Example from Section 19.3, number 2 from homework problems.
Compute the flux of the vector field $\vec{F} = y\vec{\imath} + x\vec{\jmath}$ through the surface S where S is oriented away from the z-axis and S is given by $x = 3\sin(s)$, $y = 3\cos(s)$, $z = t + 1$ for $0 \leq s \leq \pi$, $0 \leq t \leq 1$.

Solution
Note that S is a cylinder of radius 3 along the z-axis for z between 1 and 2.

We want to compute

$$\begin{aligned} \text{flux} &= \int_S \vec{F} \cdot d\vec{A} \\ &= \int_S \vec{F} \cdot \vec{n} dA \\ &= \int\int_T \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) dsdt. \end{aligned}$$

First, find $\vec{r}$ in terms of s and t using the given parameterization:

$$\begin{aligned} \vec{r} &= x\vec{\imath} + y\vec{\jmath} + z\vec{k} \\ &= 3\sin(s)\vec{\imath} + 3\cos(s)\vec{\jmath} + (t+1)\vec{k}. \end{aligned}$$

The partial derivatives of $\vec{r}$ are then

$$\frac{\partial \vec{r}}{\partial s} = 3\cos(s)\vec{\imath} - 3\sin(s)\vec{\jmath} \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial t} = \vec{k}.$$

Computing the cross product gives the normal vector:

$$\begin{aligned} \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3\cos(s) & -3\sin(s) & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \vec{\imath}(-3\sin(s) - 0) - \vec{\jmath}(3\cos(s) - 0) + \vec{k}(0 - 0) \\ &= -3\sin(s)\vec{\imath} - 3\cos(s)\vec{\jmath}. \end{aligned}$$

Note that the x and y components are negative, which means the normal vector, $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ points toward the inside of the cylinder. Since we want the normal to point outward, or away from the surface (as specified in the problem), we change the direction of the normal to get

$$\vec{n} = -\left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = 3\sin(s)\vec{\imath} + 3\cos(s)\vec{\jmath}.$$

We now need to write the vector field $\vec{F}$ in terms of s and t. So, $\vec{F}(x,y) = y\vec{\imath} + x\vec{\jmath}$ becomes $\vec{F}(\vec{r}(s,t)) = 3\cos(s)\vec{\imath} + 3sin(s)\vec{\jmath}$.

Finally,

$$
\begin{aligned}
\text{flux} &= \int_S \vec{F} \cdot d\vec{A} \\
&= \int_S \vec{F} \cdot \vec{n} dA \\
&= \int\int_T \vec{F}(\vec{r}(s,t)) \cdot \left(-\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) dsdt \quad \text{for } S \text{ to be oriented out-wards in this problem} \\
&= \int\int_T (3\cos(s)\vec{\imath} + 3sin(s)\vec{\jmath}) \cdot (3\sin(s)\vec{\imath} + 3\cos(s)\vec{\jmath}) dsdt \\
&= \int_{t=0}^{1} \int_{s=0}^{\pi} 18\sin(s)\cos(s) dsdt \\
&= \int_{t=0}^{1} 9\sin^2(s)\Big|_{s=0}^{\pi} dt \quad \text{by using substitution to integrate with respect to } s \\
&= 0.
\end{aligned}
$$