8 Optimal Defense Positioning

Stability of a conventional military balance between two nations requires either nation's defense to be assured of victory if the other nation attacks, so that there exists no incentive to attack in the first place.[1] How should a defense be positioned to achieve this objective? Following the recipe advanced in Lecture 1, we broach this question here with a highly idealized model.

We begin by making the following four assumptions. First, an attacker's objective is to maximize territory gained, and correspondingly that a defender's objective is to minimize territory lost. Second, Side 1's territory lies in the half-space where $x > 0$, Side 2's territory lies in the half-space where $x < 0$ and their territories are separated by a linear boundary or front—whose length we regard as the unit of distance—extending along the line $x = 0$ from the origin $(0, 0)$ to the point with coordinates $(0, 1)$. Third, the characteristics of the battlefield do not vary parallel to this front—if, for example, we were concerned with naval warfare, then there would be no intervening island. Fourth, each side has perfect information about positioning (of either side's forces) and other conditions on its own side of the front, but only highly imperfect or no information about conditions on the other side. These are extremely bold assumptions—but we did say highly idealized model! Let us denote by "station x" the line that extends from $(x, 0)$ to $(x, 1)$, where x may be either positive or negative, and by "station y" the line that extends perpendicularly on either side of the front from $(0, y)$, where $0 \leq y \leq 1$.

Without loss of generality—in view of the symmetry between sides—we now assume that Side 1 is the defender and Side 2 is the attacker. We also assume—for the sake of simplicity—that each side will deploy a single unit of force, Side 1 for defense, Side 2 for attack. Further assumptions are as follows. First, Side 1's force unit is positioned at (α, β). Thus optimal defense positioning means optimal choice of α and β. Second, Side 2's force unit attacks by crossing the front perpendicularly and moves into Side 1's territory at (mean) speed v_a. Third, Side 1's force unit—by virtue of having perfect information on its own side of the front—moves instantly at speed v_d directly towards Side 2's force unit to intercept it. Fourth, the station at which Side 2's force unit crosses the front, denoted by Y, is a random variable distributed between 0 and 1 with density function p. That is,

$$\text{Prob}(y_1 < Y < y_2) \;=\; \int_{y_1}^{y_2} p(y)\, dy \tag{8.1}$$

with

$$\int_0^1 p(y)\, dy \;=\; 1 \tag{8.2}$$

for any $0 \leq y_1 < y_2 \leq 1$.

Let I denote the depth of incursion by Side 2 into Side 1's territory. Then Side 2 will intercept Side 1 at the point with coordinates (I, Y), whose distance from (α, β) is $\sqrt{(I - \alpha)^2 + (Y - \beta)^2}$. The time taken by Side 2 to reach (I, Y) is I/v_a. The time taken by

[1] See Gupta (1993, p. 41), on which this lecture is based.

Side 1 to reach (I, Y) is $\sqrt{(I-\alpha)^2+(Y-\beta)^2}/v_d$. These two times must be equal. Hence

$$\frac{I}{v_a} = \frac{\sqrt{(I-\alpha)^2+(Y-\beta)^2}}{v_d} \tag{8.3}$$

implying

$$\lambda I = \sqrt{(I-\alpha)^2+(Y-\beta)^2} \tag{8.4}$$

where

$$\lambda = \frac{v_d}{v_a} \tag{8.5}$$

denotes the ratio of speed of intercepton to speed of incursion. We assume that

$$\lambda \geq 1 \tag{8.6}$$

(consistent with our assumption that Side 1 has perfect information about conditions where $x > 0$, but Side 2 has only highly imperfect information).

Two cases arise, according to whether $\lambda = 1$ or $\lambda > 1$. We deal with them separately.

8.1 Equally rapid defending and attacking force units: $\lambda = 1$

Solving (8.4) for I, in this case the depth of incursion at station Y is

$$I = I(\alpha, \beta, Y) = \frac{\alpha^2+(Y-\beta)^2}{2\alpha} = \frac{\alpha^2+\beta^2-2\beta Y+Y^2}{2\alpha} \geq \tfrac{1}{2}\alpha. \tag{8.7}$$

Hence the mean depth of incursion is

$$\begin{aligned} c(\alpha, \beta) &= \mathrm{E}[I(\alpha, \beta, Y)] = \int_0^1 I(\alpha, \beta, y) p(y)\, dy \\ &= \frac{\alpha^2+\beta^2}{2\alpha}\int_0^1 p(y)\, dy - \frac{\beta}{\alpha}\int_0^1 yp(y)\, dy + \frac{1}{2\alpha}\int_0^1 y^2 p(y)\, dy \end{aligned} \tag{8.8}$$

where E denotes expected value. It is reasonable to interpret this expression as the cost of incursion, which we therefore seek to minimize. Thus the cost of incursion is

$$c(\alpha, \beta) = \frac{\alpha^2+\beta^2}{2\alpha} - \frac{\beta}{\alpha}\mu + \frac{1}{2\alpha}\{\sigma^2+\mu^2\} \tag{8.9}$$

where

$$\mu = \mathrm{E}[Y] = \int_0^1 yp(y)\, dy \tag{8.10}$$

and

$$\sigma^2 = \mathrm{E}[(Y-\mu)^2] = \int_0^1 y^2 p(y)\, dy - \mu^2 \tag{8.11}$$

are, respectively, the mean and variance of the distribution of Y. Elementary calculus reveals that this expression is minimized where $\partial c/\partial\alpha = 0 = \partial c/\partial\beta$ or $\alpha^2 = \sigma^2 + (\mu-\beta)^2$ and $\beta = \mu$. So the optimal position, (α^*, β^*), for Side 1's force unit is

$$(\alpha^*, \beta^*) = (\sigma, \mu). \tag{8.12}$$

A Side-2 attack at $Y = y$ that penetrates to an incursion depth of $I = x$ will be intercepted by an optimally positioned Side-1 defense at the point with coordinates (x, y), where

$$x \;=\; I(\alpha^*, \beta^*, y) \;=\; \tfrac{1}{2}\alpha^* + \tfrac{1}{2\alpha^*}(y - \beta^*)^2 \;=\; \tfrac{1}{2}\sigma + \tfrac{1}{2\sigma}(y - \mu)^2 \tag{8.13}$$

by (8.7) and (8.12). The locus of all such points—the defense locus—is a parabola with vertex at $(\frac{1}{2}\sigma, \mu)$, halfway between Side 1's base and the front. The smaller the variance of Y, the nearer the front the force unit should be positioned, and the faster the defense locus will bow away from the front.[2] Moreover, the smaller the variance, the smaller the (expected) cost of incursion. Indeed cost precisely equals standard deviation, since (8.9) and (8.12) imply

$$c(\alpha^*, \beta^*) \;=\; \frac{\alpha^{*2} + \beta^{*2}}{2\alpha^*} - \frac{\beta^*}{\alpha^*}\mu + \frac{1}{2\alpha^*}\{\sigma^2 + \mu^2\} \;=\; \sigma. \tag{8.14}$$

In particular, for a uniform distribution

$$c(\alpha^*, \beta^*) \;=\; \sigma \;=\; \sqrt{\int_0^1 (y - \mu)^2\, p(y)\, dy} \;=\; \sqrt{\int_0^1 (y - \tfrac{1}{2})^2 \cdot 1\, dy} \;=\; \frac{1}{2\sqrt{3}}. \tag{8.15}$$

8.2 Defending force unit more rapid than and attacking unit: $\lambda > 1$

Solving (8.4) for I, in this case the depth of incursion at station Y is

$$I \;=\; I(\alpha, \beta, Y) \;=\; \frac{\sqrt{\lambda^2\alpha^2 + (\lambda^2 - 1)(Y - \beta)^2} - \alpha}{\lambda^2 - 1} \;\geq\; \frac{\alpha}{\lambda + 1} \tag{8.16}$$

and the cost (mean depth) of incursion is

$$c(\alpha, \beta) \;=\; \int_0^1 I(\alpha, \beta, y)\, p(y)\, dy \;=\; \int_0^1 \frac{\sqrt{\lambda^2\alpha^2 + (\lambda^2 - 1)(y - \beta)^2} - \alpha}{\lambda^2 - 1}\, p(y)\, dy. \tag{8.17}$$

We obtain

$$\frac{\partial c}{\partial \alpha} \;=\; \int_0^1 \frac{\partial I}{\partial \alpha}\, p(y)\, dy \;=\; \frac{1}{\lambda^2 - 1}\int_0^1 \left\{\frac{\lambda^2\alpha}{\sqrt{\lambda^2\alpha^2 + (\lambda^2 - 1)(y - \beta)^2}} - 1\right\} p(y)\, dy \tag{8.18}$$

and

$$\frac{\partial c}{\partial \beta} \;=\; \int_0^1 \frac{\partial I}{\partial \beta}\, p(y)\, dy \;=\; \int_0^1 \frac{(\beta - y)p(y)}{\sqrt{\lambda^2\alpha^2 + (\lambda^2 - 1)(y - \beta)^2}}\, dy \tag{8.19}$$

with

$$\frac{\partial^2 c}{\partial \alpha^2} \;=\; \lambda^2 \int_0^1 \frac{(y - \beta)^2\, p(y)}{\{\lambda^2\alpha^2 + (\lambda^2 - 1)(y - \beta)^2\}^{3/2}}\, dy \tag{8.20}$$

and

$$\frac{\partial^2 c}{\partial \beta^2} \;=\; \lambda^2\alpha^2 \int_0^1 \frac{p(y)}{\{\lambda^2\alpha^2 + (\lambda^2 - 1)(y - \beta)^2\}^{3/2}}\, dy \tag{8.21}$$

[2] As illustrated below by Figure8.2(a).

both positive. Hence c is minimized where $\partial c/\partial\alpha = 0 = \partial c/\partial\beta$ or

$$\int_0^1 \Big\{\tfrac{\lambda^2\alpha}{\sqrt{\lambda^2\alpha^2+(\lambda^2-1)(y-\beta)^2}} - 1\Big\}\, p(y)\, dy \;=\; 0 \tag{8.22a}$$

$$\int_0^1 \tfrac{(\beta-y)p(y)}{\sqrt{\lambda^2\alpha^2+(\lambda^2-1)(y-\beta)^2}}\, dy \;=\; 0. \tag{8.22b}$$

Hence, in view of (8.2), the optimal position (α^*, β^*) is determined by

$$\int_0^1 \frac{p(y)}{\sqrt{\lambda^2\alpha^{*2} + (\lambda^2-1)(y-\beta^*)^2}}\, dy \;=\; \frac{1}{\lambda^2\alpha^*} \tag{8.23a}$$

and

$$\int_0^1 \frac{y\, p(y)}{\sqrt{\lambda^2\alpha^{*2} + (\lambda^2-1)(y-\beta^*)^2}}\, dy \;=\; \frac{\beta^*}{\lambda^2\alpha^*} \tag{8.23b}$$

In the important special case where Y is uniformly distributed between 0 and 1 or $p(y) = 1$, so that uncertainty about the point of incursion is maximal, (8.22b) implies

$$\begin{aligned}\int_0^1 \frac{(\lambda^2-1)(y-\beta)}{\sqrt{\lambda^2\alpha^2 + (\lambda^2-1)(y-\beta)^2}}\, dy \;&=\; \int_0^1 \frac{\partial}{\partial y}\sqrt{\lambda^2\alpha^2 + (\lambda^2-1)(y-\beta)^2}\, dy \\ =\; \sqrt{\lambda^2\alpha^2 + (\lambda^2-1)(y-\beta)^2}\Big|_0^1 \;&=\; \sqrt{\lambda^2\alpha^2 + (\lambda^2-1)(1-\beta)^2} \\ &\quad - \sqrt{\lambda^2\alpha^2 + (\lambda^2-1)\beta^2} \;=\; 0\end{aligned} \tag{8.24}$$

and hence $(1-\beta)^2 = \beta^2$, so that the optimal β is

$$\beta^* \;=\; \tfrac{1}{2} \tag{8.25}$$

The integral on the left-hand side of (8.23a) is now readily evaluated via the substitution

$$u \;=\; \tfrac{\sqrt{\lambda^2-1}}{\lambda\alpha^*}(y - \tfrac{1}{2}) \;\Longrightarrow\; du \;=\; \tfrac{\sqrt{\lambda^2-1}}{\lambda\alpha^*}\, dy \tag{8.26}$$

to yield

$$\frac{1}{\lambda^2\alpha^*} \;=\; \frac{1}{\sqrt{\lambda^2-1}}\int_{-\frac{\{\lambda^2-1\}^{1/2}}{2\lambda\alpha^*}}^{\frac{\{\lambda^2-1\}^{1/2}}{2\lambda\alpha^*}} \frac{du}{\sqrt{1+u^2}} \;=\; \frac{1}{\sqrt{\lambda^2-1}}\,\mathrm{arcsinh}(u)\Big|_{-\frac{\{\lambda^2-1\}^{1/2}}{2\lambda\alpha^*}}^{\frac{\{\lambda^2-1\}^{1/2}}{2\lambda\alpha^*}} \tag{8.27}$$

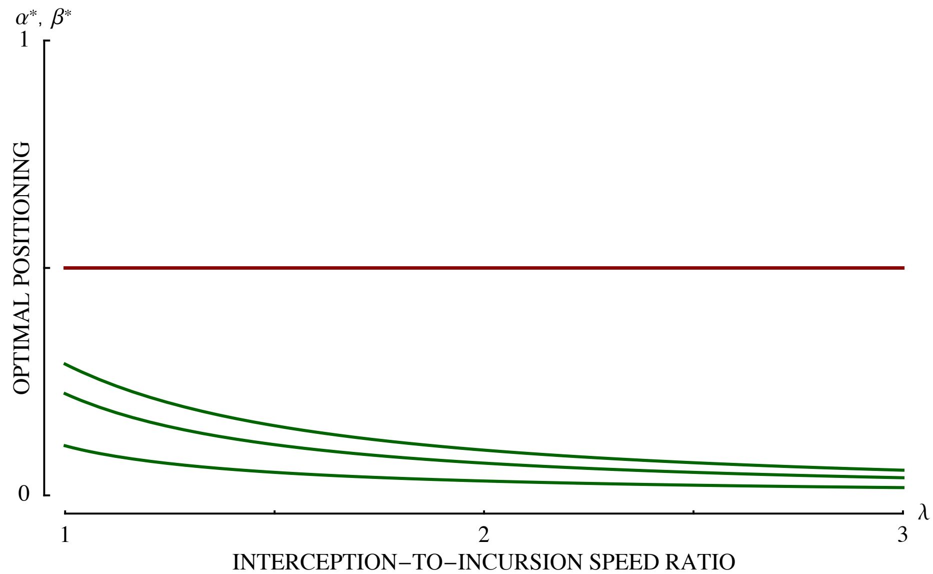
or

$$\frac{2\lambda^2\alpha^*}{\sqrt{\lambda^2-1}}\,\mathrm{arcsinh}\Big(\frac{\sqrt{\lambda^2-1}}{2\lambda\alpha^*}\Big) \;=\; 1. \tag{8.28}$$

This is a transcendental equation, which cannot be solved analytically to obtain α^*. Nevertheless, it is readily solved numerically, for any value of λ, and the resultant plot of α^* against λ is the topmost curve in Figure 8.1.

More generally, (8.23) are simultaneous nonlinear equations for α^* and β^*, which cannot be solved analytically, but are readily solved numerically, for any distribution of Y,

Figure 8.1: Optimal defense position (α^*, β^*) as a function of λ for a symmetric Beta distribution of incursion point with parameter a. The green curves show how α^* varies with λ for $a = 1$ (uniform distribution, uppermost curve, variance $\sigma^2 \approx 0.0833$), $a = 2$ (variance $\sigma^2 \approx 0.05$) and $a = 10$ (lowermost curve, variance $\sigma^2 \approx 0.0119$). The red curve indicates the optimal value of β^* for every value of λ.



for any value of λ. Suppose, for example, that Y has a symmetric Beta distribution on $[0, 1]$ with density defined by

$$p(y) = \frac{\Gamma(2a)}{\{\Gamma(a)\}^2} y^{a-1} (1-y)^{a-1} \tag{8.29}$$

where Γ denotes the Euler gamma function, i.e., $\Gamma(\eta) = \int_0^\infty e^{-\xi} \xi^{\eta-1}\, d\xi$. For $a = 1$ this distribution is uniform, because (8.29) then reduces to $p(y) = 1$; and for $a > 1$ the distribution is unimodal, with variance

$$\sigma^2 = \frac{1}{4(1+2a)} \tag{8.30}$$

The resultant dependence of α^* on λ is plotted in Figure 8.1 for three different values of a. We see that, the smaller the variance of incursion point, or the faster the speed of interception relative to that of incursion, the nearer the position of the force unit to the front. Numerical solutions strongly suggest that (8.25) continues to hold for any value of a, for every value of λ, as indicated by the red line in Figure 8.1. Indeed it seems reasonable to conjecture that (8.23) implies (8.25) not only for a symmetric Beta distribution of Y with $a > 1$, but also for any symmetric distribution on $[0, 1]$ such that $p(0) = 0 = p(1)$.

The corresponding defense locus is now obtained from (8.16) as

$$x = I(\alpha^*, \beta^*, y) = \frac{\sqrt{\lambda^2 \alpha^{*2} + (\lambda^2 - 1)(y - \beta^*)^2} - \alpha^*}{\lambda^2 - 1} \tag{8.31}$$

and is plotted in Figure 8.2 for four values of λ for three representative variances of Y, a low value (red curves), an intermediate value (blue) and a high value (green). Again we see that, the smaller the variance of incursion point, the nearer the front the force unit should be positioned, and the faster the defense locus will bow away from the front.

Gupta (1993, pp. 64-75) supplements this basic model with a verbal argument that a border between nations can be regarded as composed of a finite number of contiguous fronts separated by geographical features, and that for each such front the N-unit defense positioning problem reduces to N identical 1-unit problems, each of which has the solution we have just obtained. Thus, for example, in the case where $N = 10$, if Side 2 attacks at three different points—say, $y = y_1$, $y = y_2$ and $y = y_3$—with forces of 2, 2 and 3 units, respectively, then Side 1 should respond by sending two units from (α^*, β^*) to $(I(y_1, \alpha^*, \beta^*), y_1)$ to intercept the first incursion, another two to $(I(y_2, \alpha^*, \beta^*), y_2)$ to intercept the second incursion and three units to $(I(y_3, \alpha^*, \beta^*), y_3)$ to intercept the third—and keep the remaining three force units in reserve at (α^*, β^*) to deal with any possible later incursion by Side 2. Furthermore, it does not matter whether the three separate incursions are simultaneous or occur at different times.

The logic is sound if the assumptions hold. But as we have noted, the assumptions are extremely bold. According to Gupta (1993, p. 75), "The problems associated with our force-positioning algorithm are twofold: the vulnerability to preemption of any large force localization and the dangerously generous incursions permitted by the defense locus. These problems are a consequence of the fact that our optimization of defensive force location has not imposed any upper bound on the local concentration of forces" and potentially leaves a high-density headquarters in the same location highly vulnerable to a "surreptitious, preemptive, selective air raid" (Gupta, 1993, p. 71).

Gupta proceeds to apply his model with constraints on concentration of force units. The simplest possibility is that $\lambda = 1$ (interception speed equals incursion speed), $p(y) = 1$ (uniform distribution of Y) and the total defense force is constrained to be positioned at two separate locations (as opposed to more than two separate locations), with half of all total force units at each location defending one half of the front. The optimization formula can now be applied separately to each half of the front. Appropriate rescaling by a factor of 2—both parallel and perpendicular to the front—moves the optimal position from (α^*, β^*) to $(\frac{1}{2}\alpha^*, \frac{1}{2}\beta^*)$ for the force units defending the lower half of the front and to $(\frac{1}{2}\alpha^*, \frac{3}{2}\beta^*)$ for those defending the upper half. Correspondingly, we substitute $\frac{1}{2}\alpha^*$ and $\frac{1}{2}\beta^*$ or $\frac{3}{2}\beta^*$ for α^* and β^* in (8.13) to obtain the defense locus as

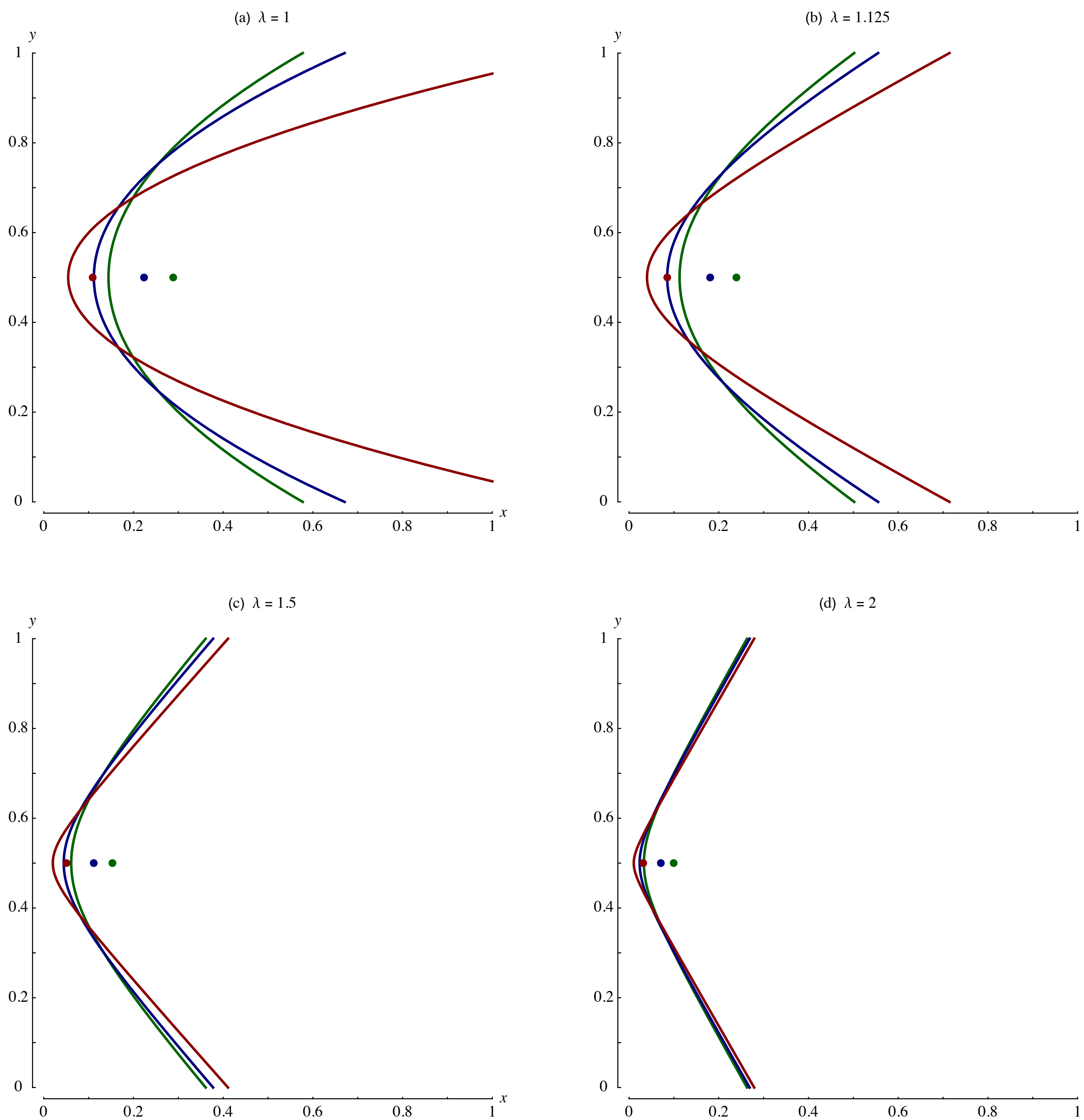
$$\begin{aligned} x &= I\left(\tfrac{1}{2}\alpha^*, \tfrac{1}{2}\beta^*, y\right) = \tfrac{1}{4}\alpha^* + \tfrac{1}{\alpha^*}\left(y - \tfrac{1}{2}\beta^*\right)^2 = \tfrac{1}{4}\sigma + \tfrac{1}{\sigma}\left(y - \tfrac{1}{2}\mu\right)^2 \\ &= \tfrac{1}{8\sqrt{3}} + 2\sqrt{3}\left(y - \tfrac{1}{4}\right)^2 \end{aligned} \tag{8.32a}$$

for $0 \le y \le \frac{1}{2}$ and

$$\begin{aligned} x &= I\left(\tfrac{1}{2}\alpha^*, \tfrac{3}{2}\beta^*, y\right) = \tfrac{1}{4}\alpha^* + \tfrac{1}{\alpha^*}\left(y - \tfrac{3}{2}\beta^*\right)^2 = \tfrac{1}{4}\sigma + \tfrac{1}{\sigma}\left(y - \tfrac{3}{2}\mu\right)^2 \\ &= \tfrac{1}{8\sqrt{3}} + 2\sqrt{3}\left(y - \tfrac{3}{4}\right)^2 \end{aligned} \tag{8.32b}$$

for $\frac{1}{2} \le y \le 1$. So, on using (8.15), we find that the (expected) cost of incursion is halved

Figure 8.2: Defense locus given by (8.13) for Panel (a) and (8.31) for Panels (b)—(d) for a symmetric Beta distribution of incursion point with parameter values $a = 1$ (uniform distribution, green curves, variance $\sigma^2 \approx 0.0833$), $a = 2$ (blue curves, variance $\sigma^2 \approx 0.05$) and $a = 10$ (red curves, variance $\sigma^2 \approx 0.0119$) for four different values of the ratio λ of interception to incursion speed. The correspondingly colored dots show the optimal defense position (α^*, β^*).



from $\frac{1}{2\sqrt{3}}$ for the green curve in Figure 8.3 to

$$\int_0^1 I(y)p(y)\,dy \;=\; \int_0^{\frac{1}{2}} \left\{\tfrac{1}{8\sqrt{3}} + 2\sqrt{3}\left(y - \tfrac{1}{4}\right)^2\right\} dy + \int_{\frac{1}{2}}^1 \left\{\tfrac{1}{8\sqrt{3}} + 2\sqrt{3}\left(y - \tfrac{3}{4}\right)^2\right\} dy \;=\; \tfrac{1}{4\sqrt{3}} \tag{8.33}$$

for the blue curve in Figure 8.3. Even if force unit totals for Side 1 equal those for Side 2, however, because we assume that enemy forces may readily be moved up and down the entire front, either half of the defending force units may now have to contend with all of the attacking ones (at least until the other half arrives at a later time).

With further verbal and graphical reasoning, Gupta (1993) is able to tease out several further insights towards a theory of defense positioning and geometry from little more than the above analysis. Indeed it is quite remarkable how much mileage he gets from so simple a model. If any of you is sufficiently interested, perhaps his work could form the basis of a more in-depth investigation leading to an end-of-term presentation.

For the rest of us, however, this is as far as we go. Gupta's model is an example of a so-called "decision-theoretic" model, which does not explicitly allow for interdependence between the best option for one decision-maker and the best options for other decision-makers, because only one decision-maker is given options to choose from within the model. Models that *do* allow for such interdependence—and hence the possibility of a more complete analysis—are known as "game-theoretic" models, and will be explored in Lecture 10 and later lectures. In between, however, there is a kind of mushy region in which it is hard to know whether what you are dealing with is truly a game-theoretic model or a de facto decision-theoretic model, or some combination of the two—which may be why some political scientists and others will tend to refer to either type as a "choice-theoretic" model.

Such a model will be the subject of Lecture 9.

Figure 8.3: Unconstrained (green) and constrained (blue) defense locus for a uniform distribution of incursion point with equal interception and incursion speeds ($\lambda = 1$). Colored dots indicate the corresponding optimal defense positions.