HOMOLOGY MANIFOLDS

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The study of the local-global geometric topology of homology manifolds has a long history. Homology manifolds were introduced in the 1930s in attempts to identify local homological properties that implied the duality theorems satisfied by manifolds [25, 57]. Bing's work on decomposition space theory opened new perspectives. He constructed important examples of 3-dimensional homology manifolds with non-manifold points, which led to the study of other structural properties of these spaces, and also established his *shrinking criterion* that can be used to determine when homology manifolds obtained as decomposition spaces of manifolds are manifolds [4]. In the 1970s, the fundamental work of Cannon and Edwards on the double suspension problem led Cannon to propose a conjecture on the nature of manifolds, and generated a program that culminated with the Edwards-Quinn characterization of higher-dimensional topological manifolds as ENR homology manifolds satisfying a weak general position property known as the disjoint disks property [17, 26, 23]. Starting with the work of Quinn [45, 47], a new viewpoint has emerged. Recent advances [10] use techniques of controlled topology to produce a wealth of previously unknown homology manifolds and to extend to these spaces the Browder-Novikov-Sullivan-Wall surgery classification of compact manifolds [54], suggesting a new role for these objects in geometric topology, and tying together two strands of manifold theory that have developed independently. In this article, we approach homology manifolds from this perspective. We present a summary of these developments and discuss some of what we consider to be among the pressing questions in the subject. For more detailed treatments, we refer the reader to article [10] by Bryant, Ferry, Mio and Weinberger, and the forthcoming lecture notes by Ferry [28]. The survey papers by Quinn [46] and Weinberger [55] offer overviews of these developments.

1. Early developments

Localized forms of global properties of topological spaces and continuous mappings often reveal richer structures than their global counterparts alone. The identification of these local properties and the study of their influence on the large scale structure of spaces and mappings have a history that dates back to the beginning of this century. Wilder's work [57] reflects the extensive study of local homology conducted by many authors, a line of investigation that has its roots in the search, started by Čech [22] and Lefschetz [39], for local homological conditions that implied the duality and separation properties known to be satisfied by triangulable manifolds.

Partially supported by NSF grant DMS-9626624.

Definition 1.1. A *topological n-manifold* is a separable metrizable space that is locally homeomorphic to euclidean n-space $\mathbb{R}^n$.

Early proofs that a closed oriented manifold M^n satisfies Poincaré duality assumed the existence of a triangulation of M [38, 43]. Orientability was defined as a global property of the triangulation, and the Poincaré duality isomorphism

$$\cap[M]\colon H^*(M;\mathbb{Z}) \to H_{n-*}(M;\mathbb{Z})$$

was established by analysing the pattern of intersection of simplices with "cells" of the dual block structure on M obtained from the triangulation.

If M is an n-manifold and $x \in M$, then x has arbitrarily small n-disk neighborhoods which have $(n-1)$-dimensional spheres as boundaries. By excision, homologically this local structure can be expressed as $H_*(M, M \setminus \{x\}) \cong H_*(D^n, S^{n-1})$, for every $x \in M$.

Definition 1.2. X is a *homology n-manifold* if for every $x \in X$

$$H_i(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$

The local homology groups $H_*(X, X \setminus \{x\})$ of these *generalized manifolds* can be used to define and localize the notion of orientation for these spaces, and to formulate proofs (at various degrees of generality) that compact oriented generalized manifolds satisfy Poincaré and Alexander duality. The reader may consult [25] for a historical account of these developments.

Topological manifolds are homology manifolds; however, the latter form a larger class of spaces. (As we shall see later, there are numerous homology manifolds without a single manifold point.) Spaces X satisfying the Poincaré duality isomorphism with respect to a fundamental class $[X] \in H_n(X)$ are called *Poincaré spaces* of formal dimension n. We thus have three distinct classes of spaces related by forgetful functors:

$$\begin{Bmatrix} \text{Topological} \\ \text{manifolds} \end{Bmatrix} \Longrightarrow \begin{Bmatrix} \text{Homology} \\ \text{manifolds} \end{Bmatrix} \Longrightarrow \begin{Bmatrix} \text{Poincaré} \\ \text{spaces} \end{Bmatrix}.$$

Classical surgery theory studies topological-manifold structures on Poincaré spaces [54]. Our discussion will be focused on the differences between topological and homology manifolds, a problem that is usually treated in two stages:

(i) determine whether or not a given homology manifold X is a "fine" quotient space of a topological manifold (we shall elaborate on this later), and
(ii) exhibit conditions under which a quotient space X of a manifold M is homeomorphic to M.

The latter is a central question in decomposition space theory, an area that originated with the work of Moore [41]. He proved that if X is Hausdorff and $f\colon S^2 \to X$ is a surjection such that $S^2 \setminus f^{-1}(x)$ is non-empty and connected, for every $x \in X$, then X is homeomorphic to S^2. This result is a precursor to the characterization of the 2-sphere in terms of separation properties obtained by Bing. If X is a compact, connected, locally connected metrizable space with more than one point, then X is homeomorphic to S^2 if and only if the complement of any two points in X is connected and the complement of any subspace of X homeomorphic to a circle is disconnected [3].

Bing's work on decompositions of 3-manifolds defined an important chapter in decomposition theory. While focused on the geometry of decompositions of low dimensional manifolds, his work was influential in subsequent developments in higher dimensions. Given a quotient map $f\colon M \to X$, exploiting the interplay between the local structure of X near points $x \in X$ and the local geometry of the embeddings $f^{-1}(x) \subseteq M$ of the corresponding point inverses, he constructed examples of generalized 3-manifolds with non-manifold points, which led to the first considerations of general position properties of generalized manifolds. Conversely, Bing's *shrinking criterion* uses the geometry of the point inverses of f to provide conditions under which the quotient space X is homeomorphic to M [4]. For metric spaces, the criterion can be stated as follows.

Theorem 1.3 (R. H. Bing). *A surjection $f\colon M \to X$ of compact metric spaces can be approximated by homeomorphisms if and only if for any $\epsilon > 0$, there is a homeomorphism $h\colon M \to M$ such that:*

(i) $d(f \circ h, f) < \epsilon$.
(ii) *diam* $h(f^{-1}(x)) < \epsilon$, *for every* $x \in X$.

Generalized manifolds also arise in the study of dynamics on manifolds. Applications of the shrinking criterion in low dimensions include the construction of a $\mathbb{Z}_2$-action on S^3 which is not topologically conjugate to a linear involution [4]. Smith theory [51, 7] implies that fixed points of topological semifree circle actions on manifolds are generalized manifolds, giving further early evidence of the relevance of these spaces in geometric topology.

2. The recognition problem

How can one decide whether or not a given topological space X is a manifold? A reference to the definition of manifolds simply reduces the question to a characterization of euclidean spaces, a problem of essentially the same complexity. The proposition that a characterization of higher dimensional manifolds in terms of their most accessible properties might be possible evolved from groundbreaking developments in decomposition space theory in the 1970s. We begin our discussion of the recognition problem with a list of basic characteristic properties of topological manifolds. For simplicity, we assume that X is compact, unless otherwise stated.

(i) *Manifolds are finite dimensional.*

Definition 2.1. The (covering) dimension of a topological space X is $\leq n$, if any open covering $\mathcal{U}$ of X has a refinement $\mathcal{V}$ such that any subcollection of $\mathcal{V}$ containing more than $(n+1)$ distinct elements has empty intersection. The *dimension* of X is n, if n is the least integer for which dimension of X is $\leq n$. If no such integer exists, X is said to be infinite dimensional.

Topological n-manifolds, and euclidean n-space $\mathbb{R}^n$ in particular, are examples of n-dimensional spaces.

(ii) *Local contractibility.*

Every point in a manifold has a contractible neighborhood. The following weaker notion of local contractibility is, however, a more manageable property.

Definition 2.2. X is *locally contractible* if for any $x \in X$ and any neighborhood U of x in X, there is a neighborhood V of x such that $V \subseteq U$ and V can be deformed to a point in U, i.e., the inclusion $V \subseteq U$ is nullhomotopic.

Absolute neighborhood retracts (ANR) are important examples of locally contractible spaces. (Recall that X is an ANR if there is an embedding of X as a closed subspace of the Hilbert cube I^∞ such that some neighborhood N of X retracts onto X.) Conversely, if X is finite dimensional and locally contractible, then X is an ANR [6]. Since any n-dimensional space can be properly embedded in $\mathbb{R}^{2n+1}$ [33], it follows that X is a finite dimensional locally contractible space if and only if X is an euclidean neighborhood retract (ENR). The definition of ENR is analogous to that of ANR with the Hilbert cube replaced by some euclidean space. Hence, conditions (i) and (ii) above can be elegantly summarized in the requirement that X be an ENR.

(iii) *Local homology.*

Topological n-manifolds are homology n-manifolds. The assumption that X is an ENR homology manifold encodes all separation properties satisfied by closed manifolds, since compact oriented ENR homology manifolds satisfy Poincaré and Alexander duality. (As usual, in the nonorientable case we twist homology using the orientation character.) Moreover, since the dimension of finite dimensional spaces can be detected homologically, ENR homology n-manifolds are n-dimensional spaces.

An ENR homology n-manifold X is an n-dimensional locally contractible space in which points have homologically spherical "links". Thus, to this hypothesis, it is necessary to incorporate a local fundamental group condition that will guarantee that "links" of points in X are homotopically spherical, as illustrated by the following classical example.

Let H^n be a homology n-sphere (i.e., a closed manifold such that $H_*(H;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$) with nontrivial fundamental group, and let $X = \Sigma H$ be the suspension of H. X is a simply connected homology manifold, but arbitrarily close to the suspension points there are loops α that are nontrivial in the complement of the suspension points. Therefore, X is not a manifold since any small punctured neighborhood of a suspension point is non-spherical. Nonetheless, an important result of Cannon establishes that the double suspension of H is a topological manifold [16]. Since any bounding disk D^2_α for the loop α must intersect one of the suspension points, the presence of a nontrivial local fundamental group can be interpreted as a failure of general position, if $n \geq 5$. D^2_α cannot be moved away from itself by small deformations.

(iv) *The disjoint disks property.*

Manifolds satisfy general position. If P^p and Q^q are complexes tamely embedded in a manifold M, under arbitrarily small perturbations, we can assume that $P \cap Q$ is tamely embedded in M and that $\dim(P \cap Q) \leq p+q-n$. In particular, if $n \geq 5$, 2-dimensional disks can be positioned away from each other by small moves.

Definition 2.3. X has the *disjoint disks property* (DDP) if for any $\epsilon > 0$, any pair of maps $f, g\colon D^2 \to X$ can be ϵ-approximated by maps with disjoint images.

The fact that the DDP is the appropriate general position hypothesis for the recognition problem became evident in Cannon's work on the double suspension problem. Later, Bryant showed that if X^n is an ENR homology manifold with the DDP, $n \geq 5$, then tame embeddings of complexes into X can be approximated by maps in general position [8].

ENR homology manifolds with the DDP have the local-global algebraic topology and general position properties of topological manifolds. In 1977, motivated largely by his solution of the double suspension problem, Cannon formulated the following conjecture [16, 17].

The characterization conjecture. *ENR homology n-manifolds with the disjoint disks property, $n \geq 5$, are topological n-manifolds.*

Definition 2.4. A mapping $f\colon M \to X$ of ENRs is *cell-like* (CE), if f is a proper surjection and for every $x \in X$, $f^{-1}(x)$ is contractible in any of its neighborhoods. A CE-map f is a *resolution* of X if M is a topological manifold.

All examples of ENR homology manifolds known at the time these developments were taking place could be obtained as cell-like quotients of topological manifolds. In addition, if M is a manifold and f is cell-like, then X is a homology manifold [37]. The fact that suspensions of homology spheres are resolvable follows from a theorem of Kervaire that states that homology spheres bound contractible manifolds [35].

The following result, of which the double suspension theorem is a special case, is a landmark in decomposition space theory [23, 26].

Theorem 2.5 (R. D. Edwards). *Let X^n be an ENR homology manifold with the DDP, $n \geq 5$. If $f\colon M \to X$ is a resolution of X, then f can be approximated by homeomorphisms.*

In light of Edwards' theorem, the completion of the manifold characterization program is reduced to the study of the following conjecture.

The resolution conjecture. *ENR homology manifolds of dimension ≥ 5 are resolvable.*

Early results supporting this conjecture assumed that the homology manifolds under consideration contained many manifold points. Cannon and Bryant-Lacher showed that X is resolvable if the dimension of the singular set of X is in the stable range [18]. Galewski and Stern proved that polyhedral homology manifolds are resolvable, so that non-resolvable homology manifolds, if they exist, must not be polyhedral [31].

A major advance toward the solution of the resolution conjecture is due to F. Quinn. He showed that the existence of resolutions can be traced to a single locally defined integral invariant that can be interpreted as an index [45, 47].

Theorem 2.6 (Quinn). *Let X be a connected ENR homology n-manifold, $n \geq 5$. There is an invariant $I(X) \in 8\mathbb{Z}+1$ such that:*

(a) *If $U \subseteq X$ is open, then $I(X) = I(U)$.*
(b) *$I(X \times Y) = I(X) \times I(Y)$.*
(c) *$I(X) = 1$ if and only if X is resolvable.*

Remark. The local character of Quinn's invariant implies that if X is connected and contains at least one manifold point, then X is resolvable. Thus, a non-resolvable ENR homology n-manifold, $n \geq 5$, cannot be a cell complex, since the interior of a top cell would consist of manifold points.

Combined, Theorems 2.5 and 2.6 yield the celebrated characterization of higher dimensional topological manifolds.

Theorem 2.7 (Edwards-Quinn). *Let X be an ENR homology n-manifold with the DDP, $n \geq 5$. X is a topological manifold if and only if $I(X) = 1$.*

The resolution conjecture, however, remained unsolved. Are there ENR homology manifolds with $I(X) \neq 1$?

3. Controlled surgery

This is a brief review of results of simply-connected controlled surgery theory needed in our discussion of the resolution problem. Proofs and further details can be found in [29, 30].

In classical surgery theory one studies the existence and uniqueness of manifold structures on a given Poincaré complex X^n of formal dimension n. Controlled surgery addresses an estimated form of this problem, when X is equipped with a map to a control space B. For simplicity, we assume that $\partial X = \emptyset$, although even in this case bounded versions are needed in considerations of uniqueness of structures.

Definition 3.1. Let $p\colon X \to B$ be a map to a metric space B and $\epsilon > 0$. A map $f\colon Y \to X$ is an *ϵ-homotopy equivalence* over B, if there exist a map $g\colon X \to Y$ and homotopies H_t from $g \circ f$ to 1_Y and K_t from $f \circ g$ to 1_X, respectively, such that the tracks of H and K are ϵ-small in B, i.e., $\operatorname{diam}(p \circ f \circ H_t(y)) < \epsilon$ for every $y \in Y$, and $\operatorname{diam}(p \circ K_t(x)) < \epsilon$, for every $x \in X$. The map $f\colon Y \to X$ is a *controlled equivalence* over B, if it is an ϵ-equivalence over B, for every $\epsilon > 0$.

In order to use surgery theory to produce ϵ-homotopy equivalences, we need the notion of ϵ-Poincaré spaces. Poincaré duality can be estimated by the diameter of cap product with a fundamental class as a chain homotopy equivalence.

Definition 3.2. Let $p\colon X \to B$ be a map, where X is a polyhedron and B is a metric space. X is an *ϵ-Poincaré complex of formal dimension n* over B if there exist a subdivision of X such that simplices have diameter $\ll \epsilon$ in B and an n-cycle y in the simplicial chains of X so that $\cap y\colon C^{\sharp}(X) \to C_{n-\sharp}(X)$ is an ϵ-chain homotopy equivalence in the sense that $\cap y$ and the chain homotopies have the property that the image of each generator σ only involves generators whose images under p are within an ϵ-neighborhood of $p(\sigma)$ in B.

The next definition encodes the fact that the local fundamental group of X is trivial from the viewpoint of the control space B.

Definition 3.3. A map $p\colon X \to B$ is UV^1 if for any $\epsilon > 0$, and any polyhedral pair (P, Q) with $\dim(P) \leq 2$, and any maps $\alpha_0\colon Q \to X$ and $\beta\colon P \to B$ such that $p \circ \alpha_0 = \beta|_Q$,

$$\begin{tikzcd} Q \arrow[r, "\alpha_0"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ P \arrow[r, "\beta"'] \arrow[ur, dashed, "\alpha"] & B \end{tikzcd}$$

there is a map $\alpha\colon P \to X$ extending α_0 so that $d(p \circ \alpha, \beta) < \epsilon$.

Remark. When both X and B are polyhedra and p is PL, this is the same as requiring that $p^{-1}(b)$ be simply connected, for every $b \in B$.

Definition 3.4. Let $p\colon X \to B$ be an ϵ-Poincaré complex over the metric space B, where p is UV^1. An ϵ-surgery problem over $p\colon X \to B$ is a degree-one normal map

$$\begin{tikzcd} \nu_M \arrow[r, "F"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

where ξ is a bundle over X, ν_M denotes the stable normal bundle of M, and F is a bundle map covering f.

Theorem 3.5. *Let B be a compact metric ENR and $n \geq 5$. There exist an $\epsilon_0 > 0$ and a function $T\colon (0, \epsilon_0] \to (0, \infty)$ satisfying $T(t) \geq t$ and $\lim_{t \to 0} T(t) = 0$, such that for any ϵ, $0 < \epsilon \leq \epsilon_0$, if $f\colon M \to X$ is an ϵ-surgery problem with respect to the UV^1 map $p\colon X \to B$, associated to the normal bordism class of f, there is an obstruction $\sigma_f \in H_n(B; \mathbb{L})$ which vanishes if and only if f is normally bordant to a $T(\epsilon)$-equivalence over B. Here, $H_n(B; \mathbb{L})$ denotes the nth generalized homology group of B with coefficients in the simply-connected periodic surgery spectrum.*

Theorem 3.5 requires that X be a polyhedron. Nonetheless, if X is an ENR homology n-manifold, a normal map $f\colon M \to X$ has a well-defined controlled surgery obstruction over B. Let U be a mapping cylinder neighborhood of X in a large euclidean space $\mathbb{R}^N$ with projection $\pi\colon U \to X$ [56, 44]. For any $\epsilon > 0$, U is an ϵ-Poincaré complex of formal dimension n over B under the control map $p \circ \pi\colon U \to B$ [11]. Hence, the composition $M \xrightarrow{f} X \subseteq U$ can be viewed as an ϵ-surgery problem f' over B. By Theorem 3.5, for each ϵ, $0 < \epsilon \leq \epsilon_0$, f' has a well-defined $T(\epsilon)$-surgery obstruction $\sigma_{f'} \in H_n(B; \mathbb{L})$ over B, where $\lim_{t \to 0} T(t) = 0$. The *controlled surgery obstruction* of f is defined by $\sigma_f = \sigma_{f'}$.

Theorem 3.6. *Let $p\colon X \to B$ be a UV^1 map, where X is a compact ENR homology n-manifold, $n \geq 5$. The controlled surgery obstruction $\sigma_f \in H_n(B; \mathbb{L})$ of $f\colon M \to X$ is well-defined, and σ_f vanishes if and only if, for any $\epsilon > 0$, f is normally bordant to an ϵ-homotopy equivalence over B.*

Definition 3.7. Let X be a compact ENR homology manifold, and let $p\colon X \to B$ be a control map. An ϵ-structure on $p\colon X \to B$ is an ϵ-homotopy equivalence $f\colon M \to X$ over B, where M is a closed manifold. Two ϵ-structures $f_i\colon M_i \to X$, $i \in \{1, 2\}$, are ϵ-equivalent if there is a homeomorphism $h\colon M_1 \to M_2$ such that

f_1 and $f_2 \circ h$ are ϵ-homotopic over B. The collection of ϵ-equivalence classes of ϵ-structures is denoted by $S_\epsilon \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix}$.

Given a Poincaré space X of formal dimension n, let $\mathcal{N}_n(X)$ denote the collection of normal bordism classes of degree-one normal maps to X.

Theorem 3.8. *Let X be a compact ENR homology n-manifold, $n \geq 5$, and let $p\colon X \to B$ be a UV^1 control map, where B is a compact metric ENR. There exist an $\epsilon_0 > 0$ and a function $T\colon (0, \epsilon_0] \to (0, \infty)$ that depends only on n and B such that $T(t) \geq t$, $\lim\limits_{t \to 0} T(t) = 0$, and if $S_{\epsilon_0} \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \neq \emptyset$, there is an exact sequence*

$$\cdots \longrightarrow H_{n+1}(B; \mathbb{L}) \longrightarrow \mathcal{S}_\epsilon \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \longrightarrow \mathcal{N}_n(X) \longrightarrow H_n(B; \mathbb{L}),$$

for each $0 < \epsilon \leq \epsilon_0$, where

$$\mathcal{S}_\epsilon \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} = im \left(S_\epsilon \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \longrightarrow S_{T(\epsilon)} \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \right).$$

Moreover, $\mathcal{S}_\epsilon \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix} \cong \mathcal{S}_{\epsilon_0} \begin{pmatrix} X \\ \downarrow \\ B \end{pmatrix}$ if $\epsilon \leq \epsilon_0$.

4. The resolution obstruction

In this section we discuss various geometric aspects of Quinn's work on the resolution conjecture that lead to the invariant $I(X)$, adopting a variant of his original formulation. For simplicity, we assume that X is a compact oriented ENR homology n-manifold, $n \geq 5$.

Resolutions are fine homotopy equivalences that desingularize homology manifolds. A map $f\colon M \to X$, M a topological manifold, is a resolution if and only if $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is a homotopy equivalence, for every open set $U \subseteq X$ [37]. This implies that $f\colon M \to X$ is a resolution if and only if f is a controlled homotopy equivalence with the identity map of X as control map.

In [29], Ferry and Pedersen showed that there is a degree-one normal map $f\colon M \to X$. Our goal is to understand the obstructions to finding a controlled homotopy equivalence over X within the normal bordism class of f. Notice that if an obstruction is encountered, we can try to eliminate it by changing the normal map to X. Therefore, in trying to construct resolutions, it is more natural to consider the collection $\mathcal{N}_n(X)$ of all normal bordism classes of n-dimensional degree-one normal maps to X. Recall that there is a one-to-one correspondence between $\mathcal{N}_n(X)$ and (stable) topological reductions of the Spivak normal fibration ν_X of X. A topological reduction of ν_X corresponds to a fiber homotopy class of lifts to $BTop$ of the map $\nu_X : X \to BG$ that classifies the Spivak fibration of X.

$$\begin{tikzcd} & BTop \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "\nu_X"] & BG \end{tikzcd}$$

Any two reductions differ by the action of a unique element of $[X, G/Top]$, where G/Top is the homotopy fiber of $BTop \to BG$. Hence, $[X, G/Top]$ acts freely and transitively on $\mathcal{N}_n(X)$, since $\mathcal{N}_n(X) \neq \emptyset$. When X is a manifold, this action induces a canonical identification $\eta\colon \mathcal{N}_n(X) \to [X, G/Top]$ since there is a preferred element of $\mathcal{N}_n(X)$, namely, the bordism class of the identity map of X, which corresponds to the (stable) Top reduction of ν_X given by the normal bundle of an embedding of X in a large euclidean space. We refer to $\eta(f) \in [X, G/Top]$ as the *normal invariant* of f.

To motivate our discussion, we first consider the case where X is a closed manifold, although this assumption trivializes the problem from the standpoint of existence of resolutions. Siebenmann's CE-approximation theorem states that cell-like maps of closed n-manifolds, $n \geq 5$, can be approximated by homeomorphisms [50]. Hence, if X is a manifold, we are to consider the obstructions to finding a homeomorphism in the normal bordism class of f. Such homeomorphism exists if and only if the normal invariant $\eta(f)$ vanishes [54].

Sullivan's description of the homotopy type type of G/Top [53] shows that, rationally, the normal invariant is detected by the difference of the rational $\mathcal{L}$-classes of M and X, respectively. Let

$$L_X = 1 + \ell_1 + \ell_2 + \ldots \in H^{4*}(X; \mathbb{Q})$$

be the total $\mathcal{L}$-class of X^n. The ith class $\ell_i \in H^{4i}(X; \mathbb{Q})$ is determined (after stabilizing X by crossing it, say, with a sphere if $4i \geq \frac{n-1}{2}$) by the signature of $4i$-dimensional submanifolds $N^{4i} \subseteq X$ with framed normal bundles. Hence, up to finite indeterminacies, the normal invariant of f is detected by the difference of the signatures of these characteristic submanifolds and their transverse inverse images. Notice that when X is a manifold, we can disregard 0-dimensional submanifolds, since the transverse inverse image of a point under a degree-one map can be assumed to be a point.

Carrying out this type of program for studying the existence of resolutions involves, among other things, defining (at least implicitly) characteristic classes for ENR homology manifolds. This has been done in [19], but following [10, 29] we take a controlled-surgery approach to the problem and argue that the Spivak normal fibration of an ENR homology manifold has a canonical Top reduction.

By Theorem 3.6, associated to a normal map $f\colon M \to X$ there is a controlled surgery obstruction $\sigma_f \in H_n(X; \mathbb{L}) \cong [X, G/Top \times \mathbb{Z}]$ such that $\sigma_f = 0$ if and only if, for any $\epsilon > 0$, f is normally bordant to an ϵ-homotopy equivalence. Under the natural (free) action of $[X, G/Top]$ on $H_n(X; \mathbb{L}) \cong [X, G/Top \times \mathbb{Z}]$, controlled surgery obstructions induce a $[X, G/Top]$-equivariant injection

$$\mathcal{N}_n(X) \longrightarrow H_n(X; \mathbb{L}).$$

Let $f\colon M \to X$ be a normal map. Letting $[X, G/Top]$ act on f, we can assume that the image of σ_f under the projection $H_n(X; \mathbb{L}) \cong [X, G/Top \times \mathbb{Z}] \to [X, G/Top]$ vanishes, so that $\sigma_f \in [X, \mathbb{Z}] \subseteq [X, G/Top \times \mathbb{Z}]$. Hence, if X is connected, σ_f is an integer. The local index of X is defined by

$$I(X) = 8\sigma_f + 1 \in 8\mathbb{Z} + 1.$$

Since the $\mathbb{Z}$-component of σ_f is persistent under the action of $[X, G/Top]$, this is the closest we can get to a resolution. This construction yields a preferred normal bordism class of normal maps to X (and therefore, a canonical Top reduction of ν_X) and induces an identification $\eta\colon \mathcal{N}_n(X) \to [X, G/Top]$. Rationally, the action of $[X, G/Top]$ on f can be interpreted as the analogue of changing the normal map f so that the signatures of the transverse preimages $f^{-1}(N)$ and N be the same for (stable) framed submanifolds $N^{4i} \subseteq X$, $i > 0$, when X is a manifold. This suggests that the $\mathbb{Z}$-component of σ_f be interpreted as a difference of signatures in dimension zero and that $I(X)$ be viewed as the 0-dimensional $\mathcal{L}$-class of X. This is the approach taken by Quinn in [45, 47], which explains the local nature of the invariant.

If $I(X) = 1$, there is a normal map $f\colon M \to X$ such that $\sigma_f = 0$. Let $\epsilon_i \to 0$ be a decreasing sequence. Theorem 3.6 implies that, for each $i > 0$, there is an ϵ_i-structure $f_i\colon M_i \to X$, so that $\mathcal{S}_{\epsilon_i} \neq \emptyset$. Under the identification $\mathcal{N}_n(X) \cong [X, G/Top] \cong H_n(X; G/Top)$, the controlled surgery sequence of Theorem 3.8 can be expressed as

$$H_{n+1}(X; G/Top) \to H_{n+1}(X; \mathbb{L}) \to \mathcal{S}_\epsilon\begin{pmatrix} X \\ \downarrow \\ X \end{pmatrix} \to H_n(X; G/Top) \to H_n(X; \mathbb{L}).$$

It follows from the Atiyah-Hirzebruch spectral sequence that $H_i(X; G/Top) \to H_i(X; \mathbb{L})$ is injective if $i = n$, and an isomorphism if $i = n+1$. This shows that $\mathcal{S}_{\epsilon_i}\begin{pmatrix} X \\ \downarrow \\ X \end{pmatrix} = 0$, if ϵ_i is small enough. Then, viewing f_i and f_{i+1} as equivalent ϵ_i-structures on X, we obtain homeomorphisms $h_i\colon M_i \to M_{i+1}$ such that $f_{i+1} \circ h_i$ and f_i are $T(\epsilon_i)$-homotopic over X. Consider the sequence

$$f_i^* = f_i \circ h_{i-1} \circ \cdots \circ h_1\colon M_1 \to X.$$

For each $i > 0$, f_i^* is an ϵ_i-equivalence over X and

$$d(f_{i+1}^*, f_i^*) = d(f_{i+1} \circ h_i, f_i) < T(\epsilon_i).$$

If $\epsilon_i > 0$ is so small that $\sum T(\epsilon_i) < \infty$, the sequence $\{f_i^*\}$ converges to a resolution of X.

5. Periodicity in manifold theory

A beautiful periodicity phenomenon emerges from the surgery classification of compact manifolds. All essential elements in the theory exhibit an almost 4-periodic behavior with respect to the dimension n. Siebenmann periodicity is the most geometric form of this phenomenon.

Definition 5.1. Let X be a compact manifold. A *structure* on X is a simple homotopy equivalence $f\colon M \to X$ that restricts to a homeomorphism $f\colon \partial M \to \partial X$ on the boundary, where M is a topological manifold. The structures $f_i\colon M_i \to X$, $i \in \{1, 2\}$, are equivalent if there is a homeomorphism $h\colon M_1 \to M_2$ making the diagram

$$\begin{tikzcd} M_1 \arrow[dd, "h"'] \arrow[dr, "f_1"] & \\ & X \\ M_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

homotopy commute rel(∂). The *structure set* $\mathcal{S}(X)$ is the collection of all equivalence classes of structures on X.

The following theorem is proved in [36], with a correction by Nicas in [42].

Theorem 5.2 (Siebenmann periodicity). *If X^n is a compact connected manifold of dimension ≥ 5, there is an exact sequence*

$$0 \longrightarrow \mathcal{S}(X) \xrightarrow{\wp} \mathcal{S}(X \times D^4) \xrightarrow{\sigma} \mathbb{Z}.$$

Moreover, $\wp$ is an isomorphism if $\partial X \neq \emptyset$.

The map σ associates to a structure $f\colon W \to X \times D^4$, the signature of the transverse inverse image of $\{*\} \times D^4$. Siebenmann's construction of the map $\wp$ was indirect. In [20], Cappell and Weinberger describe a geometric realization of the periodicity map $\wp\colon \mathcal{S}(M) \to \mathcal{S}(M \times D^4)$ using the Casson-Sullivan embedding theorem and branched circle fibrations.

The structure set $\mathcal{S}(S^n)$ of the n-sphere, $n \geq 4$, contains a single element, by the generalized Poincaré conjecture. However, it can be shown that $\mathcal{S}(S^n \times D^4) \cong \mathbb{Z}$, so that periodicity does fail for closed manifolds. This suggests that there may be "unidentified manifolds" that yield a fully periodic theory of manifolds.

Quinn's work on the resolution problem shows that the local index that obstructs the existence of resolutions and the $\mathbb{Z}$-factor that prevents periodicity from holding for closed manifolds have the same geometric nature, a fact to our knowledge first observed by Cappell. This indicates that the non-resolvable homology manifolds in the recognition problem are the same as the missing manifolds in Siebenmann periodicity, and creates an interesting link between the classification theory of manifolds and the resolution conjecture.

6. Classification of ENR homology manifolds

The first examples of nonresolvable ENR homology manifolds were produced in 1992 by Bryant, Ferry, Mio and Weinberger using techniques of controlled topology [9]. In this section, we outline the construction of examples modeled on simply-connected PL manifolds, where the central ideas are already present. For a more general discussion, we refer the reader to [10].

Theorem 6.1 (BFMW). *Let M^n be a simply-connected closed PL manifold, $n \geq 6$. Given $\sigma \in 8\mathbb{Z}+1$, there exists a closed ENR homology n-manifold X homotopy equivalent to M such that $I(X) = \sigma$.*

Variants of the methods employed in the construction yield an s-cobordism classification of ENR homology n-manifolds within a fixed simple homotopy type and an identification of the (simple) types realized by closed homology manifolds of dimension ≥ 6 in terms of Ranicki's total surgery obstruction [48]. We only state the classification theorem [10, 11], whose proof requires relative versions of the arguments to be presented.

Definition 6.2. Let M^n be a compact manifold. A *homology manifold structure* on M is a simple homotopy equivalence $f\colon (X, \partial X) \to (M, \partial M)$, where X is an ENR homology n-manifold with the DDP and f restricts to a homeomorphism on the boundary. The *homology structure set* $\mathcal{S}^H(M)$ of M is the set of all s-cobordism classes of homology manifold structures on M.

Remark. We consider s-cobordism classes of structures since the validity of the s-cobordism theorem in this category is still an open problem.

Since a structure $f\colon X \to M$ restricts to a homeomorphism on the boundary, if $\partial M \neq \emptyset$ we have that ∂X is a manifold. Adding a collar $\partial X \times I$ to X gives a homology manifold Y containing manifold points. Since Quinn's index is local, $I(X) = I(Y) = 1$ and X is a manifold. By the manifold s-cobordism theorem, $\mathcal{S}^H(M) = \mathcal{S}(M)$, so that $\mathcal{S}^H(M)$ consists entirely of manifold structures if $\partial M \neq \emptyset$.

Theorem 6.3 (BFMW). *If M^n is a closed manifold, $n \geq 6$, there is an exact sequence*

$$\ldots \to H_{n+1}(M; \mathbb{L}) \to L_{n+1}(\mathbb{Z}\pi_1(M)) \to \mathcal{S}^H(M) \to H_n(M; \mathbb{L}) \xrightarrow{\mathcal{A}} L_n(\mathbb{Z}\pi_1(M)),$$

where L_i is the ith Wall surgery obstruction group of the group $\pi_1(M)$, $\mathbb{L}$ is the simply-connected periodic surgery spectrum, and $\mathcal{A}$ denotes the assembly map.

This classification implies that homology manifold structures produce a fully periodic manifold theory.

Corollary 6.4. *The Siebenmann periodicity map $\wp\colon \mathcal{S}^H(M) \to \mathcal{S}^H(M \times D^4)$ is an isomorphism, if M^n is a compact manifold, $n \geq 6$.*

Sketch of the proof of Theorem 6.1. We perform a sequence of cut-paste constructions on the manifold M to obtain a sequence $\{X_i\}$ of Poincaré complexes that converges (in a large euclidean space) to an ENR homology manifold X with the required properties. There are two properties of the sequence that must be carefully monitored during the construction:

(i) *Controlled Poincaré duality.*

As pointed out earlier, homology manifolds satisfy a local form of Poincaré duality. Therefore, the approximating complexes are constructed so that X_i, $i \geq 2$, are Poincaré complexes with ever finer control over X_{i-1}.

(ii) *Convergence.*

We need the limit space X to inherit the fine Poincaré duality and the local contractibility of the complexes X_i. This is achieved by connecting successive stages of the construction via maps $p_i\colon X_{i+1} \to X_i$ which are fine homotopy equivalences over X_{i-1}.

If the maps $p_i\colon X_{i+1} \to X_i$ are fine equivalences, they are, in particular, finely 2-connected. Control improvement theorems imply that once enough control has been obtained at the fundamental group level, arbitrarily fine control can be achieved under a small deformation [1, 27]. Hence, throughout the construction we require that all maps be UV^1 (see Definition 3.3) so that the construction of controlled homotopy equivalences can be reduced to homological estimates via appropriate forms of the Hurewicz theorem [44].

Constructing X_1. Gluing manifolds by a homotopy equivalence of their boundaries, we obtain Poincaré spaces. We use a controlled version of this procedure to construct ϵ-Poincaré spaces. Let C_1 be a regular neighborhood of the 2-skeleton of a triangulation of M, D_1 be the closure of the complement of C_1 in M, and $N_1 = \partial C_1$.

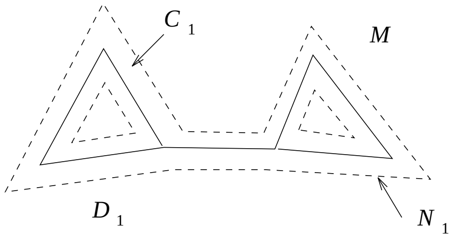

FIGURE 6.1

If the triangulation is fine enough, there is a small deformation of the inclusion $N_1 \hookrightarrow M$ to a UV^1 map $q\colon N_1 \to M$. A controlled analogue of Wall's realization theorem (Theorem 5.8 of [54]) applied to the control map $q\colon N_1 \to M$ gives a degree-one normal map $F_\sigma\colon (V, N_1, N_1') \to (N_1 \times I, N_1 \times \{0\}, N_1 \times \{1\})$ satisfying:

(a) $F_\sigma|_{N_1} = id$.
(b) $f_\sigma = F_\sigma|_{N_1'}$ is a fine homotopy equivalence over M.
(c) The controlled surgery obstruction of F_σ rel ∂ over M is $\sigma \in H_n(M; \mathbb{L})$.

Since the image of σ under the surgery forget-control map $H_n(M;\mathbb{L}) \to L_n(e)$ is trivial, doing surgery on V we can assume that $V = N_1 \times I$, and in particular that $N_1' = N_1$. Using $f_\sigma\colon N_1 \to N_1$ as gluing map, form the complex $X_1 = D_1 \cup_{f_\sigma} C_1$ as indicated in Figure 6.2.

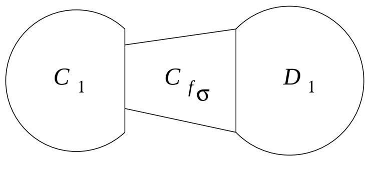

FIGURE 6.2

Here, C_{f_σ} is the mapping cylinder of f_σ. The construction of X_1 allows us to extend the control map $q\colon N_1 \to M$ to a UV^1 homotopy equivalence $p_0\colon X_1 \to M$ such that the restrictions of p_0 to C_1 and D_1 are close to the respective inclusions. In a large euclidean space $\mathbb{R}^L$, gently perturb p_0 to an embedding. This defines a metric on X_1 and completes the first stage of the construction. Notice that the control on the Poincaré duality of X_1 over M is only constrained by the magnitude of the controlled equivalence $f_\sigma\colon N_1 \to N_1$, which can be chosen to be arbitrarily fine.

Constructing X_2. Starting with a UV^1 homotopy equivalence $M \to X_1$, we perform a similar cut-paste construction on M along the boundary N_2 of a regular neighborhood C_2 of the 2-skeleton of a much finer triangulation of M. As in the construction of $p_0 \colon X_1 \to M$, we obtain a UV^1 homotopy equivalence $p_1' \colon X_2' \to X_1$. The difference in this step is that we modify p_1' to a fine equivalence over M, with a view toward fast convergence. By construction, the controlled surgery obstruction of p_1' with respect to the control map $p_0 \colon X_1 \to M$ is zero. Surgery on X_2' can be done as in the manifold case, by moving spheres off of the 2-dimensional spine of C_2 and pushing them away from the singular set under small deformations. This gives a fine UV^1 homotopy equivalence $p_1 \colon X_2 \to X_1$ over M. Control on p_1 is only limited by the Poincaré duality of X_1 over M, since X_2 can be constructed to be a much finer Poincaré space than X_1.

Mildly perturb $p_1 \colon X_2 \to X_1$ to an embedding of X_2 into a small regular neighborhood V_1 of $X_1 \subseteq \mathbb{R}^L$. By the thin h-cobordism theorem [44], we can assume that the region between V_1 and a small regular neighborhood V_2 of X_2 in V_1 admits a fine product structure over M.

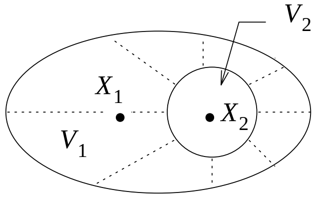

FIGURE 6.3

Iterating the construction, we obtain fine homotopy equivalences $p_i \colon X_{i+1} \to X_i$ over X_{i-1}. The control on p_{i+1} depends only on the Poincaré duality of X_i over X_{i-1} which can be chosen to be so fine that the region between small regular neighborhoods V_i and V_{i+1} of X_i and X_{i+1}, respectively, admits a controlled product structure over X_{i-1}. As before, the Poincaré duality of X_{i+1} over X_i can be assumed to be as fine as necessary in the next stage of the construction.

Let $X = \bigcap_{i=1}^{\infty} V_i$ be the intersection of the nested sequence V_i of regular neighborhoods of X_i. Concatenating the product structures on $V_i \setminus \operatorname{int}(V_{i+1})$, $i \geq 1$, gives a deformation retraction $p \colon V_1 \to X$, provided that the product structures are sufficiently fine. This shows that X is an ENR. The map p actually defines a mapping cylinder structure on the neighborhood V_1 of X.

In order to show that X is a homology manifold, we first reinterpret controlled Poincaré duality in terms of lifting properties, via a controlled analogue of Spivak's thesis [52]. Let $\rho = p|_{\partial V_1} \colon \partial V_1 \to X$, and let $\rho_i \colon \partial V_i \to X_i$ denote the restriction of the regular neighborhood projection $V_i \to X_i$ to ∂V_i. Proposition 4.5 of [10] implies that given $\delta > 0$, ρ_i has the δ-homotopy lifting property, provided that i is large enough. Hence, the projection $\partial V_1 \to X_i$ obtained from the product structure connecting ∂V_1 to ∂V_i also has the δ-homotopy lifting property, for i large enough. Since the homotopy equivalences $X_i \to X$ become finer as $i \to \infty$, it follows that $\rho \colon \partial V_1 \to X$ has the ϵ-homotopy lifting property, for every $\epsilon > 0$, i.e., ρ is a manifold approximate fibration over X. This implies that X is a homology manifold [24].

The approximate homology manifolds X_i were constructed to carry the resolution obstruction σ, in the sense that there is a normal map $\phi_i\colon M \to X_i$ with controlled surgery obstruction $\sigma \in H_n(X_{i-1};\mathbb{L})$. Since the sequence $\{X_i\}$ converges to X, a change of control space argument implies that $I(X) = \sigma$. This concludes the construction. $\square$

7. Concluding remarks

The existence of nonresolvable ENR homology manifolds raises numerous questions about the geometric topology of these spaces. In [10], we summarized several of these questions in a conjecture.

Conjecture (BFMW). *There exist spaces $\mathbb{R}^4_k$, $k \in \mathbb{Z}$, such that every connected DDP homology n-manifold X with local index $I(X) = 8k+1$, $n \geq 5$, is locally homeomorphic to $\mathbb{R}^4_k \times \mathbb{R}^{n-4}$. ENR homology n-manifolds with the DDP are topologically homogeneous, the s-cobordism theorem holds for these spaces, and structures on closed DDP homology manifolds X^n are classified (up to homeomorphisms) by a surgery exact sequence*

$$\cdots \to H_{n+1}(X;\mathbb{L}) \to L_{n+1}(\mathbb{Z}\pi_1(X)) \to \mathcal{S}^H(X) \to H_n(X;\mathbb{L}) \to L_n(\mathbb{Z}\pi_1(X)).$$

Remark. This exact sequence has been established in [10] up to s-cobordisms of homology manifolds.

Recall that a topological space X is homogeneous if for any pair of points $a, b \in X$, there is a homeomorphism $h\colon X \to X$ such that $h(a) = b$. The topological homogeneity of DDP homology manifolds seems to be a problem of fundamental importance. A positive solution would strongly support the contention that DDP homology manifolds form the natural class in which to develop manifold theory in higher dimensions and would also settle the long standing question "*Are homogeneous ENRs manifolds?*", proposed by Bing and Borsuk in [5].

The validity of Edward's CE-approximation theorem in this class of spaces is a recurring theme in the study of the topology of homology manifolds. Can a cell-like map $f\colon X \to Y$ of DDP homology n-manifolds, $n \geq 5$, be approximated by homeomorphisms? Homogeneity and many other questions can be reduced to special cases of (variants of) this approximation problem.

Homology manifolds are also related to important rigidity questions. For example, the existence of a nonresolvable closed aspherical ENR homology n-manifold X, $n \geq 5$, would imply that either the integral Novikov conjecture or the Poincaré duality group conjecture are false for the group $\pi_1(X)$. Indeed, if the assembly map

$$\mathcal{A}\colon H_*(X;\mathbb{L}) \longrightarrow L_*(\mathbb{Z}\pi_1(X))$$

is an isomorphism, the homology-manifold structure set $\mathcal{S}^H(X)$ contains a single element. Therefore, if M is a closed manifold homotopy equivalent to X, then X is s-cobordant to M. This implies that $I(X) = 1$, contradicting the assumption that X is not resolvable. Hence, $\pi_1(X)$ would be a Poincaré duality group which is not the fundamental group of any closed aspherical manifold [29].

Can a map of DDP homology manifolds be made transverse to a codimension q tamely embedded homology manifold? In her thesis, Johnston established map transversality (up to s-cobordisms) in the case the homology submanifolds have bundle neighborhoods [34]. Although the existence of such neighborhoods is, in general, obstructed (since indices satisfy a product formula), it seems plausible that there exist an appropriate notion of normal structure for these subobjects that yield general map transversality. When the ambient spaces are topological manifolds, $q \geq 3$, and the homology submanifolds have dimension ≥ 5, mapping cylinders of spherical manifold approximate fibrations appear to provide the right structures [44]. This is consistent with the fact that, for manifolds, Marin's topological transversality is equivalent to the neighborhood transversality of Rourke and Sanderson [40]. In [12], approximate fibrations are used to extend to homology manifolds the classification of tame codimension q manifold neighborhoods of topological manifolds, $q \geq 3$, obtained by Rourke and Sanderson in [49]. In [14], a notion of submanifold transversality is defined, and results are obtained in the metastable range [13, 14].

Smith theory and the work of Cappell and Weinberger on propagation of group actions [21] indicate that nonstandard homology manifolds may occur as fixed sets of semifree periodic dynamical systems on manifolds. Homology manifolds also arise as limits of sequences of riemannian manifolds in Gromov-Hausdorff space [32]. Results of Bestvina [2] show that boundaries of Poincaré duality groups are homology manifolds, offering further evidence that exotic ENR homology manifolds may become natural geometric models for various phenomena.

Acknowledgements. I wish to thank Andrew Ranicki for many comments and suggestions, and John Bryant for numerous discussions during several years of collaboration.

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