1. a) \(n = 2, a = 8, f(x) \approx T_2(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{9(2)^3}(x - 8)^2\)

b) For \(n = 2\), \(|R_n2(x)| = |R_2(x)| \leq \frac{M}{(3)!}|x - a|^{(3)}\) where \(M \geq |f^{(n+1)}(x)| = |f^3(x)| = \frac{10}{27x^{8/3}}\).

For \(7 \leq x \leq 9\), \(M\) is a maximum at \(x = 7\), giving

\[
M = |f'''(7)| = \frac{10}{27(7)^{8/3}} \quad \text{and so} \quad |R_2(x)| \leq \frac{10}{27(7)^{8/3}} \cdot \frac{1}{3!}|x - 8|^3.
\]

Observe that for \(7 \leq x \leq 9\), \(|R_2(x)|\) is a maximum at \(x = 9\) (or \(x = 7\)). Thus,

\[
|R_2(x)| \leq |R_2(9)| = \frac{10}{27(7)^{8/3}} \cdot \frac{1}{3!}|9 - 8|^3 \approx 3.442629 \times 10^{-4} \quad \text{for} \quad 7 \leq x \leq 9.
\]

2. After applying ratio test, convergence occurs when \(|x| < 5\). Radius of convergence = 5.

Thus, the series converges (at least) for the interval \(-5 < x < 5\). Test the interval endpoints for convergence/divergence:

At \(x = -5\), the series converges because it is a \(p\)-series with \(p = 2\).

At \(x = 5\), the series converges by the alternating series test. (Reminder: be sure to show the conditions for the alternating series test hold.)

Therefore, the interval of convergence is \([-5, 5]\).

3. a) The series diverges by the root test. Alternatively, the series diverges by the test for divergence.

b) The series diverges by the test for divergence.

4. Let \(f(x) = \frac{1}{x(\ln x)^2}\).

a) Verify the conditions for the integral test hold and test convergence.

Condition i): \(f(x)\) is continuous for all \(x > 1\).

Condition ii): \(f(x)\) is positive as \(f(x) > 0\) for all \(x > 1\),

Condition iii): \(f'(x) = -\frac{(\ln x)^2 + 2\ln x}{x^2(\ln x)^4} < 0\) for all \(x > 1\) and so \(f(x)\) is decreasing for all \(x > 1\).

All conditions are satisfied. Therefore the integral test and its remainder estimate applies.

\[
R_n \leq \int_n^\infty f(x) \, dx \leq 0.1, \quad \text{and so} \quad \frac{1}{\ln n} \leq 0.1, \quad \text{giving} \quad n \geq e^{10} \quad \Rightarrow n = 22027.
\]

5. \(A = \int_0^{\pi/2} \cos x \, dx = 1; \quad \overline{x} = \frac{1}{A} \int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1; \quad \overline{y} = \frac{1}{A} \int_0^{\pi/2} \frac{1}{2}(\cos x)^2 \, dx = \frac{\pi}{8}.
\)

Thus, \((\overline{x}, \overline{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)\).
6. a) \[ A = \int_0^{1/2} \sqrt{1 - 4x^2} \, dx = \frac{\pi}{8} \]

b) \[ A \approx S_6 = \left( \frac{1}{8} \right) \left( \frac{1}{3} \right) \left[ f(0) + 4f\left( \frac{1}{8} \right) + 2f\left( \frac{1}{4} \right) + 4f\left( \frac{3}{8} \right) + f\left( \frac{1}{2} \right) \right] \]
where \( f(x) = \sqrt{1 - 4x^2} \)
\[ = \frac{1}{24} \left( 1 + \sqrt{15} + \sqrt{3} + \sqrt{7} \right) \]
\[ \approx 0.385449 \]

7. \[ \frac{y}{1-y} = Ke^x \] where \( K = \pm e^C \). Using \( y(0) = \frac{1}{2} \) gives \( K = 1 \). Thus, \[ \frac{y}{1-y} = e^x \]. One bonus point was awarded if the solution was rearranged to express \( y \) as \[ y = \frac{e^x}{1 + e^x} \].

8. a) \[ L = \int_0^\pi \sqrt{(2 \cos \theta)^2 + (-2 \sin \theta)^2} \, d\theta = 2\pi \]

b) Equation of the curve is \( x^2 + (y - 1)^2 = 1 \), i.e. a circle centered at \((0, 1)\) with radius 1.

Bonus:
Expand \( e^{-x^2} \) as an infinite series by manipulating the Maclaurin series of \( e^x \), then integrate term by term.

By definition of a Maclaurin series for \( e^x \),
\[ e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots. \quad (1) \]

To obtain a series for \( e^{-x^2} \), substitute \(-x^2\) for \( x \) into Equation (1), giving
\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots. \quad (2) \]

Integrating Equation (2) gives
\[ \int e^{-x^2} = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \, dx \]
\[ = \int \left( 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots \right) \, dx \]
\[ = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots + C \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n + 1)} + C. \]