n both approach zero—until one of them actually reaches zero and the other one stops decreasing.

These equations can be interpreted as saying that armies are reduced according to the rate of incoming fire from the opposing army,[4] but are perhaps easier to interpret when

[1] See, e.g., Cioffi-Revilla (1989); McNeilly (2001); Howard (2002).

[2] Although the equation $4x = 15y$ appears—in the context of what is arguably a simple mathematical model—in Chapter 2 of Book 14 of Tolstoy's *War and Peace*, which was published in 1869.

[3] Mindful that a Russian might rather call them Osipovian models.

[4] See Karr (1983). More specifically, two interpretations of (2.1) are consistent with the assumption of concentrated fire. The first is that targets must be destroyed individually but are either sufficiently numerous or sufficiently easy to identify that each surviving attacker locates them at a constant rate. The second is that although (surviving) targets may be destroyed more than one at a time, they are dispersed over an area that decreases in proportion to their number, and each attacker destroys all targets in a certain "lethal" area per unit time. If the constant of proportionality and lethal area per weapon are A_n and a_n, respectively, so that n surviving Group 2 attackers generate lethal area $a_n n$ per unit time for m surviving Group 1 defenders occupying area $A_n m$, then we would expect the per capita death rate $-\frac{1}{m}\frac{dm}{dt}$ for Group 1 to equal the ratio of $a_n n$ to $A_n m$, which produces (2.2a) with $\alpha_n = a_n / A_m$ (Karr, 1983, p. 92).

first rewritten as

$$-\frac{1}{m}\frac{dm}{dt} \;=\; \alpha_n\,\frac{n}{m} \tag{2.2a}$$

$$-\frac{1}{n}\frac{dn}{dt} \;=\; \alpha_m\,\frac{m}{n} \tag{2.2b}$$

Then each equation states that the instantaneous per capita death rate or attrition rate experienced by a fighting group equals the ratio of the number of opponents to the number of comrades, multiplied by the individual fighting ability of each opponent. Either way, integration of $-\alpha_m m\,dm = -\alpha_n n\,dn$ leads to the state equation

$$\alpha_m(m_0^2 - m^2) \;=\; \alpha_n(n_0^2 - n^2) \tag{2.3a}$$

or

$$\alpha_m m_0^2 + \alpha_n n^2 \;=\; \alpha_m m^2 + \alpha_n n_0^2, \tag{2.3b}$$

which is satisfied at any time during the battle. Moreover, because (2.1) are linear ODEs, they are readily solved to yield[5]

$$\begin{aligned} m(t) \;&=\; \tfrac{1}{2}\left\{\left(m_0 - \sqrt{\frac{\alpha_n}{\alpha_m}}n_0\right)e^{\sqrt{\alpha_m\,\alpha_n}\,t} + \left(m_0 + \sqrt{\frac{\alpha_n}{\alpha_m}}n_0\right)e^{-\sqrt{\alpha_m\,\alpha_n}\,t}\right\} \\ n(t) \;&=\; \tfrac{1}{2}\left\{\left(n_0 - \sqrt{\frac{\alpha_m}{\alpha_n}}m_0\right)e^{\sqrt{\alpha_m\,\alpha_n}\,t} + \left(n_0 + \sqrt{\frac{\alpha_m}{\alpha_n}}m_0\right)e^{-\sqrt{\alpha_m\,\alpha_n}\,t}\right\} \end{aligned} \tag{2.4}$$

Let us now suppose that

$$\alpha_m m_0^2 \;>\; \alpha_n n_0^2. \tag{2.5}$$

Then (2.3b) implies $\alpha_n n^2 < \alpha_m m^2$ or

$$-\frac{\alpha_n n}{m} \;>\; -\frac{\alpha_m m}{n} \tag{2.6}$$

and hence

$$\frac{1}{m}\frac{dm}{dt} \;>\; \frac{1}{n}\frac{dn}{dt} \tag{2.7a}$$

or

$$\left|\frac{1}{m}\frac{dm}{dt}\right| \;<\; \left|\frac{1}{n}\frac{dn}{dt}\right| \tag{2.7b}$$

by (2.2). So, in a fully escalated fight, n will reach zero before m. In fact, at time

$$t_f \;=\; \frac{1}{2\sqrt{\alpha_m\,\alpha_n}}\ln\!\left(\frac{\sqrt{\alpha_m}\,m_0 + \sqrt{\alpha_n}\,n_0}{\sqrt{\alpha_m}\,m_0 - \sqrt{\alpha_n}\,n_0}\right) \tag{2.8}$$

[5]Differentiation of (2.1a) with respect to time and substitution from (2.1b) yields the homogeneous second-order linear ODE $\frac{d^2m}{dt^2} - \alpha_m\alpha_n\,m = 0$ for m, whose characteristic equation has roots $\pm\{\alpha_m\alpha_n\}^{1/2}$, and so its solution is a linear combination of $e^{\pm\{\alpha_m\alpha_n\}^{1/2}\,t}$. Differentiation and division by $-\alpha_m$ yields n, and the values of the two coefficients are determined by the initial conditions.

Group 2 will be eliminated, while Group 1 still has

$$m_0 \sqrt{\frac{\alpha_m}{\alpha_n} - \frac{n_0^2}{m_0^2}} \tag{2.9}$$

survivors.[6] We therefore interpret the left-hand side of (2.5) as the (initial) fighting ability of Group 1, the right-hand side as the fighting ability of Group 2, and (2.5) itself as stating that the fighting strength of Group 1 is higher—because its attrition rate is lower. Thus, group fighting ability is proportional to the square of the size of the fighting group but is only linearly related to individual fighting ability. It is therefore more important to enter battle with a large army than with fighters of high prowess. Although a smaller force can still win,[7] if Group 1 is smaller than Group 2, that is, if $m_0 < n_0$, then a win for Group 1 requires its relative fighting ability α_m/α_n to exceed not only its relative numerical disadvantage n_0/m_0, but also the square of that number, because with $m_0 < n_0$ we require

$$\frac{\alpha_m}{\alpha_n} > \left(\frac{n_0}{m_0}\right)^2 \tag{2.10}$$

for (2.5) to hold. Hence the phrase "square law." For example, defeating an adversary that is three times as numerous requires fighters to be at least nine times as effective.

Under Lanchester's second model, death rates are proportional to the product of the sizes of the two armies:

$$\frac{dm}{dt} = -\alpha_n \, mn \tag{2.11a}$$

$$\frac{dn}{dt} = -\alpha_m \, mn \tag{2.11b}$$

More transparently, per capita death rate is proportional to size of enemy army:

$$-\frac{1}{m}\frac{dm}{dt} = \alpha_n \, n \tag{2.12a}$$

$$-\frac{1}{n}\frac{dn}{dt} = \alpha_m \, m \tag{2.12b}$$

This model was intended for circumstances in which "there is no direct value in concentration" (Lanchester, 1916, p. 30), which can result if it becomes more difficult to acquire a target in the opposing group as the size of the opposing group is reduced.[8] Integration

[6] Where (2.8) follows from (2.4).

[7] And often has—Epstein (1997, p. 21) cites a dozen examples from the 19th and 20th centuries alone.

[8] See Karr (1983). More specifically, two interpretations of (2.12) are consistent with the assumption that it is difficult or impossible to concentrate fire. The first is that targets must be destroyed individually but are either sufficiently few or sufficiently difficult to identify that each surviving attacker locates them at a rate proportional to the number of targets still present—as opposed to at a constant rate, as in the case of the square law (Footnote 4). The second is that although (surviving) targets may be destroyed more than one at a time, they are dispersed over an area that does not vary over time, targets "redisperse between shots" (Karr, 1983, p. 94), and each attacker destroys all targets in a certain "lethal" area per unit time. If the area over which shots redisperse and the lethal area per weapon are A_n and a_n, respectively, so that n surviving Group 2 attackers generate lethal area $a_n n$ per unit time for m surviving Group 1 defenders occupying area A_n, then we would expect the per capita death rate $-\frac{1}{m}\frac{dm}{dt}$ for Group 1 to equal the ratio of $a_n n$ to A_n, which produces (2.12a) with $\alpha_n = a_n/A_m$.

of $-\alpha_m\, dm = -\alpha_n\, dn$ now leads to the state equation

$$\alpha_m(m_0 - m) \;=\; \alpha_n(n_0 - n) \tag{2.13a}$$

or

$$\alpha_m m_0 + \alpha_n n \;=\; \alpha_m m + \alpha_n n_0 \tag{2.13b}$$

Moreover, even though (2.11) are nonlinear ODEs, they can be solved to yield[9]

$$\begin{aligned} m(t) &= \frac{\beta m_0}{\beta + \alpha_n n_0(1 - e^{-\beta t})} \\ n(t) &= \frac{\beta n_0}{\beta + \alpha_m m_0(e^{\beta t} - 1)} \end{aligned} \tag{2.14}$$

where

$$\beta \;=\; \alpha_m m_0 - \alpha_n n_0 \tag{2.15}$$

whenever $\beta \neq 0$ (or

$$m(t) \;=\; \frac{m_0}{1 + m_0 \alpha_m t}, \qquad n(t) \;=\; \frac{n_0}{1 + n_0 \alpha_n t} \tag{2.16}$$

in the unlikely event that $\beta = 0$, in which case, $\alpha_m m = \alpha_n n$ for all $t \geq 0$). Note that, in theory, the losers no longer die out in finite time; rather, by (2.14), $m(\infty) > 0, n(\infty) = 0$ if $\beta > 0$, whereas $m(\infty) = 0, n(\infty) > 0$ if $\beta < 0$.

Group 1 has the greater fighting ability and is expected to win a fully escalated fight if

$$\alpha_m m_0 \;>\; \alpha_n n_0 \tag{2.17}$$

because then (2.13b) implies $\alpha_n\, n < \alpha_m\, m$ so that (2.12) implies (2.7). In these conditions, a smaller Group 1 will win—with

$$m(\infty) \;=\; m_0\left(1 - \frac{\alpha_n}{\alpha_m}\frac{n_0}{m_0}\right) \tag{2.18}$$

survivors—if its relative fighting ability α_m/α_n merely exceeds its relative numerical disadvantage n_0/m_0, because with $m_0 < n_0$ we obtain

$$\frac{\alpha_m}{\alpha_n} \;>\; \frac{n_0}{m_0} \tag{2.19}$$

in place of (2.10). Thus, group strength is equally sensitive to the size of the army and to individual fighting abilities.

Lanchester's laws are most frequently stated in terms of the initial balance that would yield a stalemate, that is, as

$$\alpha_m m_0^2 \;=\; \alpha_n n_0^2 \tag{2.20}$$

[9]Using (2.11a) to write n in terms of m and $\frac{dm}{dt}$ and substituting into (2.11b) yields $\frac{d}{dt}\{\frac{1}{m}\frac{dm}{dt} + \alpha_m m\} = 0$, which readily integrates to $\frac{1}{m}\frac{dm}{dt} + \alpha_m m = C =$ constant. Division by m and the substitution $u = \frac{1}{m}$ now yield $\frac{du}{dt} + Cu = \alpha_m$, which readily integrates to $u = \alpha_m/C + De^{-Ct}$ for $C \neq 0$ or to $u = \alpha_m t + D$ for $C = 0$, where D is another constant. The values of the two constants are determined by the initial conditions.

for his square law and

$$\alpha_m m_0 \;=\; \alpha_n n_0 \tag{2.21}$$

for his linear law.[10] Lanchester (1916, p. 48) himself expressed his square law by saying that "the fighting strengths of the two forces are equal when the *square of the numerical strength multiplied by the fighting value of the individual units are equal*" (the emphasis being his). For an intuitive sense of what it means, it is helpful to ask how many machine guns are equivalent to a given number of rifles. Let us follow Bellany (2002) in comparing 1000 rifles on one side with 200 machine guns having 25 times the firing power of a rifle on the other side. It might at first be thought that the 200 machine guns should overpower the rifles by 5 to 1. If the rifles can concentrate their fire, however, then 5 rifles attack each machine gun, which therefore lasts on average only a fifth as long as a rifle and hence does only five times the damage of a rifle—instead of 25 times, as supposed at first. So in fact the fighting strength of 1000 rifles is precisely equal to that of 200 machine guns, as follows from (2.20) with $\alpha_m = 1$, $m_0 = 1000$, $\alpha_n = 25$ and $n_0 = 200$.

Indeed we can obtain (2.20) for any values of m_0, n_0, α_m and α_n via a generalization of the above argument. Compare m_0 rifles to n_0 machine guns and let

$$k \;=\; \frac{\alpha_n}{\alpha_m} \tag{2.22}$$

be the firepower of a machine gun relative to that of a rifle. Then fire from m_0/n_0 rifles is concentrated on each machine gun, reducing its relative advantage in firepower from k to $k \cdot n_0/m_0$. So the effective fighting power of a machine gun is not $\alpha_m \cdot k$ but rather $\alpha_m \cdot k \cdot n_0/m_0$. Hence the total effective firepower of the rifles is $m_0 \cdot \alpha_m$, that of the machine guns is $n_0 \cdot \alpha_m \cdot k \cdot n_0/m_0$, and these two quantities are equal when (2.20) holds. If, on the other hand, fire cannot be concentrated, perhaps because the two sides are too far apart, then the total firepower of the machine guns is not reduced from $n_0 \cdot \alpha_m \cdot k$ and equals the total firepower $m_0 \cdot \alpha_m$ of the rifles when (2.21) holds instead. The two sides are then said to engage in "positional fire"—each fires into a region believed occupied by the other side, but without aiming at specific targets.

Lanchester's second model is not, however, the only one that yields the linear law. A second such model derives from assuming that all fighting is in one-to-one contests:

$$\frac{dm}{dt} \;=\; -\alpha_n \,\min(m, n) \tag{2.23a}$$

$$\frac{dn}{dt} \;=\; -\alpha_m \,\min(m, n) \tag{2.23b}$$

(Franks and Partridge, 1993). For both groups, mortality rates are proportional to number of survivors in the smaller group, since excess members of the larger group do not participate until there is an opportunity to replace a member of their own army—because, e.g., the battlefield geometry does not allow simultaneous attacks of many against one. Now

$$-\frac{1}{m}\frac{dm}{dt} \;=\; \frac{\alpha_n n}{m} \tag{2.24a}$$

$$-\frac{1}{n}\frac{dn}{dt} \;=\; \alpha_m \tag{2.24b}$$

[10] Which is not what Lanchester called it—the word "linear" does not appear in his book.

if $m > n$ but

$$-\frac{1}{m}\frac{dm}{dt} = \alpha_n \tag{2.24c}$$

$$-\frac{1}{n}\frac{dn}{dt} = \frac{\alpha_m}{n} \tag{2.24d}$$

if $n > m$. Because (2.23) or (2.24) still yields $-\alpha_m\, dm = -\alpha_n\, dn$, the state equation is still (2.13) and the condition under which Group 1 wins is still (2.17), because (2.17) still implies $\alpha_n\, n < \alpha_m\, m$, so that (2.24) implies (2.7), regardless of whether $m < n$ or $m > n$.[11]

Collectively, the three Lanchestrian models indicate that concentrated fire favors a square-law outcome, whereas positional fire and one-to-one combat favor a linear-law outcome. How should an invading army use these results to choose or seek to influence mode of battle? Let us suppose that Side 1 is the attacker and Side 2 is the defender. Then, from Side 1's perspective, the fighting-ability ratio is α_m/α_n and the numerical-disadvantage ratio is n_0/m_0. Winning $(n_0/m_0, \alpha_m/\alpha_n)$ pairs are points in Figure 2.1 that lie above the darker shaded region (in which neither (2.10) nor (2.19) is satisfied, so that Side 1 is eliminated). Inspection of this diagram reveals that if Side 1 is heavily outnumbered, then it should favor positional fire or one-to-one combat over concentrated fire, because if it can win at all then it is only if the linear law applies. On the other hand, if Side 1 is greatly favored numerically, then it should also favor concentrated fire, because even if its fighting abilities are weak enough for α_m/α_n to be smaller than n_0/m_0, which is a small number, they will be strong enough for victory as long as α_m/α_n exceeds only $(n_0/m_0)^2$—which is very much smaller. Curiously, there are two regions—indicated by lighter shading in Figure 2.1—where although Side 1 wins either way, it has fewer casualties when the linear law applies than when the square law applies.[12]

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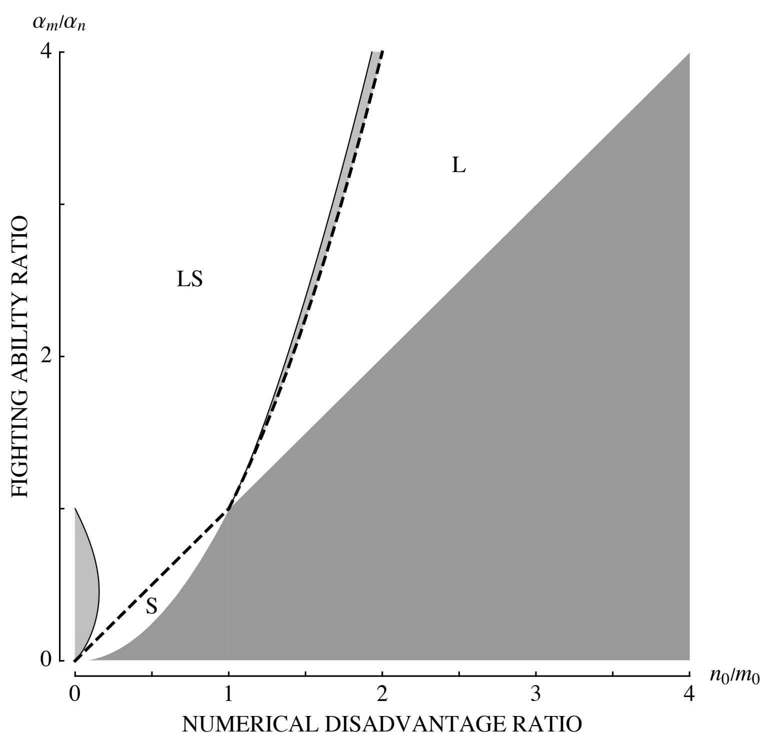
[11]Note that both models yielding the linear law involve nonlinear differential equations, whereas the square law involve linear differential equations—the phrase "linear law" refers to the linear relationship between group fighting ability and group size (as opposed to the underlying ODEs).

[12]If we temporarily set $r = n_0/m_0$, $k = \alpha_m/\alpha_n$ and use s_S, s_L to denote the survivor numbers given by (2.9) and (2.18), respectively, then Figure 2.1 becomes a quadrant of the r-k plane, (2.10) and (2.19) are both satisfied where $0 < r < \min(k, \sqrt{k})$ and $s_S > s_L$ or $s_S < s_L$ according to whether $\sqrt{k - r^2} > 1 - \frac{r}{k}$ or $\sqrt{k - r^2} > 1 - \frac{r}{k}$. Also, s_S and s_L both decrease with r, increase with k and are equal where $(1 + k^2)r = k\{k \pm \sqrt{k(k^2 - k + 1)}\}$. The upper branch of this curve forms the left-hand boundary of the long and narrow lighter shaded region for $k \geq 1$, while its lower branch forms the right-hand boundary of the shorter and wider lighter shaded region for $0 \leq k \leq 1$. We can now verify that $s_L > s_S$ within the lighter shaded regions by evaluating $\sqrt{k - r^2}$ and $1 - \frac{r}{k}$ at any suitable points within them—e.g., $(0, k)$ for $0 < k < 1$ and $(\frac{1}{2}\sqrt{2} + \frac{1}{5}(1 + \sqrt{6}), 2)$ for $k > 1$—and comparing the results.

Figure 2.1: Winning combinations of (initial) numerical disadvantage ratio n_0/m_0 and fighting ability ratio α_m/α_n for Side 1 in the (n_0/m_0)-(α_m/α_n) plane. In the darker shaded region, Side 1 is eliminated. Above the dashed curve, Side 1 wins only by the linear law in the unshaded region marked L or by the square law in the unshaded region marked S; above this curve, Side 1 wins by either law. The lighter shading indicates where Side 1 wins either way, but there are more survivors according to the linear law.



Although different armies may have different sizes and fighting abilities, all three models considered so far are symmetric with regard to role. In contrast, Deitchman (1962) has explored an asymmetric Lanchestrian model in which Group 1 has the role of ambusher while Group 2 has the role of ambushed. Deitchman (1962) begins by restating the conditions for (2.1) or (2.11) to apply as follows: (2.1) applies when each side is visible to the other, and each individual on each side is able to fire on any opposing individual, because then the loss rate on one side is proportional to the number of opponents firing, whereas (2.11) applies when each side is invisible to the other, and each fires into the area believed to be occupied by the other, because then the loss rate on one side is proportional to the number of men on the other and to the number of men occupying the area under fire. When Group 1 is an ambusher, the inherent symmetry is broken. Now Group 2 is visible to Group 1, but Group 1 is invisible to Group 2. Thus (2.1b) applies to Group 2, all of whose members are visible to Group 1, whereas (2.11a) applies to Group 1, whose members are invisible to Group 2. We obtain

$$\frac{dm}{dt} = -\alpha_n\, mn \tag{2.25a}$$

$$\frac{dn}{dt} = -\alpha_m\, m \tag{2.25b}$$

In place of (2.3) or (2.13) we obtain the state equation

$$2\alpha_m(m_0 - m) = \alpha_n(n_0^2 - n^2) \tag{2.26a}$$

or

$$2\alpha_m m_0 + \alpha_n n^2 = 2\alpha_m m + \alpha_n n_0^2, \tag{2.26b}$$

and in place of (2.10) or (2.19) the condition for Group 1 to win an all-out fight becomes

$$\frac{\alpha_m}{\alpha_n} > \frac{n_0^2}{2m_0} = \tfrac{1}{2}m_0\left(\frac{n_0}{m_0}\right)^2. \tag{2.27}$$

Even if n_0/m_0 is large, it is possible that an ambusher has sufficient advantage for (2.27) to hold. For example, if each individual in a group of 10 has a thousandfold advantage in combative effectiveness (ability to hit an opponent) over each individual in a group of 100 by virtue of being hidden from them, then the smaller group would defeat the larger group despite being ten times smaller—although only $m_0 - \frac{1}{2}n_0^2\,\alpha_n/\alpha_m = \frac{1}{2} \times 100^2 \times 10^{-3} = 5$ or half its members would survive.

According to Epstein (1985, p. 13), this "so-called ambush variant ... may well be the most plausible of all Lanchester variants." Nevertheless, he has criticized Lanchester's models on at least three grounds.[13] First, the models do not allow an army to reduce its rate of attrition by withdrawing—on the contrary, they assume a fight to the death. Second, and as a consequence of the first assumption, there is no trading of space for time—because expression (2.8) for t_f does not account for withdrawal, the duration of war does

[13]See Epstein (1985, pp. 4–13). At least to my reading, he suggests that the degree of reliance of military analysts on Lanchester's equations—see, e.g., Lepingwell (1987, p. 89)—is truly alarming (Epstein, 1985, pp. 3–4), given "little historical or empirical evidence to support their use" (Lepingwell, 1987, p. 127).

not depend on how much space is traded away. Third, the models do not account for diminishing marginal returns. For the square-law model, by (2.1), the "instantaneous casualty-exchange ratio"

$$\frac{dm}{dn} = \frac{\alpha_n}{\alpha_m}\frac{n}{m} \tag{2.28}$$

(that is, the limiting ratio of Group-1 members killed per Group-2 member killed) grows at a constant—as opposed to diminishing—rate with respect to the force ratio n/m as the force ratio grows: no crowding or force-to-space constraint ever moderates the extent to which Group 2 can concentrate its force.[14] For the linear-law models, the absence of diminishing returns is even more striking, since the instantaneous casualty-exchange ratio is just the constant α_n/α_m, by (2.11) and (2.23). Epstein (1997) has addressed these three issues by constructing an adaptive model of war, which we will discuss in Lecture 4. Meanwhile, in Lecture 3, we focus on only the third issue—diminishing returns—with a view to extending the scope of Lanchestrian models from humans to non-human animals.

Despite Epstein's criticisms, the extensive literature related to Lanchestrian models continues to grow with few signs of abating,[15] perhaps because they are "useful as a heuristic for thinking about combat interactions, albeit a heuristic with distinct limitations" (Lepingwell, 1987, p. 127). Moreover, the central insights of these models continue to guide strategic analysis. For example, Bellany (2002, p. 74) argues that in asymmetric warfare among humans, a more weakly armed but numerically strong side will seek out engagements that permit aimed fire, with a preference for daylight, engaging the enemy closely and a relative absence of cover; whereas a strongly armed but more casualty-conscious side will seek engagements that permit positional fire, with a preference for withdrawing to a longer range and engaging the enemy less closely. Similarly, Franks and Partridge—after first noting that if the attacking side "is in the majority it should try to fight a battle using its numerical superiority to minimize casualties by concentrating its attack" whereas if it "is greatly in the minority it should try to fight a series of one-to-one duels"—argue that army ants and obligate slave-making ants have evolved what might be called, respectively, square-law and linear-law strategies: "By concentrating their attack army ants are likely to minimize their casualties by dividing and conquering each nest the instant it is detected" whereas to "avoid concentrated attack from the more numerous defenders they encounter on a slave-raid . . . many slave-making ants have independently and convergently evolved propaganda substances" that "enable the slave-makers to reorganize the battle into a series of one-to-one duels as predicted by Lanchester's Linear Law" (Franks and Partridge, 1993, p. 198).

[14] More rigorously, $\frac{\partial^2}{\partial(n/m)^2}\left\{\frac{dm}{dn}\right\} = 0$, as opposed to $\frac{\partial^2}{\partial(n/m)^2}\left\{\frac{dm}{dn}\right\} < 0$.

[15] See, e.g., Kress and MacKay (2014) and Lin and MacKay (2014) for recent contributions.