Characterizing Cortical Folding Patterns Across Species using Prolate Spheroidal Harmonics

D A. Smith, M. K. Hurdal
dsmith@math.fsu.edu
Department of Mathematics, Florida State University, Tallahassee, Florida 32306-4510

Introduction
Folding patterns of the cerebral cortex have fascinated many generations of scientists. In the past ten years, with the discovery of intermediate progenitor (IP) cells, new hypotheses about how these patterns develop have been introduced. Here, we adopt the intermediate progenitor cell hypothesis [1] (IPH) and develop a new model of the production of IP cells using a Turing reaction-diffusion system. The domain for this model is a prolate spheroidal surface. We then expand the theory to be able to predict evolving patterns on a prolate spheroidal surface. Next, we examine the role that focal distance plays in pattern formation and use it to predict the evolutionary development of cortical patterns in different species.

1. Cortical Development
A founding population of radial glial cells (R) is created in the ventricular zone (VZ) and迁移到the lateral wall of the lateral ventricle (Lateral Ventricle).

Radial glial cells (R) go through cycles of asymmetric cellular divisions creating neurons (N) that travel to the cortex creating the lower layers of the cortical plate (CP).

2. Turing patterns on a one-dimensional domain

BVM Model [1] (Barros, Varela, and Maini)

$u_t = D_u \frac{\partial^2 u}{\partial x^2} + r (1 - u) v + (u - a)$
$v_t = D_v \frac{\partial^2 v}{\partial x^2} + \beta (1 - \frac{u}{\theta}) v - \gamma v$

Solutions to $u$ are of the form $u(x,t) = A \exp(d_1 x + d_2 t) + B \exp(-d_1 x - d_2 t)$.

In one dimension, where $0 < c = p$, with a periodic boundary condition the solution is $u(x,t) = u_0 + u_1 \cos(k x) + u_2 \sin(k x)$, where $n$ is an integer.

Literature cited


3. Turing Patterns on Prolate Spheroidal Surfaces
Since $\lambda(x)$ is separable in prolate spheroidal coordinates ($\xi$, $\eta$) [6], we can rewrite $u(x,t)$ in terms of

$x = \lambda(x) \eta \cos(\xi) + \lambda(x) \eta \sin(\xi)$

we obtain an equation (1) that relates $k^2$ to the surface area $\lambda(x) \eta$.

3.1. Computer Modeling of Prolate Spheroidal Surfaces

Computer modeling verification of spherical node equation (Equation (3b, a))

Graph of $k^2 \lambda(x) \eta = 4 \cos(\xi)$ for $\xi = \pi$. (black) and where $\lambda(x) \eta = 4 \sin(\xi)$ (green), $\lambda(x) \eta = 4 \cos(\xi)$ (blue), $\lambda(x) \eta = 4 \sin(\xi)$ (red) are different colors and $\lambda(x) \eta = 4$ (dashed line) for different $\lambda(x) \eta$.

4. Simulations on a Prolate Spheroidal Surface

With a fixed focal distance, $d_{f}(\pi)$, the right hand side of (1) can be plotted versus $d$ for each $\lambda(x) \eta$.

Below are two simulations on a prolate spherical with focal distance of one and parameters given below. The top simulation has a domain scaling corresponding to $d_{f} = 0.899$. Notice, $d_{f} = 0.899$ (lower star) corresponds to $A_{f}$ and verifies the pattern obtained. The bottom simulation corresponds to $d_{f} = 1$ (top star) and verifies the pattern predicted of $A_{f}$. The result of discretization of BVM system for $d_{f} = 0.899$, $A_{f} = 0.5319$, $\alpha = -0.899$, $\gamma = 0.013$ (corresponding to $d_{f} = 0.899$).

Evolutionarily the cerebral cortex and lateral ventricle have expanded greatly. The expansion of the lateral ventricle is captured with the focal distance ($d$) of the prolate spherical. Here we are plotting the surface area of $A_{f} = 4$ and $A_{f} = 0.5319$, the pattern $A_{f}$ corresponds to the pattern needed to create one sulcus sectorially (following along the major axis of the lateral ventricle) and $A_{f}$ corresponds to 2 sulcal maps.

Notice in Graph A as $k^2$ increases the first pattern it will intersect is $A_{f}$. If $d_{f}$ is increased (B), the second curve shifts and now if $k^2$ increases it will intersect $A_{f}$ before $A_{f}$, therefore a one sulcal ring could form before a sectorial sulcus. If $d_{f}$ is increased further (C), as $k^2$ increases it will intersect $A_{f}$ before $A_{f}$ meaning 2 ring sulci could form before the first sectorial sulcus. These scenarios are similar to what occurs in the cat, mouse, and human.

5. Evolutionary Development
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For further information please contact dsmith@math.fsu.edu.

www.math.fsu.edu/~mhurdal

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