6 War Duration: Insights from a Deterministic Model

As we noted in Lecture 2, Lanchester's equations exclude recruitment and reinforcement. To allow for such resupply, Bellany (1999) modified Lanchester's square-law equations by supposing that there is an upper limit to either side's force strength, and that its rate of resupply at any time is proportional to its unused capacity—that is, the difference between its upper limit or "ceiling force" and current force level. If the upper limit and constant of proportionality are K_m and ρ_m for Group 1, and if K_n and ρ_n are the corresponding parameters for Group 2, then in place of (2.1) we obtain

$$\frac{dm}{dt} = \rho_m(K_m - m) - \alpha_n n \tag{6.1a}$$

$$\frac{dn}{dt} = \rho_n(K_n - n) - \alpha_m m. \tag{6.1b}$$

Bellany (1999, p. 730) refers to ρ_m as Group 1's "coefficient of performance." It is "the fraction of the eventual ceiling force" that Group 1 "can put into the field, in unit time, at the start of the war, when the war is begun with far fewer troops at the front than the ceiling figure" (and likewise, of course, for ρ_n).

These equations are formally identical to (5.1) if we substitute $k = -\alpha_n$, $l = -\alpha_m$, $\alpha = \rho_m$, $\beta = \rho_n$, $g = \rho_m K_m$ and $h = \rho_n K_n$. So we know from (5.2)–(5.3) that there is a stable equilibrium at $(m, n) = (m_0, n_0)$ in the positive quadrant of the m-n plane when

$$\rho_m \rho_n > \alpha_m \alpha_n, \tag{6.2}$$

$$\rho_m K_m > \alpha_n K_n \tag{6.3}$$

and

$$\rho_n K_n > \alpha_m K_m, \tag{6.4}$$

where we define

$$m_0 = \frac{(\rho_m K_m - \alpha_n K_n)\rho_n}{\rho_m \rho_n - \alpha_m \alpha_n}, \qquad n_0 = \frac{(\rho_n K_n - \alpha_m K_m)\rho_m}{\rho_m \rho_n - \alpha_m \alpha_n} \tag{6.5}$$

Using data from Voevodsky (1971) to suggest that these inequalities hold, Bellany (1999, p. 729) argues that this steady-state solution accurately reflects "the nature of the major wars of the 20th century," which have an "apparent tendency . . . to be prolonged beyond the duration time anticipated by a least one of the major participants" (Bellany, 1999, p. 731). Thus "Prolongation and stalemate are seen as the default state of modern war" (Bellany, 1999, p. 729).

Then how do wars end? Indeed if wars are bound to end in stalemate, why are they ever begun? Bellany has two broad answers. The first is that the adverse consequences of a stalemate "will generally present themselves to one side first" (Bellany, 1999, p. 734); in particular, a country with limited goals and low cost tolerance, especially to casualties, will be in a relatively weak position as a stalemated war drags on. However, determining which side will require a separate model.

Bellany's second answer is, in essence, that wars are not bound to end in stalemate because (6.2)–(6.4) may not be satisfied. For example, the Falklands/Malvinas war in

1982 was short, perhaps because "the capacity on both sides to resupply was very restricted, partly by geographical considerations"—making $\rho_m \rho_n$ too small for (6.2) to hold. This much follows from the model itself. Yet Bellany goes further to argue that the very tendency towards stable equilibrium implies that "some effort can be diverted from the front-line with comparative impunity. If this effort in turn can be channelled into weakening the enemy's supply infrastructure, the military situation can be turned from a stable one or potentially stable one into an unstable one in favour of the side which is the more successful, relatively, at undermining the enemy's supply infrastructure." So, for example, Side 1 could work at reducing Side 2's ρ_n and K_n by economic, military or political means—away from the battlefield itself.

Bellany emphasizes that infrastructure includes not only lines of supply, but also sources of supply within an enemy's economy and society. "It is a characteristic of modern warfare that it is often possible, at least in principle, to disrupt the enemy's supply infrastructure before defeating the enemy in the field (i.e. whilst the situation in the field is a stable one) by virtue of the existence of air and missile power ..." so that, in effect, modern wars "... are won not so much on the battlefield where they have tendencies to stalemate, but off the battlefield" (Bellany, 1999, pp. 735–736). In this regard, Bellany suggests that his model offers an explanation for the shifts in military strategy in the 20th century towards waging war off the battlefield as well as on it, with the culmination of these shifts being arguably seen in the Gulf War of 1990-91 and the Kosovo War of 1999, "where the battlefield employment of force was in one case a formality and in the other unnecessary" (Bellany, 1999, p. 737). Would you or I have drawn the same conclusions from the same pair of differential equations? Perhaps not—illustrating again that clever interpretation is just as big a part of clever modelling as clever formulation and analysis.

Bellany also used his model to suggest a theoretical underpinning for a result that Voevodsky (1971) had reached on purely empirical grounds. Voevodsky used data from five US wars[1] to show that a nation's battle strength, S, varies with time t according to

$$S(t) \;=\; S_\infty\bigl(1 - e^{-t/\tau}\bigr) \tag{6.6}$$

where τ and S_∞ are constants. He obtained this result by plotting data for $\ln(S_\infty - S)$ against data for time and showing that a line whose slope is taken to be $-1/\tau$ yields an excellent fit (Voevodsky, 1971, p. 158).[2] Bellany rationalized by noting that *if*

$$\rho_m \;=\; \rho_n \;=\; \rho \tag{6.7}$$

and

$$\alpha_m \;=\; \alpha_n \;=\; \alpha \tag{6.8}$$

(which are two big ifs), then adding (6.1a) to (6.1b) yields

$$\frac{d\{m+n\}}{dt} + (\rho+\alpha)(m+n) \;=\; \rho(K_m + K_n) \tag{6.9}$$

[1]The Civil War, World War I, World War II, the Korean War and the Vietnam War.

[2]Although in the first instance this line corresponds to $S = S_\infty - (S_\infty - S_0)e^{-t/\tau}$ where $S_0 = S(0)$, in practice the asymptotic value S_∞ is so much larger than the initial value S_0—exceeding it by a factor of at least 10—that (6.6) is an adequate approximation (Voevodsky, 1971, p. 162).

and hence

$$m(t) + n(t) \;=\; (m_0 + n_0)e^{-(\rho+\alpha)t} + \frac{\rho(K_m + K_n)}{\rho + \alpha}\{1 - e^{-(\rho+\alpha)t}\}, \tag{6.10}$$

which is identical to (6.6) if we set $S(t) = m(t) + n(t)$ and hence $S_0 = m_0 + n_0$, $S_\infty = \rho(K_m + K_n)/(\rho + \alpha)$, $\tau = 1/(\rho + \alpha)$ and assume that $S_0 \ll S_\infty$, which is not unreasonable. However, Voevodsky found (6.6) to hold for each side separately, whereas (6.9) implies only that (6.6) holds for both sides together—and that after two big ifs!