CORTICAL SURFACE FLATTENING USING LEAST SQUARE CONFORMAL MAPPING WITH MINIMAL METRIC DISTORTION

Lili Ju\textsuperscript{1}, Josh Stern\textsuperscript{2}, Kelly Rehm\textsuperscript{3}, Kirt Schaper\textsuperscript{2}, Monica Hurdal\textsuperscript{4}, and David Rottenberg\textsuperscript{2,3}

\textsuperscript{1}Institute for Math. and its Applications, University of Minnesota, Minneapolis, MN 55455, USA
\textsuperscript{2}Department of Neurology and \textsuperscript{3}Radiology, University of Minnesota, Minneapolis, MN 55455, USA
\textsuperscript{3}Department of Mathematics, Florida State University, Tallahassee, FL 32306, USA

ABSTRACT

Although flattening a cortical surface necessarily introduces metric distortion due to the non-constant Gaussian curvature of the surface, the Riemann Mapping Theorem states that continuously differentiable surfaces can be mapped without local angular distortion. We applied the so-called least-square conformal mapping method to flatten cortical patches into planar regions and also to generate the spherical conformal map of the entire cortex while minimizing metric distortion within the class of conformal maps. Our method, which preserves angular information and controls metric distortion, only involves the solution of a linear system and a nonlinear minimization problem with three parameters and is a very fast approach.

I. INTRODUCTION

The human cortex is a highly convoluted surface – making it difficult to view functional brain activity in a meaningful way. For example, functional foci can be buried in cortical sulci and appear in a number of discrete slices or foci that are on opposite walls of a sulcus may appear to be close together. Additionally, it also makes it difficult to compare the locations and patterns of functional activity in humans across subjects because of individual differences in cortical folding. The surface-based approach is a useful tool to address these problems.

A number of techniques have been proposed to flatten cortical surfaces. An approach that purports to substantially minimize the metric distortion (FreeSurfer) was suggested by Fischl et al. [4]; another one (CARET) by Drury and Van Essen et al. [3] attempts to obtain minimum areal distortion. Both methods have been successful in comparative and functional investigation studies. On the other hand, prospects for managing angular distortion are better since the Riemann Mapping Theorem [1] states that continuously differentiable surfaces can be mapped onto each other without local angular distortion. Hurdal et al. [6] proposed a method using CirclePack for quasi-conformal flattening of cortical surfaces that can handle user-defined patches (topologically equivalent to discs) as well as an entire cortical surface (topological spheres) and yields an upper bound for distortion, however, it runs slowly. Angenent et al. [2] proposed another PDE-based method (Laplace-Beltrami operator) that is much faster than CirclePack but cannot handle cortical patches except by mapping them into a rectangle. Gu and Yau [5] recently suggested a new method for computing conformal structures by minimizing the harmonic energy iteratively but it is still computationally expensive.

In this paper, we first apply the so-called least square conformal mapping (LSCM) method introduced in [7] to flatten cortical patches, then generalize it to produce spherical conformal maps of the entire cortex while minimizing metric distortion within the class of conformal maps. We will also compare the performance of our method with FreeSurfer and CARET.

II. DISCRETE CONFORMAL MAPPING

A conformal mapping of a Riemannian surface to another one is a continuous one-to-one function that preserves all angular measures locally, i.e., locally isotropic.

II-A. Planar conformal map using LSCM

The least square conformal mapping is a planar quasi-conformal parameterization method based on a least-square approximation of the Cauchy-Reimann equation. We give a brief description below, see [7] for details.

Let $\mathcal{K}$ represent a simply-connected triangulated surface of topological disc

$$\left\{ (v_1)^n_{i=1}, T = \{ (v_{i1}, v_{i2}, v_{i3}) \}^n_{i=1} \right\}$$

where $\{v_i\}^n_{i=1}$ is a set of $n$ vertices with $n \geq 3$ and $T$ is a set of $m$ triangles consisting of triples of vertices. We assume that $\mathcal{K}$ is consistently oriented, then each triangle of $T$ has a uniquely defined normal. Furthermore, each triangle can be imposed a local orthonormal basis $(x, y)$ with the normal along the $z$-axis.
Now we consider a smooth mapping \( \mathcal{U} \) from \( \mathcal{K} \) to \( \mathbb{R}^2 \). When restricting \( \mathcal{U} \) on one of the triangles of \( \mathcal{T} \), say \( T \), according to the above assumptions, we could write \( \mathcal{U} \) in the following form:

\[
\mathcal{U}|_T : (x, y) \rightarrow (u, v),
\]

e.g., \( U(x + iy) = u + iv \). The Cauchy-Riemann equation says that \( \mathcal{U} \) is conformal on \( T \) if and only if the following equality

\[
\frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} = 0
\]

holds true on the whole \( T \). Clearly, this conformal condition generally cannot be strictly satisfied on the whole triangulated surface \( \mathcal{K} \), so the minimization of the violation of this condition was suggested in [7] to construct the quasi-conformal map in the least square sense:

\[
\min_{\mathcal{U}} C(\mathcal{K}) = \sum_{T \in \mathcal{T}} \int_T \left( \frac{\partial U}{\partial x} + i \frac{\partial U}{\partial y} \right)^2 dA
\]

(2)

II-B. Some measurements of distortion

Since our final goal is to analyze the brain imaging data using cortical surface flattening as a tool, we have to consider the quality of resulting flat maps. Consequently, we need a uniform way to measure the distortion between the original cortical surface and its corresponding flat map.

The angular distortion is defined by the following:

\[
F_{\text{ang}}(\mathcal{U}) = \frac{1}{3m} \sum_{F \in \text{faces}(\mathcal{K})} \left( |\theta_{ijk}^U - \theta_{ijk}| + |\theta_{jki}^U - \theta_{jki}| + |\theta_{ikj}^U - \theta_{ikj}| \right)
\]

(6)

where \( \theta_{ijk} \) and \( \theta_{ijk}^U \) denote the angles \( \angle v_i v_j v_k \) and \( \angle \mathcal{U}(v_i) \mathcal{U}(v_j) \mathcal{U}(v_k) \) respectively. Although a conformal mapping from our piecewise flat surface to a planar region preserves the “market share” of angles at vertices [6], our definition for angular distortion is still valid since the cortical surface is almost flat within local neighborhood of triangles.

Before defining the metric distortion, we must deal with the computation of geodesic distances on the triangulated cortical surface \( \mathcal{K} \). For each vertex, we label each of its nearest neighbors as a 1-neighbor, then we label each neighbor of a 1-neighbor that is not already labeled as a 2-neighbor. Repeating this process, we could define \( k \)-neighbors for each vertex. Denote by \( d(v_i, v_j) \) and \( d_{il}(v_i, v_j) \) the geodesic distances between the vertex \( v_i \) and \( v_j \) on \( \mathcal{K} \) and its flat map \( \mathcal{U}(\mathcal{K}) \) respectively. There are many practical algorithms to compute \( d(v_i, v_j) \), we take the one proposed in [4], which employs the Dijkstra algorithm and requires dynamic programming due to the memory restriction.

Then, the metric distortion is defined by the following:

\[
F_{\text{met}}(\mathcal{U}) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{N(i)} \sum_{j \in N(i)} |d_{ij} - d_{ij}|/d_{ij} \right)
\]

(7)

where \( N(i) \) denotes the set of vertices which are predefined neighbors of vertex \( i \), \( d_{ij} \) and \( d_{ij}^U \) are geodesic distances between \( v_i \) and \( v_j \) normalized by the sum of length of all edges on \( \mathcal{K} \) and its flat map \( \mathcal{U}(\mathcal{K}) \) respectively to avoid the influence of similarity transformations.

II-C. Spherical conformal map

In the former Section II-A, the LSCM approach for flattening a surface of topological disc into a planar region has been discussed, we are now left with another problem: how to generate its discrete spherical conformal map if the triangulated surface \( \mathcal{K} \) is a topological sphere.

Our spherical conformal mapping \( \mathcal{U} \) from \( \mathcal{K} \) to \( S^2 \) proceeds via the map for discs using a trick. First, an
arbitrary vertex \( \mathbf{v}^* \) chosen from \( \{ \mathbf{v}_i \}_{i=1}^n \) and all edges containing it are removed from the input triangulated mesh \( \mathcal{K} \), clearly, the pruned mesh \( \mathcal{K}' \) becomes a topological disc. Then we generate the conformal planar map \( \mathcal{U}' \) of the \( \mathcal{K}' \) using the LSCM approach. Finally, the map \( \mathcal{U}' \) is stereographically projected onto the unit sphere \( S^2 \) while \( \mathbf{v}^* \) is mapped to the “north pole” of \( S^2 \); i.e., \( \forall \mathbf{v} \in \{ \mathbf{v}_i \}_{i=1}^n, \)

\[
\mathcal{U}(\mathbf{v}) = \begin{cases} 
(0,0,1), & \mathbf{v} = \mathbf{v}^*, \\
\mathcal{P}(\mathcal{U}'(\mathbf{v})), & \text{otherwise}
\end{cases}
\]

where \( \mathcal{P} \) is the stereographic projection. While such a process does give us a spherical conformal map of \( \mathcal{K} \), the spherical conformal maps of \( \mathcal{K} \) are not unique since the sphere \( S^2 \) has a rich group of automorphisms, i.e., one-to-one conformal maps from \( S^2 \) to itself. The automorphism group of \( S^2 \), \( \text{Aut}(S^2) \) is the group of all Mobius transformations of \( S^2 \), i.e.,

\[
\text{Aut}(S^2) = \{ \psi : \psi(z) = \frac{az + b}{cz + d} | a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \}.
\]

Since the above process may also produce a highly distorted spherical map, our idea here is to normalize the map \( \mathcal{U} \) by an automorphism \( \psi \in \text{Aut}(S^2) \) to minimize the metric distortion defined in (7), i.e., we need to solve the following minimization problem:

\[
\min_{\psi \in \text{Aut}(S^2)} \mathcal{F}_{\text{met}}(\mathcal{U}^*(\mathcal{K})) = \mathcal{F}_{\text{met}}(\mathcal{P}\psi\mathcal{P}^{-1}\mathcal{U}(\mathcal{K})) \tag{8}
\]

This a very small-scale nonlinear optimization problem, and we also want to mention several points:

- If we fix \( \mathcal{U}'(\mathbf{v}^*) = (0,0,1) \) and get rid of the rotation influence, then \( \psi \) could be simplified to: \( \psi(z) = \frac{az + b}{cz + d} \) where \( a \in \mathbb{R}, b \in \mathbb{C} \). Then it is only a three-parameter minimization problem.
- No derivative information is available since it is very complicated to compute. The Powell Method can be used to solve this minimization problem. Some global searching methods may produce better results.
- Since our conformal maps preserve angles locally, it is very reasonable to use only 1-neighbor to define \( N(i) \) in \( \mathcal{F}_{\text{met}}, \) that will greatly reduce the computation time and not affect the resulting map much.

III. CORTICAL FLAT MAPS

We illustrate our method by flattening a cerebellar cortex and a cerebral hemisphere, and comparing the maps with those produced by the popular flattening softwares, FreeSurfer and CARET.

III-A. Cerebellar cortex

We first chose to flatten a cerebellum which was extracted from a high-resolution T1-weighted MRI volume. Cortical regions defined by different lobes and fissures were colored for identification purposes. The parcellated surface consists of 56,676 triangles and 28,340 vertices and is equivalent to a topological sphere, see Fig. 1-A. As discussed in Section II-C, for each vertex \( \mathbf{v}_i \), \( N(i) \) was set to be all its 1-neighbors during the minimization process (8). The spherical conformal map of the entire cerebellar cortex was then created using our method and shown in Fig. 1-B.

The lobe IV and V patch (Fig.1-C) was cut off from the cerebellar cortex forming a topological disc, then its planar conformal map was generated with only two vertices pre-pinned (Fig.1-D). This patch has 7,482 triangles and 3,903 vertices with 322 vertices on the boundary. It is easy to see that the shape of the patch seems to be well-preserved. We also would like to point out that the patch size in general should not be too large in order to obtain a planar map with low metric distortion; otherwise, artificial cuts should be made for the patch.

In addition, we also pre-decided a planar region, in particular, we chose an ellipse such as \( \mathcal{E} = \{ (u,v) \in \mathbb{R}^2 | u^2 + v^2/4.2^2 = 1 \} \) and fixed a homeomorphism \( g : \partial \mathcal{K} \rightarrow \partial \mathcal{E} \) using the following procedure: The boundary vertices of \( \mathcal{U}(\mathcal{K}) \), were positioned on the boundary of \( \mathcal{E} \) such that the sides subtend angles basically proportional to the length of the boundary edges of \( \mathcal{K} \) joining the corresponding corners. Then we flattened the lobe IV and V patch in the region \( \mathcal{E} \) (Fig.1-E). Of course, some conformality will be lost, but we gain control of the shape. The CPU time and measurements of distortion for the above flat maps are reported in Table I. Note that the metric distortion is computed by setting \( N(i) \) to be all neighbors up to the 5th level.

III-B. Cerebral hemisphere

The parcellated surface of a left cerebral hemisphere was created with 383,444 triangles and 191,724 vertices, see Fig. 2 (left) for the colored lobar map. The corresponding spherical conformal map is shown in Fig. 2 (right). The occipital lobe patch (Fig.3-A) was cut off from the cerebral cortex which is a topological disc, having 11,670 triangles and 5,918 vertices with 290 vertices on the boundary. As before, we first flattened the occipital lobe patch with only two vertices pinned in a plane (Fig.3-B), and then flattened it on a pre-defined planar region (Fig.3-C), in particular, the unit circle \( \mathcal{D} = \{ (u,v) \in \mathbb{R}^2 | u^2 + v^2 = 1 \} \). The CPU time and measurements of distortion are reported in Table I.

III-C. Comparisons with other approaches

FreeSurfer and CARET are two commonly used softwares for flattening cortical surfaces currently. In order to evaluate the performance of our LSCM method, we
IV. DISCUSSION AND CONCLUSION

Our discrete conformal method can be applied to either user-defined patches or to an entire cortical surface and it truly preserves local angular information in both cases. Metric distortion is a little worse than that produced by FreeSurfer and CARET, but computation time is greatly reduced relative to both.

V. REFERENCES


Table I. CPU time and angular and metric distortion produced by the LSCM method. A. spherical map of the cerebellar cortex or the cortex of the left cerebral hemisphere; B. planar map of a cortical patch when two vertices are fixed; C. planar map of the same patch when all boundary vertices are pre-defined.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Cerebellar Cortex</th>
<th>Cerebral Cortex</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPU Time (hr.)</td>
<td>FreeSurfer</td>
<td>CARET</td>
</tr>
<tr>
<td>Angular Distortion</td>
<td>15.33°</td>
<td>21.22°</td>
</tr>
<tr>
<td>Metric Distortion</td>
<td>21.59%</td>
<td>31.72%</td>
</tr>
</tbody>
</table>

Table II. CPU time and angular and metric distortion of spherical maps produced by CARET and FreeSurfer.

also flattened the above cerebellar and cerebral cortices using FreeSurfer and CARET. The computational results are reported in Table II.