Quasi-Conformally Flat Mapping The Human Cerebellum

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Abstract

We present a novel approach to creating flat maps of the brain. Our approach attempts to preserve the conformal structure between the original cortical surface in 3-space and the flattened surface. We demonstrate this with data from the human cerebellum. Our maps exhibit quasi-conformal behavior and offer several advantages over existing approaches.

Introduction

- The convoluted surface of the brain, fold complexity and anatomical variability make it difficult to compare anatomical and functional information within and between subjects.
- Current visualization techniques (such as projecting functional data onto a rendered cortical surface) make it difficult to compare the location and extent of activated foci. For example, foci buried in deep sulci may appear on the cortical surface and widely separated foci on opposite walls of a sulcus may appear to be close together.

Surface Flattening

- The surface representing the cortical grey matter is topologically equivalent to a two-dimensional sheet.
- Thus, it is possible to unfold or flatten this surface to create a 2D flat map of the cortex.
- This surface-based approach can assist in visualizing and comparing cortical folding patterns and help to resolve some of the problems which exist in traditional visualization techniques.
- However, it is impossible to flatten a curved surface in 3D space without metric and areal distortion.
- The Riemann Mapping Theorem [1] implies that it is theoretically possible to preserve conformal (angular) information under flattening.

Quasi-Conformal Flattening

- Begin with a a piecewise flat triangulated surface in 3D space that is topologically a 2D disk:
  - ⇨ each edge of the triangluated mesh is either an interior edge contained in exactly two triangles or a boundary edge contained in exactly one triangle;
  - ⇨ there is one boundary component which is a single closed chain of boundary edges forming the boundary of the surface.
- A piecewise flat triangulated surface representing the cortex carries the structure of a Riemann surface [2]:
  - ⇨ the measure of an angle based at a point other than one of the triangle vertices is the Euclidean measure of that angle;
  - ⇨ the measure of an angle based at a vertex is the Euclidean measure linearly rescaled so the total angle measure is 2π;
  - ⇨ a triangle vertex *v* belongs to k triangle faces, giving k angles at *v*;
  - ⇨ the angle sum $\Theta(v)$ about the vertex *v* is the sum of these these Euclidean angles about *v*;
  - ⇨ the scale factor used at v for measuring angles is then $2\pi/\Theta(v)$;
  - ⇨ a vertex *v* with Euclidean measure θ has measure $2\pi\theta/\Theta(v)$ in the complex atlas of the Riemann surface.

The Riemann Mapping Theorem (1854):

If *D* is any simply-connected open set on a surface with a distinguished point $a \in D$ and a specified direction (tangent vector) through *a*, then there is ***a UNIQUE conformal map*** (1-1 bijection) which takes *D* to the interior of the unit circle Ω in the plane, with $a \longrightarrow 0$ and the specified direction pointed in the positive *X* direction [2] (see Figure 1).

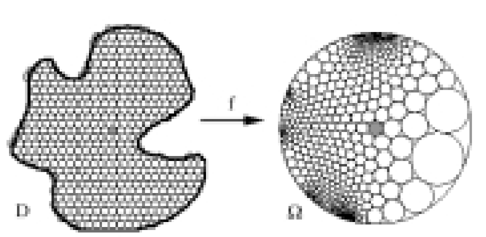

Figure 1: Example of a conformal map.

CONFORMAL MAPPINGS EXIST: a conformal map preserves angle proportions.

Approximating Conformal Mappings using Circle Packings

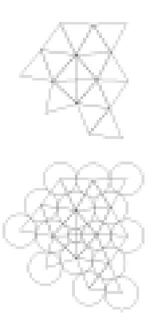

Figure 2: A triangulation (top) and its resulting circle packing (bottom).

- A circle packing ***is a configuration of circles with a specified pattern of tangencies*** (see Figure 2).
- Given a piecewise flat triangulation *K* of a surface, a circle packing for *K* is a collection of circles in the plane where:
  - ⇨ there is one circle *C(v)* for each vertex *v* of *K*;
  - ⇨ circles *C(v)* and *C(w)* are tangent when the vertices *v* and *w* form an edge of *K*.

The Circle Packing Theorem [3] states that given such a triangulation of *K* and any assignment of positive numbers $r(v_1),\dots,r(v_n)$ to the n boundary vertices $v_1,\dots,v_n$ of *K*, then there is a unique (up to Euclidean isometry) circle packing in the plane with boundary circle $C(v_i)$ have radius $r(v_i)$, for $i = 1,\dots,n$.

The Ring Lemma [4] guarantees that this circle packing mapping is quasi-conformal, meaning there is a bounded amount of angular distortion.

- An initial circle packing of a surface is obtained by using the combinatorial data from the surface contained in its abstract triangulation.
- A set of circles can be "flattened" so that the circles fit together in the plane if $\Theta(v) = 2\pi$.
- To "flatten" a surface at the interior vertices:
  - ⇨ when $\Theta(v) < 2\pi$ there is positive curvature (cone point) which is represented by a gap between the circles around *v*, so shrink the circles around v until all the circles around *v* are tangent;
  - ⇨ when $\Theta(v) = 2\pi$ there is zero curvature and the map is already flat at *v*;
  - ⇨ when $\Theta(v) > 2\pi$ there is negative curvature (saddle point) which is represented by overlapping circles in the plane, so expand the circles around *v* until all the circles around *v* are tangent (see Figure 3).

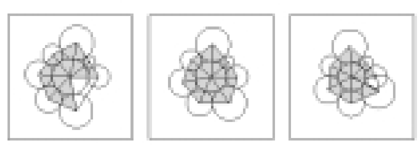

Figure 3: Example of how positive curvature (left) and negative curvature (right) are represented in the plane and how the circles need to be adjusted to achieve a circle packing (middle).

- Iteratively repeat this procedure at each vertex to obtain a circle packing.
- Closer approximations to the actual conformal mappings can be obtained by:
  - ⇨ if vertices *v* and *w* are connected by an edge, then the distance between circles *C(v)* and *C(w)* is given by the inversive distance (a conformal invariant of circle pairs) between *v* and *w* so that inversive distance rather than circle tangency is preserved;
  - ⇨ refining the triangles in the surface by subdividing each triangle into smaller triangles and repeating the above procedure until the map converges to the unique conformal map [5].
- In a similar manner, hyperbolic radii, lines and triangles can be used to create maps in the hyperbolic plane.
  - ⇨ Hyperbolic maps near the origin have little distortion and regions near the map border are greatly distorted.
  - ⇨ The focus of the hyperbolic map can be interactively moved to change the regions which are in focus.

Flattening the Cerebellum

- We chose to flatten the cerebellum in an effort to facilitate the description of activated cerebellar foci in functional neuroimaging.
- The cerebellum was extracted from a high resolution MRI scan [6] and a topologically correct triangulated surface representing the cerebellar surface was produced (see Figures 4 and 5). Regions were colored to correspond to various lobes and fissures.

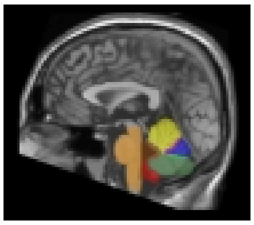

Figure 4: Cerebellum extracted from MRI scan.

- A single closed boundary was introduced around the brain stem and along the walls and apex of the fourth ventricle to act as the boundary of the flattened maps.
- Our quasi-conformal flattening procedure was applied to this surface to produce a number of flat maps [7] (see Figure 5).
- PET data from a target interception experiment can be displayed on the maps.

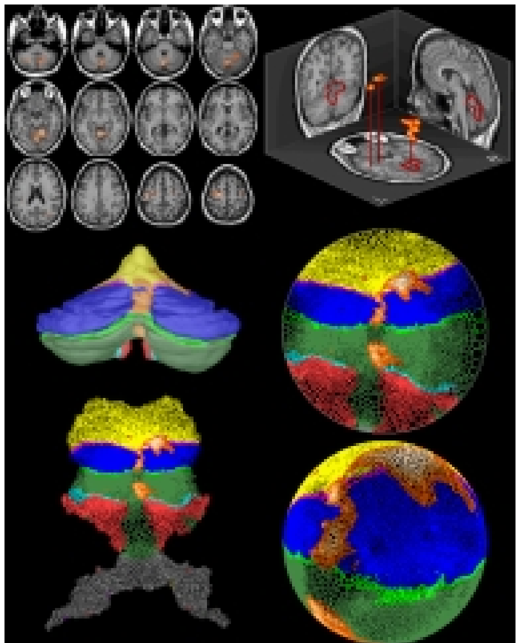

Figure 5: A triangulated surface representing the cerbellum was produced from a high-resolution MRI (middle left). This surface was flattened using our quasi-conformal approach to produce maps in the Euclidean plane (bottom left), hyperbolic plane (middle right) and on a sphere (bottom right). Functional PET data from a target interception experiment (top) was superimposed on these to display the activation on the maps.

Discussion: Advantages of Our Approach

- We have introduced a novel approach for unfolding and flattening the cortical surface which attempts to preserve the conformal structure of the original surface.
- Conformal mappings control and minimize angular distortion.
- Conformal mappings are canonical and hence mathematically unique.
- Extraneous cuts in the surface are not required to reduce distortion.
- Flattening can be done in the Euclidean and hyperbolic planes and mapping to a sphere is also possible.
- A coordinate system can be easily imposed on the maps by specifying two anatomical landmarks.
- The map origin of the hyperbolic maps can be transformed interactively to alter the location of map distortion.
- Fast compution times (minutes) with real time user interaction.

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