TRANSVERSALITY IN GENERALIZED MANIFOLDS

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ABSTRACT. Suppose that X is a generalized n-manifold, $n \geq 5$, satisfying the disjoint disks property, and M and Q are topological m- and q-manifolds, respectively, 1-LCC embedded in X, with $n - m \geq 3$ and $n - q \geq 3$. We define what it means for M to be stably transverse to Q in X. In the metastable range, $3m \leq 2(n-1)$ and $3(m+q) < 4(n-1)$, we show that there is an arbitrarily small homotopy of M to a 1-LCC embedding that is stably transverse to Q.

1. INTRODUCTION

In this paper we introduce a notion of transversality for submanifolds of a generalized n-manifold. One of the major difficulties in arriving at suitable criteria for transversality is that a (generalized) submanifold M of a generalized manifold X may not have a stable euclidean normal (micro)bundle neighborhood in X. This situation occurs, for example, when M is a topological manifold, which has Quinn index [22] $\iota(M) = 1$, and X is a generalized manifold with $\iota(X) \neq 1$. Examples of generalized manifolds X with $\iota(X) \neq 1$ were constructed in [4]. An embryonic form of transversality was established in [5] for codimension three topological submanifolds M and Q of a generalized manifold X having complementary dimensions in X. Specifically, it was shown that if $m \leq q \leq n-3$, $m+q = n \geq 6$, and M and Q are orientable topological manifolds of dimensions m and q, respectively, tamely embedded in an orientable generalized n-manifold X with the disjoint disks property, then there is an arbitrarily small homotopy of M to a tame embedding $f\colon M \to X$ such that $f(M) \cap Q$ is a finite set and the intersection number of $f(M) \cap Q$ at each point of intersection is ± 1. Assuming the metastable codimension restriction $3m \leq 2(n-1)$, $3(m+q) < 4(n-1)$, we find a small homotopy of M to a tame embedding $f\colon M \to X$ such that $f(M)$ and Q are stably transverse, in an sense to be described. In fact, we need only assume that Q is a generalized q-manifold with the disjoint disks property. In particular, $f(M) \cap Q$ will be a tame topological submanifold of $f(M)$ and Q of the expected dimension, $m+q-n$. The proof makes use of the transversality theorems of Kirby-Siebenmann [15] and Marin [16], the Main Construction of [5], and a splitting theorem of [7]. Map transversality, which can be obtained from submanifold transversality, has beeen studied by Johnston [14] in the special case where the homology submanifold has a bundle neighborhood .

2. DEFINITIONS

A *generalized n-manifold* (*n-gm*) without boundary is a locally compact euclidean neighborhood retract (ENR) X such that for each $x \in X$,

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$$H_k(X, X \setminus \{x\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Following Mitchell [19] we say that an ENR X is an *n-gm with boundary* if the condition $H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ is replaced by $H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$ or 0, and if $\mathrm{bd}X = \{x \in X : H_n(X, X \setminus \{x\}; \mathbb{Z}) \cong 0\}$ is an $(n-1)$-gm embedded in X as a Z-set. (In [19] Mitchell shows that $\mathrm{bd}X$ is a homology $(n-1)$-manifold.) Recall that Y is a *Z-set* in X if, for each open set U in X, the inclusion $U \setminus Y \to U$ is a homotopy equivalence. A n-gm X, $n \geq 5$, has the *disjoint disks property* (*DDP*) if every pair of maps of the 2-cell B^2 into X can be approximated arbitrarily closely by maps that have disjoint images. A subset A of X is 1-*LCC* in X if for each $x \in A$ and neighborhood U of x in X, there is a neighborhood V of x in X lying in U such that the inclusion induced homomorphism $\pi_1(V \setminus A) \to \pi_1(U \setminus A)$ is trivial. An ENR A in X of codimension at least three will be called *tame* in X if it is 1-*LCC* in X.

Given an n-gm X, a *manifold approximate fibration with fiber F* (*MAF*) over X is an approximate fibration $p : N \to X$, where N is a topological manifold and the homotopy fiber of p is homotopy equivalent to F. (Equivalently, each $p^{-1}(x)$ has the shape of the space F.) (See [8], [13].) If Q is a (topological or generalized) manifold in X and $p : N \to X$ is a *MAF*, then p is said to be *split over Q* if $p|p^{-1}(Q) : p^{-1}(Q) \to Q$ is also a *MAF*.

Suppose that M_p is the mapping cylinder of a *MAF* $p : N \to X$ with fiber a sphere and mapping cylinder projection $\pi : M_p \to X$. If M_p is a topological manifold, then we will call $\pi : M_p \to X$ (or, sometimes, just M_p) a *manifold stabilization* of X. As the following proposition shows, this last condition is almost always satisfied.

Proposition 2.1. *Suppose that N is a topological n-manifold, X is a generalized manifold, and M_p is the mapping cylinder of a MAF $p : N \to X$ with fiber a k-sphere and mapping cylinder projection $\pi : M_p \to X$. If $n \geq 5$, then M_p is a topological manifold. If, in addition, $k \geq 2$, then X is 1-LCC embedded in M_p.*

Proof. That M_p is a homology manifold follows easily from results of Gottlieb [11] and Quinn [20]. Since M_p has manifold points, M_p has a resolution [22], and, hence, by a theorem of Edwards (see [9]), it suffices to observe that M_p has the *DDP*. We consider three cases.

Case 1. $k \geq 2$. In this case it enough to show that X is 1-*LCC* in M_p, since we can then use ordinary general position in $M_p \setminus X$. Suppose then that $f : B^2 \to M_p$ and T is a fine triangulation of B^2. By Alexander duality, X is 0-*LCC* in M_p; hence, we may assume that, if $T^{(1)}$ denotes the 1-skeleton of T, then $f(T^{(1)}) \cap X = \emptyset$. Let Δ be a 2-simplex of T with boundary Σ, such that $f(\Delta) \cap X \neq \emptyset$. By a small homotopy of $f|\Sigma$ in $M_p \setminus X$, we can assume that $f(\Sigma)$ lies in some t-level N_t of the mapping cylinder near X. Since $\pi|\Sigma$ is null-homotopic in X, we can use the approximate lifting property of p to assume that $f(\Sigma)$ lies near a fiber of p (in N_t). Since the fibers have the shape of S^k, $k \geq 2$, we can homotope $f|\Sigma$ to a constant in a neighborhood of a fiber in N_t. Thus there is a small homotopy of $f|\Delta$ to a map of Δ into $M_p \setminus X$.

Case 2. $k = 1$. Since X is 0-*LCC* in M_p, we can begin as in Case 1. Given $f : B^2 \to M_p$, we can assume that $f(T^{(1)}) \cap X = \emptyset$, where T is a fine triangulation of B^2. If $f(\Delta) \cap X \neq \emptyset$, for some 2-simplex Δ of T with boundary Σ, then we

may assume that $f(\Sigma)$ lies near a fiber of p in some t-level N_t of M_p, as above. Thus, there is a small homotopy of $f|\Delta$ to $f'\colon \Delta \to M_p$ such that $f'(\Delta)\cap X$ is a single point. This process gives a small homotopy of f to $f'\colon B^2 \to M_p$ such that $f'(B^2)\cap X$ is a finite set. Given another mapping $g\colon B^2 \to X$, we can get a small homotopy of g to g' such that $g(B^2)\cap X$ is a finite set disjoint from $f'(B^2)\cap X$. We can then use general position in $M_p \setminus X$ to get $f'(B^2)$ and $g'(B^2)$ disjoint.
Case 3. $k = 0$. In this case X locally separates M_p, and the approximate lifting property of p implies that X is 1-*LCC* in M_p. If $f\colon B^2 \to M_p$, and T is a fine triangulation of B^2, then it is easy to get a small homotopy of f to f' such that $\dim f'(B^2)\cap X \le 1$. Since $\dim X \ge 4$, $f'(B^2)\cap X$ is 0-*LCC* in X. Thus, if $g\colon B^2 \to M_p$ is another mapping, then there is a small homotopy of g to g' such that $g'(B^2)\cap(f'(B^2)\cap X) = \emptyset$. We can then use general position in $M_p\setminus X$ to get $f'(B^2)$ and $g'(B^2)$ disjoint as before. □

Suppose $M, Q \subseteq N$ are topological manifolds without boundary of dimensions m, q, and n, respectively. Let $p = m+q-n$. Then M and Q are *locally transverse* if, for each $x \in M\cap Q$, there is a neighborhood W of x in N, with $W\cap M = U$ and $W \cap Q = V$, such that

$$(W, U, V, U\cap V) \cong (\mathbb{R}^n, \mathbb{R}^{m-p}\times\mathbb{R}^p\times 0, 0\times\mathbb{R}^p\times\mathbb{R}^{q-p}, 0\times\mathbb{R}^p\times 0).$$

This implies, in particular, that $P = M\cap Q$ is a p-dimensional submanifold of both M and Q. If M (or Q) has boundary, and $x \in \mathrm{bd}M$ (or $x\in\mathrm{bd}Q$), then local transversality at x can be described by replacing $\mathbb{R}^m$ by $\mathbb{R}^{m-1}\times\mathbb{R}_+$, (or $\mathbb{R}^q$ by $\mathbb{R}_+\times\mathbb{R}^{q-1}$), and $\mathbb{R}^p$ by the appropriate intersection. Following [15], we say that M is *stably microbundle transverse* to Q in N if M and Q are locally transverse and, for some integer $s\ge 0$, there exists a normal microbundle ξ to $Q\times 0$ in $N\times\mathbb{R}^s$ so that $M\times\mathbb{R}^s$ is embedded microbundle transverse to ξ in $N\times\mathbb{R}^s$. That is, $M\cap Q$ has a normal microbundle ν in M each of whose fibers lies in a fiber of ξ. Marin shows that this relation is symmetric [16] and, with help from Scharlemann [23] when $p=4$, that local transversality implies stable microbundle transversality, provided $n-m\le 3$ and $n-q\le 3$. With these ideas in mind, we make the following definition.

Definition 2.2. Given a topological manifold M and generalized manifold Q in a generalized manifold X, Q is *stably locally transverse* to M if there is a manifold stabilization $\pi\colon M_p \to X$ of X, split over Q, such that $\pi^{-1}(Q)$ and M are locally transverse in M_p.

3. Transversality in the Metastable Range

Theorem 3.1. *Suppose that X is an n-gm with the DDP, $n\ge 5$, M is a topological m-manifold embedded in X (with or without boundary), and Q is either a topological q-manifold or a q-gm with the DDP if $q \ge 5$, 1-LCC embedded in X, such that $n-q\ge 3$, $3m\le 2(n-1)$, and $3(m+q) < 4(n-1)$. Then for every $\epsilon>0$ there is an ϵ-homotopy of the inclusion of M in X to a 1-LCC embedding $f\colon M\to X$ such that Q is stably locally transverse to $f(M)$ in X.*

The following corollary is a consequence of Theorem 3.1 and Corollary 1.3 of [5].

Corollary 3.2. *Suppose that M and Q are topological m- and q-manifolds, respectively, in an n-gm X, $n\ge 5$, with the DDP, such that $3m\le 2(n-1)$, $3q\le 2(n-1)$, $3(m+q) < 4n-4$. Then there are arbitrarily small homotopies of the inclusions* *to 1-LCC embeddings $f: M \to X$ and $g: Q \to X$ such that $f(M)$ is stably locally transverse to $g(Q)$ in X.*

The proof of Theorem 3.1 ultimately depends upon a transversality theorems of Kirby-Siebenmann [15] and Marin [16]. One of the main ingredients of the proof is the following splitting theorem proved in [7].

Theorem 3.3 ([7]). *Suppose that X is an n-gm without boundary, $n \geq 5$, and $Q \subseteq X$ is an q-gm (with or without boundary), $n - q \geq 3$, 1-LCC in X. Assume Q is a topological manifold if $q \leq 4$. Then there is a manifold stabilization $\pi: M_p \to X$ of X of dimension $\geq n+3$ that is split over Q.*

The manifold stabilization X of Theorem 3.3 is obtained in [7] by first taking a mapping cylinder neighborhood $M_{p'}$ of X is some euclidean space [18],[25], where $p': N \to X$ is a *MAF* with homotopy fiber a sphere, and then homotoping p' to a *MAF* $p: N \to X$ such that $p^{-1}(M)$ is a topological manifold. A similar argument can be found in [6], wherein X is a topological manifold.

Another important ingredient is the Main Construction of [5]. It can be summarized in the following theorem.

Theorem 3.4 ([5]). *Suppose that M is a topological m-manifold and X is an n-gm with the DDP, $n \geq 5$, $3m \leq 2(n-1)$. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $f: M \to X$ is a $(\delta, 2m-n+1)$-connected map, then f is ϵ-homotopic to a 1-LCC embedding. Moreover, the homotopy is supported in a neighborhood of a 1-LCC subset of X of dimension $\leq 2m-n+2$.*

A map $f: M \to X$ is (δ, k)-*connected* if the pair (M_f, X) is (δ, i)-connected for $0 \leq i \leq k$. If M, in 3.3 or 3.4, is not compact, then f should be a proper map and ϵ and δ should be interpreted as positive, continuous functions on M. The "moreover" part of Theorem 3.4 has the following consequence, which will be important for us here.

Addendum. *If P is a (closed) ANR in M, with $dim P < m$, such that $f|f^{-1}f(P)$ is a 1-LCC embedding, then we can arrange to have the homotopy f_t, $t \in [0,1]$, of f to an embedding satisfy $f_t|P = f|P$ and $f_t^{-1}f_t(P) = P$ for all $t \in [0,1]$.*

Proof of Theorem 3.1. Suppose that X, M, and Q are given as in the hypothesis of Theorem 3.1. By Theorem 3.3, there is a manifold stabilization $\pi: M_p \to X$ of X of dimension $n+k$, with $k \geq 3$, that is split over Q. Let $W = \pi^{-1}(Q)$. Choose k large enough so that, by 2.1, W is a topological $(q+k)$-manifold. Since Q is 1-*LCC* in X, W is 1-*LCC* in M_p, hence, locally flat [3]. Thus, by [15], [16], and [23], there is an arbitrarily small ambient isotopy of the inclusion of M in M_p to a locally flat embedding $h: M \to M_p$ such that $h(M)$ and W are locally transverse. Let $P = h(M) \cap W$. Then P is a manifold of dimension $p = m+q-n$, locally flatly embedded in $h(M)$ and in W. The next step is to push $h(M)$ down into X, sending P into Q and $h(M) - P$ into $X - Q$, to a 1-*LCC* embedding close to M. Observe that $\pi|h(M)$ has all but the last of these properties.

The first step is to observe that the inequalities $3m \leq 2(n-1)$, $3(m+q) < 4(n-1)$ imply $2p+1 \leq q$. General position then implies that $\pi|P: P \to Q$ can be approximated by a 1-*LCC* embedding. (If Q is a manifold, this is immediate. If Q is a q-gm with the *DDP*, then the general position results of [2] and [24] may be applied.) Since $k \geq 3$, there is a small ambient isotopy of W taking P to this embedding [1], which can be extended to M_p by [12]. After composing with π, we get a map $h'\colon (M, M \setminus h^{-1}(P)) \to (X, X \setminus Q)$ such that h' approximates the inclusion of M into X and $h'|P$ is a 1-*LCC* embedding into Q. Finally, as long as $\pi \circ h'$ is a sufficiently close approximation to the inclusion of M in X, it will have the desired connectivity properties to apply Theorem 3.4. Thus we can get a small homotopy of h' rel P to a 1-*LCC* embedding in X. According to Theorem 3.4, this homotopy is supported on a 1-*LCC* set of dimension $2m - n + 2$, and our dimension restrictions imply that $(2m - n + 2) + q < n$. By the general position results of [2] and [24], we can assume that these supports can be made to miss Q. Thus, the homotopy of h' to a 1-*LCC* embedding can be constructed so as not to introduce any new intersections of M with Q as guaranteed by the Addendum to Theorem 3.4. This final adjustment provides the map $f\colon M \to X$ promised in the theorem.

$\square$

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