The Uncertainty Principle in Fourier Analysis

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The Uncertainty Principle is the following vague statement: a function and its Fourier transform can not both be concentrated on small sets. The famous Heysenberg Uncertainty Principle in Quantum Mechanics is a particular version of this general principle. We consider the following type of the Uncertainty Principle: $\int |f|^2 \leq C(\int_{E^c} |f|^2 + \int_{\Sigma^c} |\hat{f}|^2)$ where C does not depend on $f \in L^2(\mathbb{R}^d)$ and E and Σ are "small" sets in $\mathbb{R}^d$ and, in particular, it implies that if f is supported on E and $\hat{f}$ is supported on Σ then $f \equiv 0$. The challenging part is to find such pair of sets E and Σ. There are three main results of this type: the Amrein-Berthier theorem where E and Σ are sets of finite Lebesgue meausre, the Logvinenko-Sereda theorem where Σ is a ball and E^c is relatively dense and Wolff's theorem where E and Σ are so called ϵ-thin sets. The last two theorems have numerous applications for PDE. We obtain a new version of the Uncertainty Principle which links the last two theorems by introducing a new notion of density for sets E and Σ. The obtained result is sharp. We apply the new result to estimate certain operators on $L^2(\mathbb{R}^d)$.