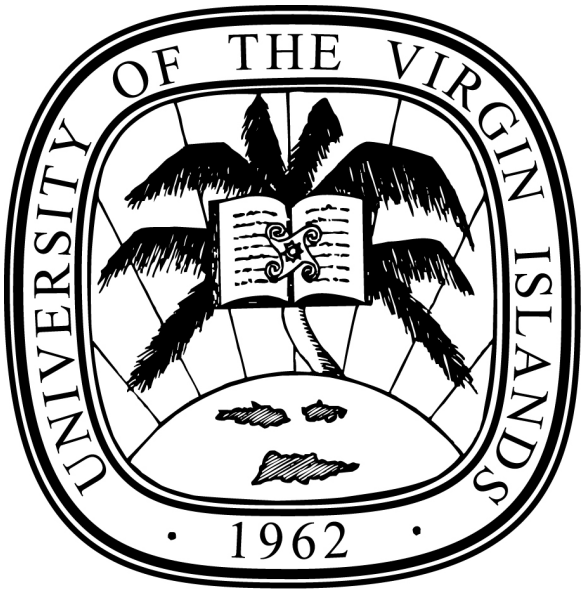


College Trigonometry

Dr. Matthew McCurdy

Summer 2026

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7 The Unit Circle: Sine and Cosine Functions

7.1 Angles

Learning Goals 7.1

After this section, I should be able to:

1. Draw angles in standard position.
2. Convert between degrees and radians.
3. Find coterminal angles.
4. Find the length of a circular arc.
5. Use linear and angular speed to describe motion on a circular path.

Properly defining an angle first requires that we define a **ray**. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the **endpoint** of the ray. We can refer to a specific ray by stating its endpoint and any other point on it with an arrow above the two point's identifiers.

An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the **sides** of the angle.

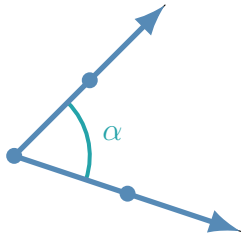
Angles can be named using a point on each ray and the vertex, such as angle DEF, or in symbol form $\angle DEF$.

In this class, we will use uppercase letters to denote points (or rays or sides!), lowercase letters for variables representing sides, and lowercase Greek letters for the measure of angles.

Example 7.1 *Use each of the vocabulary words* ray, endpoint, angle, vertex, sides *to describe items in the diagram. There may be more than one of each item!*

Then, provide labels for points in the diagram so that there is a ray denoted $\overrightarrow{OM}$ *in the left sketch and the angle* $\alpha = \angle BAC$ *in the right sketch.*





For the Greek alphabet, we have:

Letters		Name	Letters		Name	Letters		Name
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

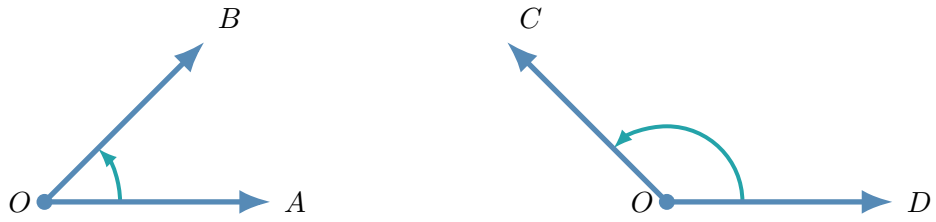
Note: *Trigonometry is the study of the relations between the sides and angles of triangles. The word "trigonometry" is derived from the Greek words* trigono *(*τρίγωνο*), meaning "triangle", and* metro *(*μετρώ*), meaning "measure".*

Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. – A.D. *200, its history is much older. For example, the Egyptian scribe Ahmes recorded some rudimentary trigonometric calculations (concerning ratios of sides of pyramids) in the famous Rhind Papyrus sometime around 1650* B.C. *However, the creation of trigonometry may precede that time; Ahmes claimed that he copied the papyrus from a work that may date as far back as 3000* B.C.

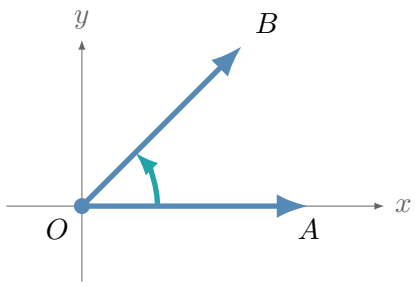
To draw an angle, we begin with two rays lying on top of one another. We leave one fixed in place, and rotate the other about the endpoint. The fixed ray $\overrightarrow{OA}$ is the **initial side**, and the rotated ray $\overrightarrow{OB}$ is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex. We denote the angle formed by this rotation as $\angle AOB$, or simply $\angle O$ if there is no ambiguity about which angle this refers to!

Example 7.2 *Label the* initial *and* terminal sides *of the two sketches below, and then write out how the labels would be referenced (using $\angle$ notation).*

Then, create a hypothesis about which angle is larger with a sentence justifying your answer.



An angle is in **standard position** if its vertex is located at the origin (0,0)– usually denoted O when labelled on a graph– and its initial side extends along the positive x-axis.



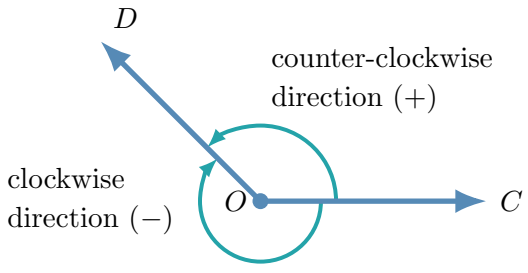
The measure of an angle is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree.

One degree is $^1/_{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. The more a ray is rotated from the initial side, the larger the angle is!

Note: *The system of measuring angles in degrees, such that* 360° *is one revolution, originated in ancient Babylonia. It is often assumed that the number* 360 *was used because the Babylonians supposedly thought that there were* 360 *days in a year– or one full revolution of the Earth around the Sun. However, there is another, perhaps more likely, explanation which says that in ancient times a person could travel* 12 Babylonian miles *in one day– or one full rotation of the Earth about its axis. The Babylonian mile was large enough (approximately* 7 *miles) to be divided into* 30 *equal parts for convenience, thus giving* $12 \times 30 = 360$ *equal parts in a full rotation. From* An Introduction to the History of Mathematics *(1983) by H. Eves.*

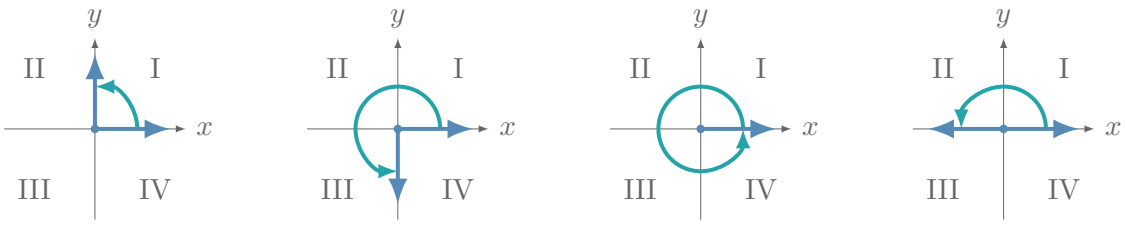
If the rotation is **counter-clockwise**, then we say that the angle is **positive**, and the angle is **negative** if the rotation is **clockwise**.

Example 7.3 *Determine if* $\angle AOB$ *– in the figure at the top of this page– is a positive or negative angle. Then, sketch an angle in standard position, labelled* $\angle EFG$ *that is negative.*



Since we define an angle in standard position by its terminal side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270°, or 360°.

Example 7.4 *Label each of the* quadrantal angles *below with how much of a full revolution there is from the (positive) x-axis and then determine their measure in terms of degrees.*



How To: Draw an angle in standard position given the measure in degrees

1. Express the angle measure as a fraction of 360.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive x-axis and moving counterclockwise for positive angles and clockwise for negative angles.

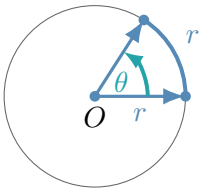
Example 7.5 *In the graphs below, sketch an angle of* 30° *in standard position and an angle of* −135° *in standard position.*



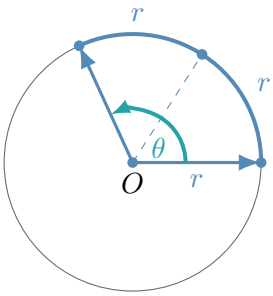
Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. However, we can (and will!) choose other ways to divide a circle.

To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an **arc**. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an *entire circle* is called the **circumference** of that circle.

One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle.



$\theta = 1$ radian



$\theta = 2$ radians

A **central angle** is an angle formed at the center of a circle by two radii (the plural of *radius*). Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians.

This tells us that 2π radians is the same as one full revolution around the circle. How could we measure that in degrees?

How would we measure a half of a revolution around the circle in terms of radians and degrees? How many degrees would 1 radian be?

How To: Convert between degrees and radians

Degrees to radians: x degrees $= \left(\frac{\pi}{180} \cdot x\right)$ radians

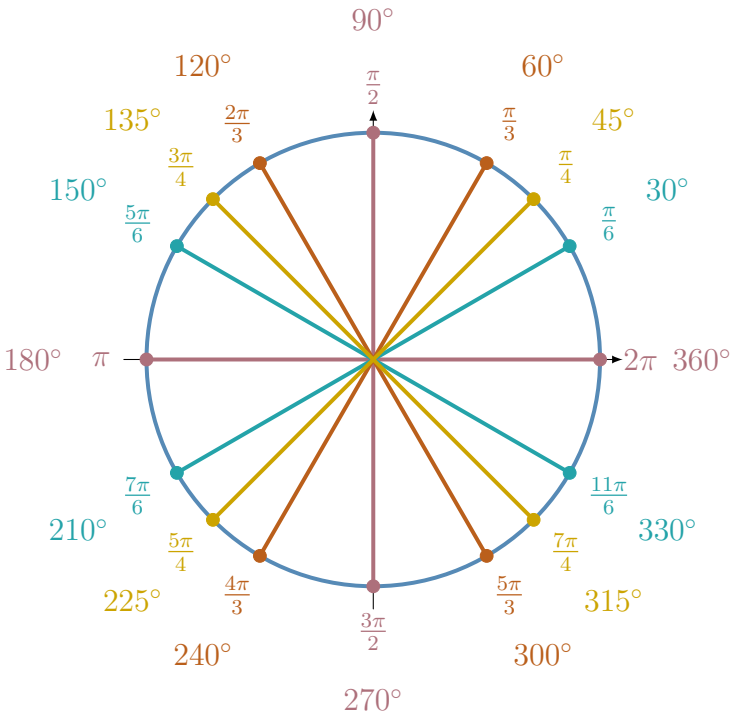
Radians to degrees: y radians $= \left(\frac{180}{\pi} \cdot y\right)$ degrees

Example 7.6 *Convert* $18°$ *to radians.*

Example 7.7 *Convert* $\frac{\pi}{9}$ *radians to degrees.*

Commonly used angles in radians and degrees

Degrees	Radians	Degrees	Radians	Degrees	Radians	Degrees	Radians
$0°$	0	$90°$	$\frac{\pi}{2}$	$180°$	π	$270°$	$\frac{3\pi}{2}$
$30°$	$\frac{\pi}{6}$	$120°$	$\frac{2\pi}{3}$	$210°$	$\frac{7\pi}{6}$	$300°$	$\frac{5\pi}{3}$
$45°$	$\frac{\pi}{4}$	$135°$	$\frac{3\pi}{4}$	$225°$	$\frac{5\pi}{4}$	$315°$	$\frac{7\pi}{4}$
$60°$	$\frac{\pi}{3}$	$150°$	$\frac{5\pi}{6}$	$240°$	$\frac{4\pi}{3}$	$330°$	$\frac{11\pi}{6}$



Commonly used angles in radians and degrees

Example 7.8 *Convert* $15°$ *to radians.*

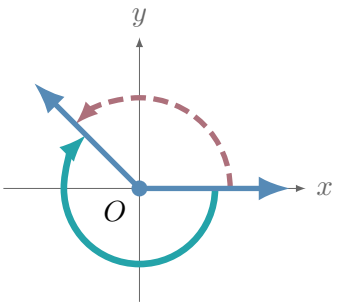
Example 7.9 *Convert* 3 *radians to degrees.*

Example 7.10 *Convert* $-\dfrac{3\pi}{4}$ *radians to degrees.*

Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of $0°$ to $360°$, or 0 to 2π radians. It is convenient sometimes to replace these out-of-range angles with a corresponding angle *within* the range of a single revolution.

If two angles in standard position have the same terminal side, they are **coterminal angles**. It is possible for more than one angle to have the same terminal side!

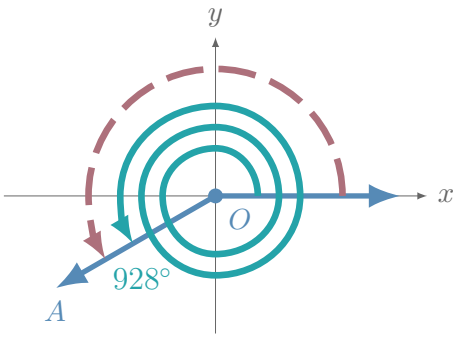
Here, we have an angle of $-225°$. How could we reference the *same* angle using an angle in between $0°$ and $360°$? Graphically, what do we see must be true about the two angles shown? Think about how much of a revolution these two angles complete!



How To: Given an angle greater than $360°$, find a coterminal angle between $0°$ to $360°$

1. Subtract $360°$ from the given angle.
2. If the result is still greater than $360°$, subtract $360°$ again until the result is between $0°$ and $360°$.
3. The resulting angle is coterminal with the original angle.

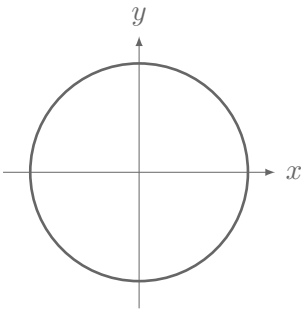
Example 7.11 *Given an angle of* $928°$*, find a coterminal angle between* $0°$ *to* $360°$*. Then, find a second coterminal angle that is between* $-360°$ *and* $0°$ *and sketch what this angle would look like on the graph.*



How To: Given an angle with measure less than $0°$, find a coterminal angle between $0°$ to $360°$

1. Add $360°$ to the given angle.
2. If the result is still less than $0°$, add $360°$ again until the result is between $0°$ and $360°$.
3. The resulting angle is coterminal with the original angle.

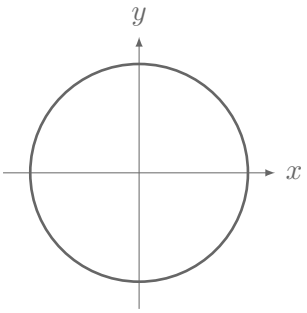
Example 7.12 *Show the angle with measure $-450°$ on a circle and find a positive coterminal angle α such that $0° \leq \alpha < 360°$.*



How To: Given an angle greater than 2π radians, find a coterminal angle between 0 to 2π radians.

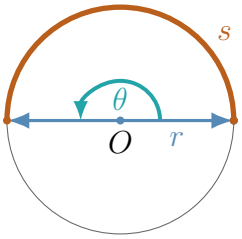
1. Subtract 2π radians from the given angle.
2. If the result is still greater than 2π radians, subtract 2π radians again until the result is between 0 and 2π radians.
3. The resulting angle is coterminal with the original angle.

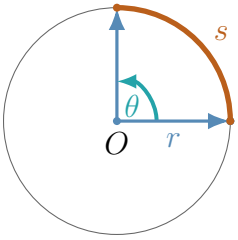
Example 7.13 *Show the angle with measure $\frac{19}{4}\pi$ on a circle and find a positive coterminal angle β such that $0 \leq \beta < 2\pi$ radians.*



An **arc length** s is the length of the curve along the arc.

In a circle of radius r, find the length of the noted portion along the outside of each circle using that the circumference of a circle is $C = 2\pi r$.

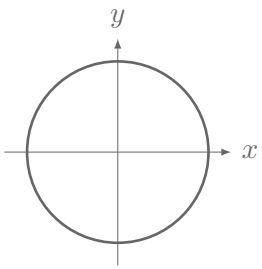




How To: Finding arc length

1. If necessary, convert θ to radians.
2. Multiply the radius r by the angle θ (in radians!) to get the arc length s: $s = r\,\theta$.

Example 7.14 *Find the arc length along a circle of radius 10 units subtended by an angle of* $215°$.



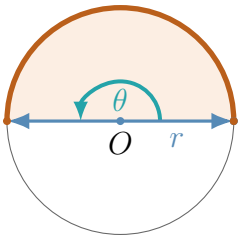
Example 7.15 *Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.*

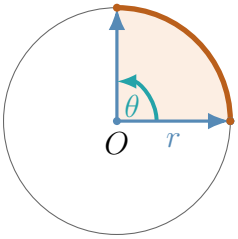
(a) In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?

(b) Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

The **area of a sector** A of a circle with radius r subtended by an angle θ, measured in radians.

In a circle of radius r, find the area of the shaded portion of each circle using that the area of a circle is $A = \pi r^2$.

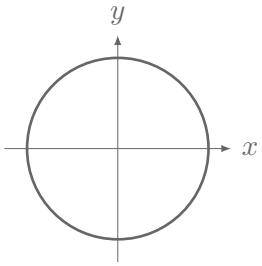




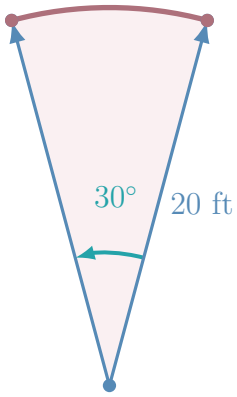
How To: Finding area of a sector

1. If necessary, convert θ to radians.
2. The area of a sector of a circle with radius subtended by an angle measured in θ radians, is $A = \frac{1}{2}\theta r^2$.

Example 7.16 *A sector of a circle with diameter* 10 *ft and an angle of* 60°. *Sketch the area, and then find the area of the sector.*



Example 7.17 *An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees. What is the area of the sector of the grass the sprinkler waters?*



Example 7.18 *In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Sketch the situation, and determine the answer in radians, measured to two decimal places.*

As a point moves along a circle of radius r, its angular speed, ω, is the angular rotation θ per unit time, t.

The **linear speed** v of the point can be found as the distance travelled per unit time. The distance travelled is the arc length s, which we defined earlier.

Definition 7.1: Linear speed along a circle

Linear speed is defined as $v = \frac{s}{t}$ where s is the distance travelled along the circle and t is the amount of time.

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time.

The **angular speed** ω of the point can be found as the angle traversed per unit time. The angle traversed is the angle θ in radians.

Definition 7.2: Angular speed

Angular speed is defined as $\omega = \frac{\theta}{t}$ where θ is the angle traversed (in radians) and t is the amount of time.

Angular speeds will be in terms of radians per time unit.

Example 7.19 *A water wheel completes 1 rotation every 5 seconds. Find the angular speed in radians per second.*

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation:

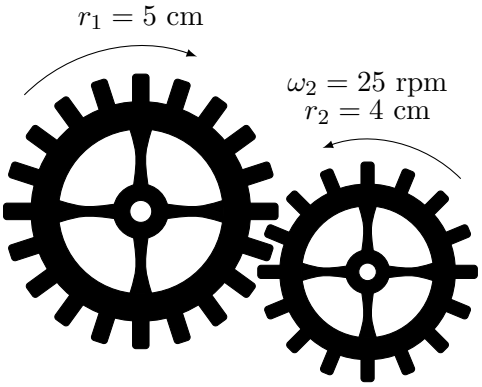
$$v = r\,\omega\,.$$

Example 7.20 *A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.*

Example 7.21 *An object travels a distance of 35 ft in 2.7 seconds as it moves along a circle of radius 2 ft. Find its linear and angular speed over that time period.*

Example 7.22 *A gear with an outer radius of $r_1 = 5$ cm moves in the clockwise direction, causing an interlocking gear with an outer radius of $r_2 = 4$ cm to move in the counterclockwise direction at an angular speed of $\omega_2 = 25$ rpm. What is the angular speed ω_1 of the larger gear?*

Hint: Imagine a particle on the outer radius of each gear. After the gears have rotated for a period of time $t > 0$, the circular displacement of each particle will be the same. How could we write this mathematically?



Section Recap

What are some take-away concepts from this section?

7.1 Exercises:

1. A ferris wheel rotates through an angle of 315°. Convert this angle to radians, then find one positive and one negative coterminal angle in radians.

2. A bicycle wheel has a radius of 0.35 meters. As the bicycle moves, the wheel rotates through an angle of 720°. Find the distance traveled by a point on the outer rim of the wheel.

3. A carousel has a radius of 8 meters and rotates at a constant rate of 3 revolutions per minute. Determine the angular speed in radians per minute, the linear speed of a rider on the outer edge of the carousel, and the distance the rider travels in 5 minutes.

7.1 Scaffolding Questions:

1. A ferris wheel rotates through an angle of 315°. Convert this angle to radians, then find one positive and one negative coterminal angle in radians.

    - Question Comprehension:
        - What information is given in the problem?
        - What does it mean to convert an angle from degrees to radians?
        - What does it mean for two angles to be coterminal?
    - Content Knowledge:
        - What formula is used to convert degrees to radians?
        - Why do coterminal angles differ by multiples of 2π radians?
        - How can you generate both positive and negative coterminal angles?
    - Solution:
        - Did you correctly convert 315° to radians?
        - Did you find one positive and one negative coterminal angle?
        - Are all of your answers expressed in radians?
    - Reflect:
        - Why do all coterminal angles represent the same terminal side?
        - How would your answers change if the Ferris wheel rotated through an additional full revolution?

2. A bicycle wheel has a radius of 0.35 meters. As the bicycle moves, the wheel rotates through an angle of 720°. Find the distance traveled by a point on the outer rim of the wheel.

   - Question Comprehension:
     - What quantity is the problem asking you to find?
     - How is the distance traveled by a point on the outer rim of the wheel related to arc length?
     - Why is the angle of rotation important?
   - Content Knowledge:
     - How do you convert 720° to radians?
     - What formula relates arc length, radius, and angle measured in radians?
     - Why must the angle be expressed in radians when using the arc length formula?
   - Solution:
     - Did you correctly convert the angle to radians?
     - Did you correctly apply the formula $s = r\,\theta$?
     - Did you include appropriate units in your final answer?
   - Reflect:
     - What does the fact that the wheel rotates through 720° tell you about the number of revolutions completed?
     - How would the distance traveled change if the radius of the wheel were doubled?

3. A carousel has a radius of 8 meters and rotates at a constant rate of 3 revolutions per minute. Determine the angular speed in radians per minute, the linear speed of a rider on the outer edge of the carousel, and the distance the rider travels in 5 minutes.

   - Question Comprehension:
     - What quantities are you being asked to find?
     - What is the difference between angular speed and linear speed?
     - How is the rider's distance traveled related to linear speed?
   - Content Knowledge:
     - How do you convert revolutions per minute into radians per minute?
     - What formula relates linear speed, angular speed, and radius?
     - How can you use speed to find the total distance traveled in a given amount of time?
   - Solution:
     - Did you correctly compute the angular speed?
     - Did you correctly use $v = r\omega$ to find the linear speed?
     - Did you correctly determine the total distance traveled in 5 minutes?
   - Reflect:
     - Why does a rider farther from the center travel faster even though every rider completes a revolution in the same amount of time?
     - If the carousel's rotation rate doubled, what would happen to the angular speed, linear speed, and distance traveled in 5 minutes?

7.2 Right Angle Trigonometry

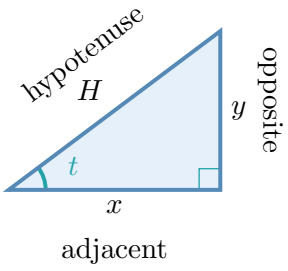
Learning Goals 7.2

After this section, I should be able to:

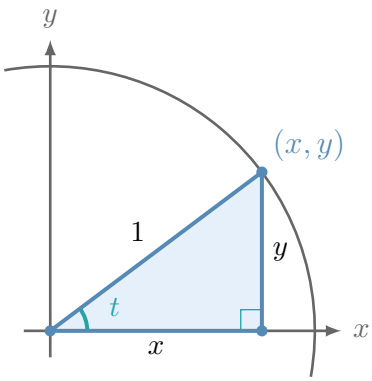
1. Use right triangles to evaluate trigonometric functions.
2. Find function values for 30° ($\frac{\pi}{6}$ rad), 45° ($\frac{\pi}{4}$ rad), and 60° ($\frac{\pi}{3}$ rad).
3. Use equal cofunctions of complementary angles.
4. Use the definitions of trigonometric functions of any angle.
5. Use right-triangle trigonometry to solve applied problems.

Below shows a right triangle with a vertical side of length y and a horizontal side has length x. The triangle is inscribed in a circle of radius 1. Such a circle, with a center at the origin and a radius of 1, is known as a **unit circle**. We can define the **trigonometric functions** in terms an angle t and the lengths of the sides of the triangle:

- The **adjacent side** is the side closest to the angle.
- The **opposite side** is the side across from the angle.
- The **hypotenuse** is the side of the triangle opposite the right angle.



Right triangle definitions:

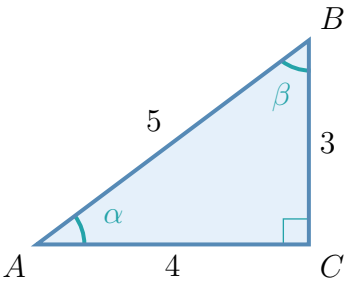


Unit circle definitions:

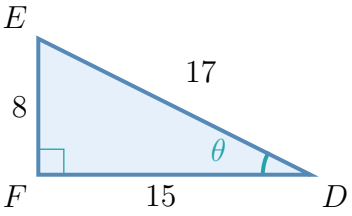
How To: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find **sine** as the ratio of the opposite side to the hypotenuse.
2. Find **cosine** as the ratio of the adjacent side to the hypotenuse.
3. Find **tangent** as the ratio of the opposite side to the adjacent side.

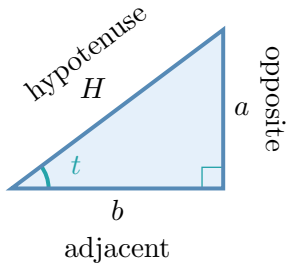
Example 7.23 *For the right triangle $\triangle ABC$ shown below, find the values of all three trigonometric functions of the angles α and β.*



Example 7.24 *For the right triangle $\triangle DEF$ shown below, find the values of all three trigonometric functions of the angle θ.*



In addition to sine, cosine, and tangent, there are three more functions. These are also defined in terms of the sides of the triangle. These functions are the reciprocals of the first three functions.



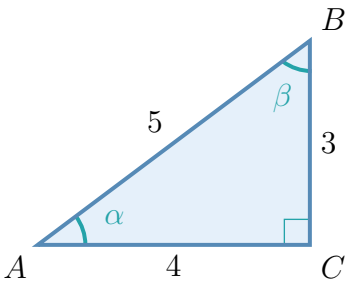
Name of function	**Abbreviation**		**Definition**	
sine t	$\sin(t)$	$=$	$\dfrac{\text{opposite side}}{\text{hypotenuse}}$	$=$
cosine t	$\cos(t)$	$=$	$\dfrac{\text{adjacent side}}{\text{hypotenuse}}$	$=$
tangent t	$\tan(t)$	$=$	$\dfrac{\text{opposite side}}{\text{adjacent side}}$	$=$
cosecant t	$\csc(t)$	$=$	$\dfrac{\text{hypotenuse}}{\text{opposite side}}$	$=$
secant t	$\sec(t)$	$=$	$\dfrac{\text{hypotenuse}}{\text{adjacent side}}$	$=$
cotangent t	$\cot(t)$	$=$	$\dfrac{\text{adjacent side}}{\text{opposite side}}$	$=$

We will usually use the abbreviated names of the functions. Notice from the table above that the pairs $\sin(t)$ and $\csc(t)$, $\cos(t)$ and $\sec(t)$, and $\tan(t)$ and $\cot(t)$ are reciprocals:

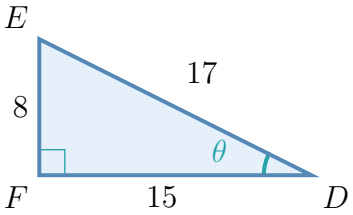
$$\csc(t) = \frac{1}{\sin(t)} \qquad \sec(t) = \frac{1}{\cos(t)} \qquad \cot(t) = \frac{1}{\tan(t)}$$

$$\sin(t) = \frac{1}{\csc(t)} \qquad \cos(t) = \frac{1}{\sec(t)} \qquad \tan(t) = \frac{1}{\cot(t)}$$

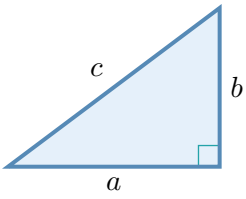
Example 7.25 *For the right triangle $\triangle ABC$ shown below, find the values of the trigonometric functions secant, cosecant, and cotangent of the angles α and β.*



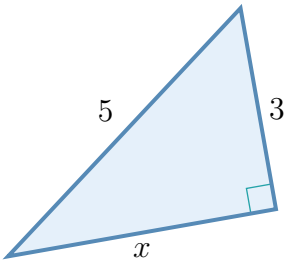
Example 7.26 *For the right triangle $\triangle DEF$ shown below, find the values of the trigonometric functions secant, cosecant, and cotangent of the angle θ.*

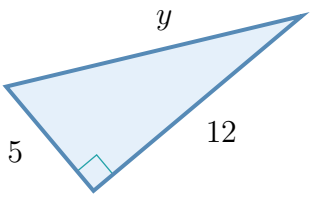


With a right triangle, we can use the **Pythagorean theorem** to relate the triangle's side lengths: the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs. Mathematically, this can be written as:

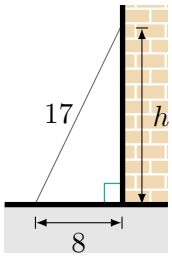


Example 7.27 *Find the missing side length of the right triangles below.*

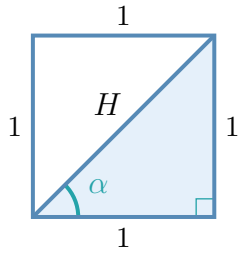




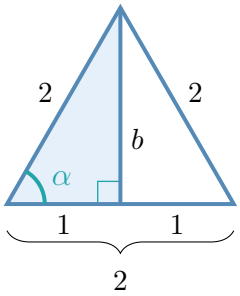
Example 7.28 *A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. How high up does the ladder reach?*



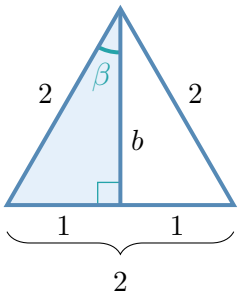
Example 7.29 *Consider a square with sides all equal to* 1 *and all interior angles equal to* 90°*. If we have a line dividing the square in half (from one corner to an opposite corner), find the angle measure* α *and the hypotenuse* H*. Then, find* $\sin(\alpha)$, $\cos(\alpha)$, *and* $\tan(\alpha)$.



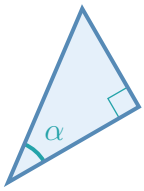
Example 7.30 *Consider an equilateral triangle with sides all equal to* 2 *and all interior angles equal to* 60°*. If we have a line dividing the triangle in half (from one corner to an opposite side), find the angle measure* α *and the side length* b*. Then, find all six trigonometric functions of* α*.*



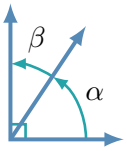
Example 7.31 *Consider an equilateral triangle with sides all equal to* 2 *and all interior angles equal to* 60°. *If we have a line dividing the triangle in half (from one corner to an opposite side), find the angle measure* β *and the side length* b. *Then, find all six trigonometric functions of* β.

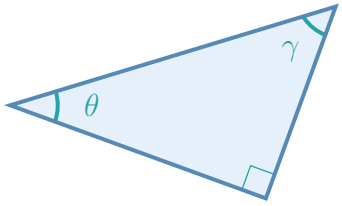


Example 7.32 *Consider a right triangle with an acute angle* α *where* $\sin(\alpha) = \frac{2}{3}$. *Find* $\cos(\alpha)$ *and* $\tan(\alpha)$.



Two angles are **complementary angles** if they add up to 90° :





With Examples 7.30 and 7.31, what do we notice about all of the trigonometric functions of α and β, two complementary angles?

Theorem: Cofunction Theorem

If α and β are two complementary angles in a right triangle, then:

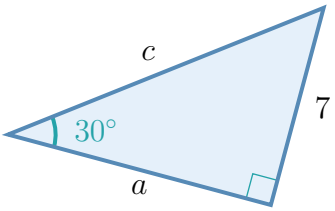
$$\sin(\alpha) = \cos(\beta) \qquad \sec(\alpha) = \csc(\beta) \qquad \tan(\alpha) = \cot(\beta)$$

$$\sin(\beta) = \cos(\alpha) \qquad \sec(\beta) = \csc(\alpha) \qquad \tan(\beta) = \cot(\alpha)$$

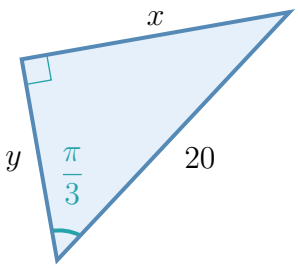
and the pairs of functions $\{\sin, \cos\}$, $\{\sec, \csc\}$, and $\{\tan, \cot\}$ are **cofunctions**.

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But, "the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides."

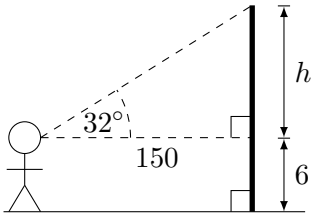
Example 7.33 *Find the unknown sides of the triangle. Determine the solutions using exact, analytical values. Then, make sure your calculator is in degree mode and use your calculator to compute the same values.*



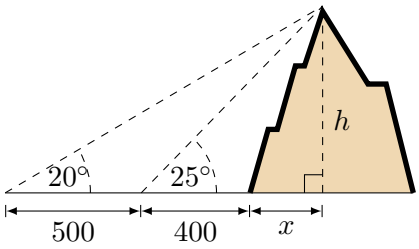
Example 7.34 *Find the unknown sides of the triangle using exact, analytical values.*



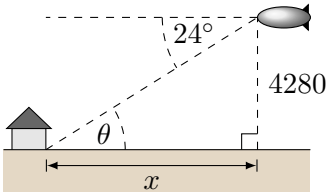
Example 7.35 *A person stands* 150 *ft away from a flagpole and measures an* **angle of elevation** *of* 32° *from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole?*



Example 7.36 *A person standing* 400 *ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be* 25°. *The person then walks* 500 *ft straight back and measures the angle of elevation to now be* 20°. *How tall is the mountain?*



Example 7.37 *A blimp* 4280 *ft above the ground measures an* **angle of depression** *of* 24° *from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?*



Section Recap

What are some take-away concepts from this section?

7.2 Exercises:

1. A right triangle has side lengths 8 cm, 15 cm, and 17 cm, where the 17 cm side is the hypotenuse. Find all six trigonometric functions of the acute angle opposite the 8 cm side.

2. A wheelchair ramp rises 3 feet vertically over a horizontal distance of 12 feet. Find the angle that the ramp makes with the ground. Round your answer to the nearest tenth of a degree.

3. A drone is flying at a constant altitude of 120 meters. An observer on the ground measures the angle of elevation to the drone to be $38°$. The drone then flies directly away from the observer while maintaining the same altitude, and the angle of elevation decreases to $25°$. How far did the drone travel horizontally?

7.2 Scaffolding Questions:

1. A right triangle has side lengths 8 cm, 15 cm, and 17 cm, where the 17 cm side is the hypotenuse. Find all six trigonometric functions of the acute angle opposite the 8 cm side.

    - Question Comprehension:
        - Which side is the hypotenuse?
        - Relative to the given angle, which side is opposite and which side is adjacent?
        - What does the problem ask you to find?
    - Content Knowledge:
        - What are the definitions of sine, cosine, and tangent in a right triangle?
        - How are cosecant, secant, and cotangent related to sine, cosine, and tangent?
        - Which side lengths should be used in each ratio?
    - Solution:
        - Did you correctly identify the opposite, adjacent, and hypotenuse sides?
        - Did you correctly compute $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$?
        - Did you correctly use reciprocals to find $\csc(\theta)$, $\sec(\theta)$, and $\cot(\theta)$?
    - Reflect:
        - Why are the values of sine and cosine both less than 1 for an acute angle?
        - How would the six trigonometric function values change if you used the other acute angle of the triangle?

2. A wheelchair ramp rises 3 feet vertically over a horizontal distance of 12 feet. Find the angle that the ramp makes with the ground. Round your answer to the nearest tenth of a degree.

   - Question Comprehension:
     - How can this situation be represented using a right triangle?
     - Which side is opposite the angle you are trying to find?
     - Which side is adjacent to the angle you are trying to find?
   - Content Knowledge:
     - Which trigonometric function relates the opposite and adjacent sides?
     - How do you use an inverse trigonometric function to find an angle?
     - What calculator operation will allow you to solve for the angle?
   - Solution:
     - Did you correctly set up a tangent ratio?
     - Did you correctly use the inverse tangent function?
     - Did you round your answer to the nearest tenth of a degree?
   - Reflect:
     - Why might it be important to know the angle of a wheelchair ramp in construction?
     - What would happen to the angle if the ramp rose the same height over a shorter horizontal distance?

3. A drone is flying at a constant altitude of 120 meters. An observer on the ground measures the angle of elevation to the drone to be 38°. The drone then flies directly away from the observer while maintaining the same altitude, and the angle of elevation decreases to 25°. How far did the drone travel horizontally?

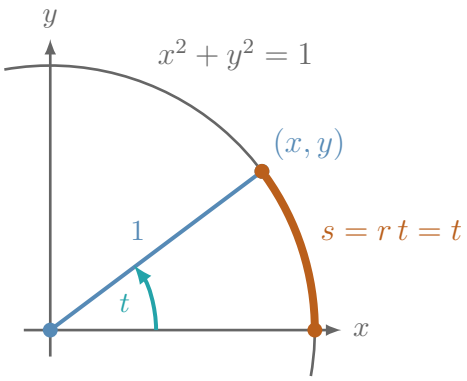
   - Question Comprehension:
     - What quantities are known at each observation?
     - How can each observation be modeled with a right triangle?
     - What quantity is represented by the drone's horizontal travel distance?
   - Content Knowledge:
     - Which trigonometric function relates an angle of elevation, the altitude, and the horizontal distance?
     - How can you find the horizontal distance from the observer to the drone for each angle?
     - How can those two distances be used to determine how far the drone traveled?
   - Solution:
     - Did you correctly find the initial horizontal distance?
     - Did you correctly find the final horizontal distance?
     - Did you subtract the distances appropriately to find the drone's horizontal travel?
   - Reflect:
     - Why does the angle of elevation decrease as the drone moves farther away while maintaining the same altitude?
     - How would your answer change if the drone had increased its altitude while moving away?

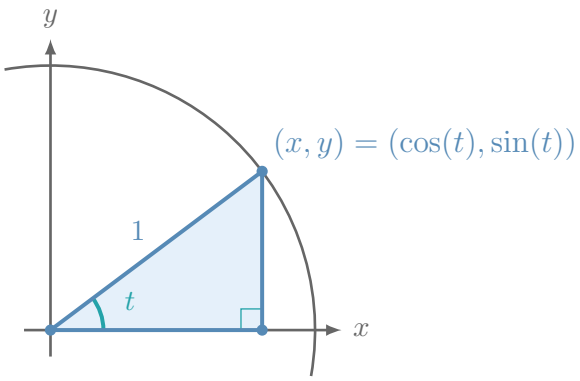
7.3 Unit Circle

Learning Goals 7.3

After this section, I should be able to:

1. Find function values for 30° ($\frac{\pi}{6}$ rad), 45° ($\frac{\pi}{4}$ rad), and 60° ($\frac{\pi}{3}$ rad).
2. Identify the domain and range of sine and cosine functions.
3. Find reference angles.
4. Use reference angles to evaluate trigonometric functions.

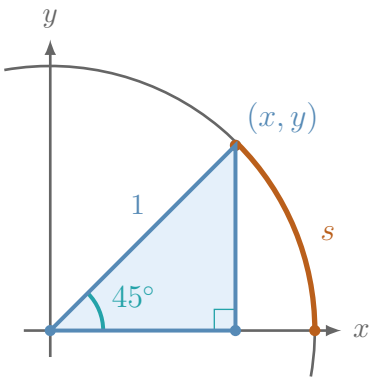




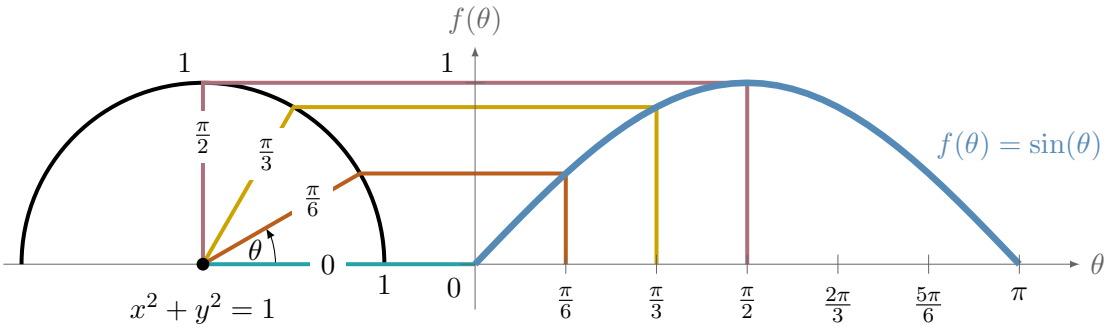
A **unit circle** has a center at $(0, 0)$ and radius of 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle t.

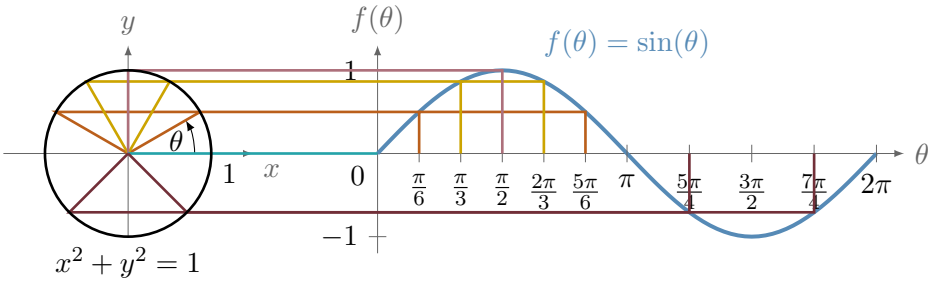
Example 7.38 *Determine the $x-$ and $y-$coordinates of the point along the unit circle with the central angle of 45°. Then, find the arc-length s.*



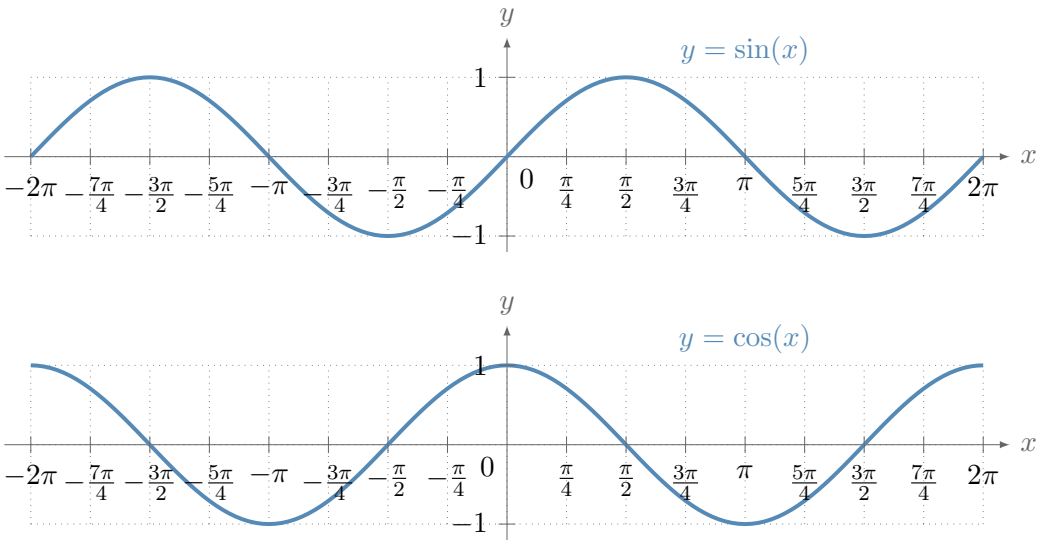
The sine function relates a real number t to the $y-$coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the $y-$value of the endpoint on the unit circle of an arc of length t. In the figure on the previous page, we see sine of t is equal to y. Like all functions, the sine function has an input and an output: its input is the measure of the angle, and its output is the $y-$coordinate of the corresponding point on the unit circle.



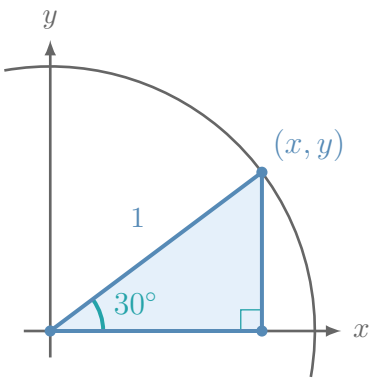
We can extend the above picture to include angles from 0 to 2π radians:



Since the trigonometric functions repeat every 2π radians ($360°$), we can follow a similar argument to get the graphs of $y = \sin(x)$ and $y = \cos(x)$ for x in the interval $[-2\pi, 2\pi]$:



Example 7.39 *Determine the $x-$ and $y-$coordinates of the point along the unit circle with the central angle of* $30°$.

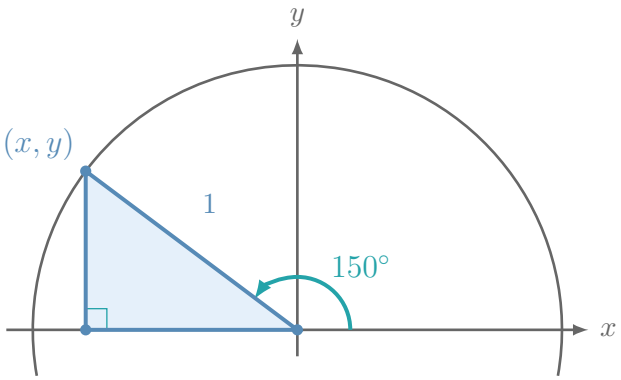


For angles not in Quadrant I, we have to shift perspective slightly to identify since the $x-$ and $y-$coordinates are not both positive. To help us shift perspective, we will use angles we *do* know how to easily work with **reference angles**.

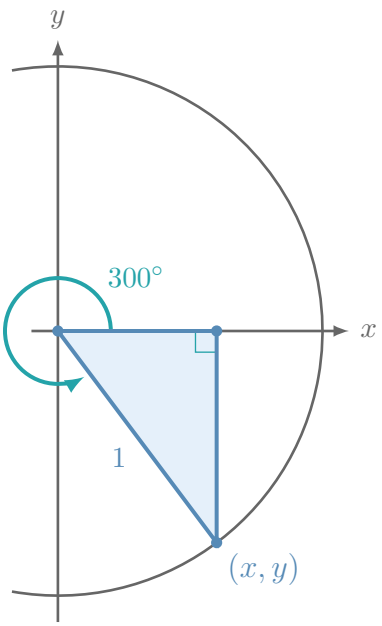
Definition 7.3: Reference Angle

An angle's **reference angle** is the size of the smallest acute angle (an angle less than $90°$), formed by the terminal side of the angle and the horizontal axis.

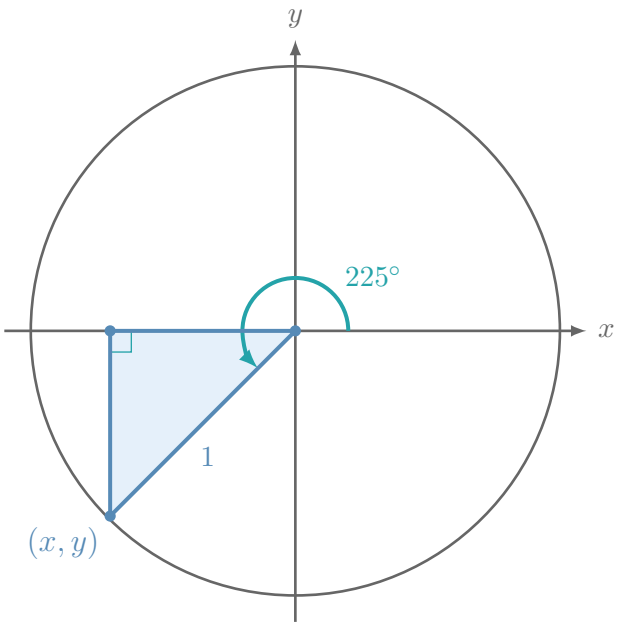
Example 7.40 *Using a reference angle, determine the $x-$ and $y-$coordinates of the point along the unit circle with the central angle of* $150°$.



Example 7.41 *Using a reference angle, determine the $x-$ and $y-$coordinates of the point along the unit circle with the central angle of* $300°$.



Example 7.42 *Using a reference angle, determine the $x-$ and $y-$coordinates of the point along the unit circle with the central angle of* $225°$.



If we are working with the unit circle $x^2 + y^2 = 1$, how can we relate $x = \cos(t)$ and $y = \sin(t)$?

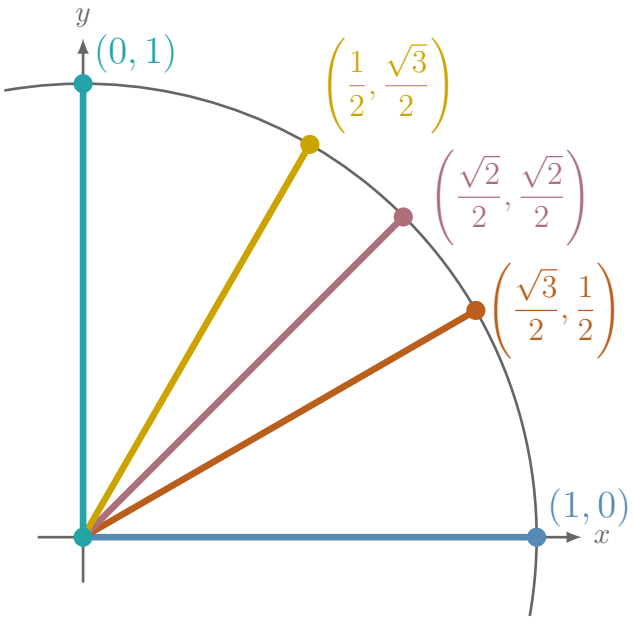
This is the **Pythagorean Identity.**

Example 7.43 *If* $\sin(t) = \dfrac{3}{7}$ *and* t *is in the second quadrant, find* $\cos(t)$.

Hint: we need to choose the solution with the appropriate sign for the x-values in the quadrant where is located.

Example 7.44 *If* $\cos(\theta) = \dfrac{24}{25}$ *and* θ *is in the fourth quadrant, find* $\sin(\theta)$.

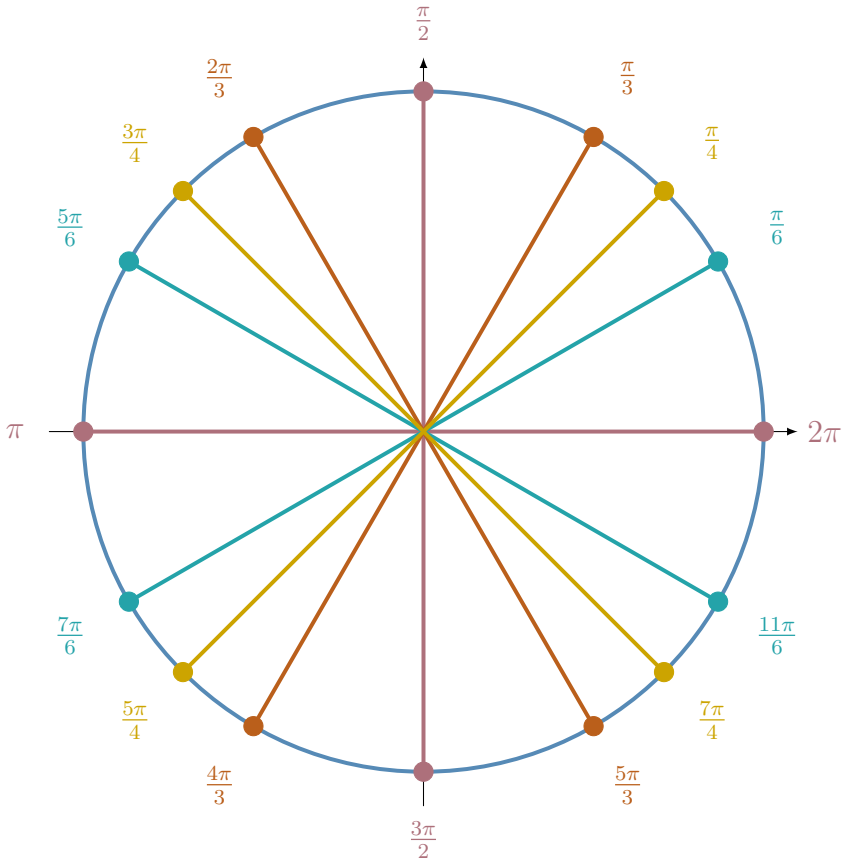
Example 7.45 *With the figure below, determine the angles (in terms of radians and degrees) that correspond with the given points.*



Example 7.46 *Using the information from the unit circle above, fill in the table below.*

$\sin(0°) = \sin(0) =$	$\cos(0°) = \cos(0) =$
$\sin(30°) = \sin(\pi/6) =$	$\cos(30°) = \cos(\pi/6) =$
$\sin(45°) = \sin(\pi/4) =$	$\cos(45°) = \cos(\pi/4) =$
$\sin(60°) = \sin(\pi/3) =$	$\cos(60°) = \cos(\pi/3) =$
$\sin(90°) = \sin(\pi/2) =$	$\cos(90°) = \cos(\pi/2) =$

Example 7.47 *Determine the* (x, y) *coordinates for* all *of the commonly used coordinates of the unit circle.*



Section Recap

What are some take-away concepts from this section?

7.3 Exercises:

1. A point moves counterclockwise around a unit circle and stops at an angle of $135°$. Find the coordinates of the point and determine the values of $\sin(135°)$ and $\cos(135°)$.

2. A Ferris wheel passenger is located at an angle of $225°$ from the positive x-axis, where the center of the wheel is at the origin and the wheel has radius 1. Find the reference angle and Then, determine the exact values of $\sin(225°)$, $\cos(225°)$, and $\tan(225°)$.

3. A weather radar rotates continuously and detects a storm cell located in the direction of $300°$. Using the unit circle, determine the exact coordinates of the point on the unit circle corresponding to this direction. Then, determine all six trigonometric functions of $300°$.

7.3 Scaffolding Questions:

1. A point moves counterclockwise around a unit circle and stops at an angle of 135°. Find the coordinates of the point and determine the values of sin(135°) and cos(135°).

   - Question Comprehension:
     - In which quadrant does an angle of 135° lie?
     - What does a point on the unit circle represent?
     - How are the coordinates of a point on the unit circle related to sine and cosine?
   - Content Knowledge:
     - What is the reference angle for 135°?
     - What are the coordinates associated with the special angle 45°?
     - Which coordinate corresponds to cosine and which corresponds to sine?
   - Solution:
     - Did you correctly identify the coordinates of the point?
     - Did you correctly determine the signs of the coordinates in Quadrant II?
     - Did you correctly identify sin(135°) and cos(135°) from the coordinates?
   - Reflect:
     - Why does cosine have a different sign than sine at 135°?
     - How would your answer change if the angle were 45° instead?

2. A Ferris wheel passenger is located at an angle of 225° from the positive x-axis, where the center of the wheel is at the origin and the wheel has radius 1. Find the reference angle and Then, determine the exact values of $\sin(225°)$, $\cos(225°)$, and $\tan(225°)$.

   - Question Comprehension:
     - In which quadrant does 225° lie?
     - What is a reference angle?
     - Why can reference angles help evaluate trigonometric functions?
   - Content Knowledge:
     - How do you find the reference angle for an angle in Quadrant III?
     - What are the exact trigonometric values associated with a 45° reference angle?
     - What signs do sine, cosine, and tangent have in Quadrant III?
   - Solution:
     - Did you correctly find the reference angle?
     - Did you correctly determine the signs of the trigonometric functions?
     - Did you simplify your exact values?
   - Reflect:
     - Why does tangent remain positive in Quadrant III even though sine and cosine are both negative?
     - How is 225° related to 45° on the unit circle?

3. A weather radar rotates continuously and detects a storm cell located in the direction of 300°. Using the unit circle, determine the exact coordinates of the point on the unit circle corresponding to this direction. Then, determine all six trigonometric functions of 300°.

   - Question Comprehension:
     - In which quadrant does 300° lie?
     - What information can be obtained from a point on the unit circle?
     - What does the problem ask you to determine?
   - Content Knowledge:
     - What is the reference angle for 300°?
     - What are the exact sine and cosine values associated with that reference angle?
     - How are secant, cosecant, and cotangent related to cosine, sine, and tangent?
   - Solution:
     - Did you correctly determine the coordinates of the point?
     - Did you correctly compute $\sin(300°)$, $\cos(300°)$, and $\tan(300°)$?
     - Did you correctly use reciprocals to find $\sec(300°)$, $\csc(300°)$, and $\cot(300°)$?
   - Reflect:
     - Why are the sine and cosine values different signs at 300°?
     - How does knowing the reference angle make it easier to evaluate trigonometric functions of larger angles?

7.4 The Other Trigonometric Functions

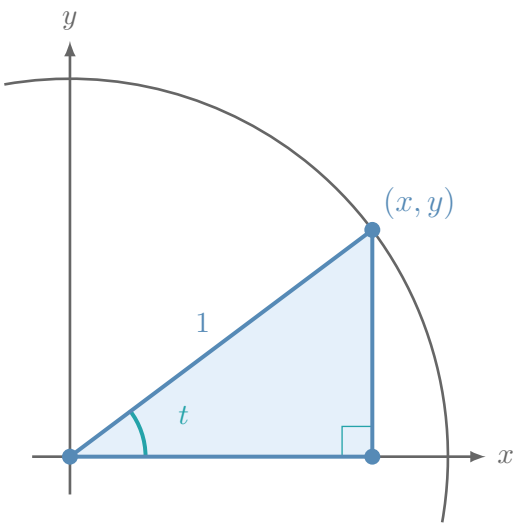
Learning Goals 7.4

After this section, I should be able to:

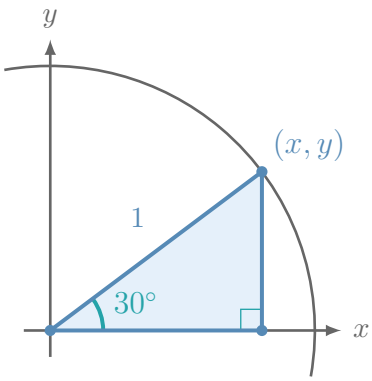
1. Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of 30° ($\frac{\pi}{6}$ rad), 45° ($\frac{\pi}{4}$ rad), and 60° ($\frac{\pi}{3}$ rad).
2. Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
3. Use properties of even and odd trigonometric functions.
4. Recognize and use fundamental identities.
5. Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $1/12$ or less, regardless of its length. A tangent represents a *ratio*; so, this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions.

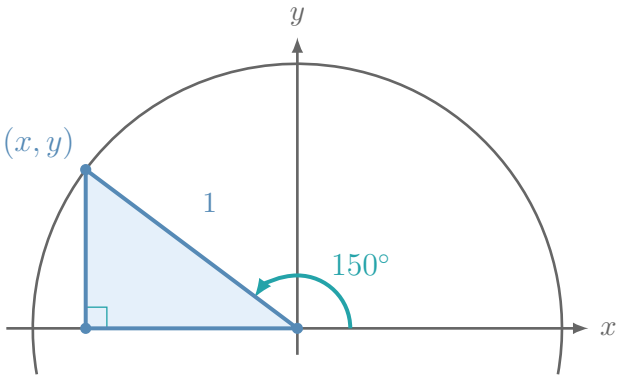
Example 7.48 *Define* $\sin(t)$, $\cos(t)$, $\tan(t)$, $\csc(t)$, $\sec(t)$, $\cot(t)$ *in terms of* x *and* y *where* (x, y) *is a point along the unit circle and* t *is the central angle.*



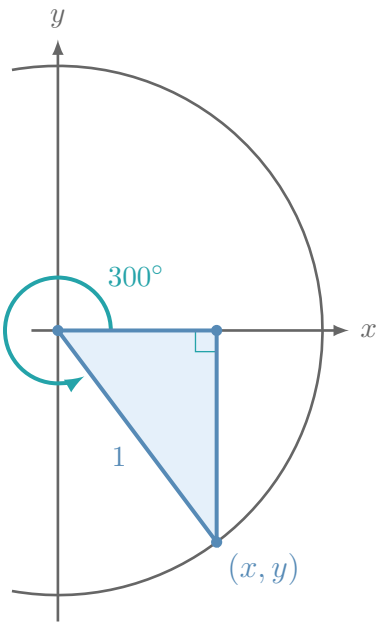
Example 7.49 *Find the exact values of all six trigonometric functions of* $30°$.



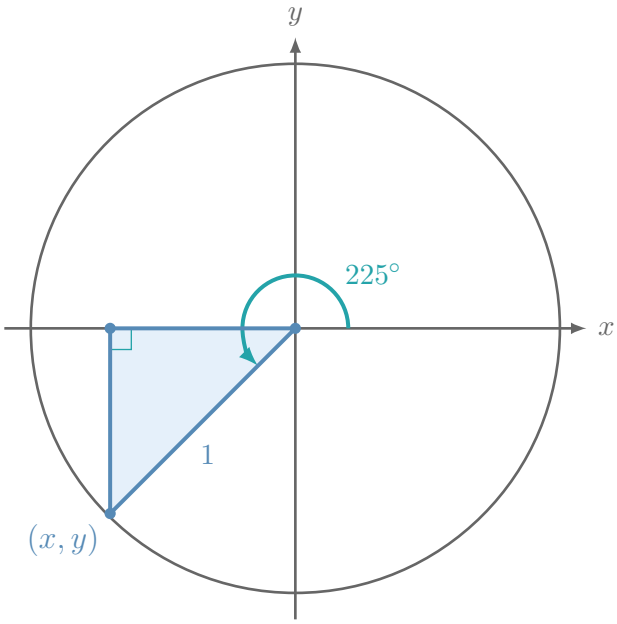
Example 7.50 *Find the exact values of all six trigonometric functions of* $150°$.



Example 7.51 *Find the exact values of all six trigonometric functions of* $300°$.

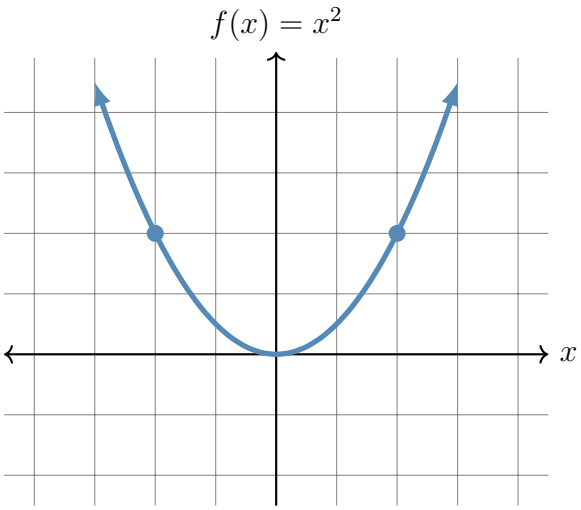


Example 7.52 *Find the exact values of all six trigonometric functions of* $225°$.

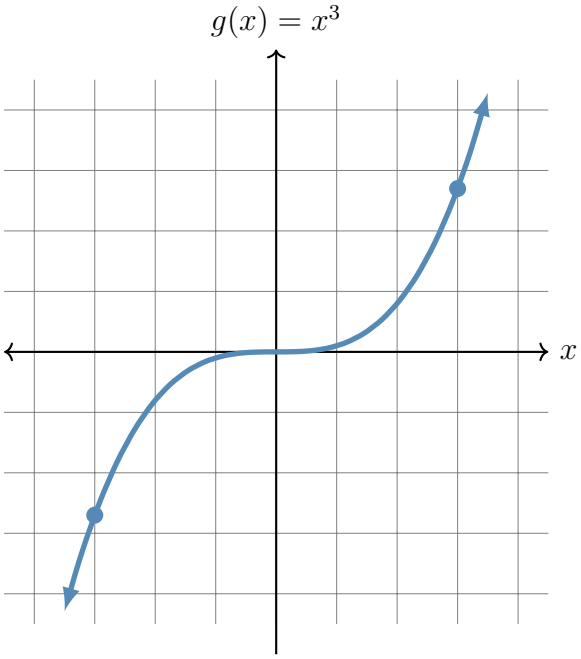


Even and Odd Functions

The graph of the function $f(x) = x^2$ below is symmetrical about the y-axis. All along the curve, any two points with opposite x-values have the same function value. So, $f(x) = x^2$ is an **even function**. An **even function** is a function such that two inputs *that are opposites* have the same output.

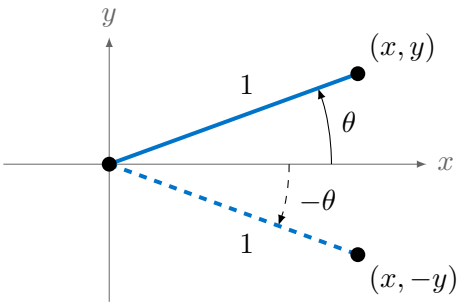


The graph $g(x) = x^3$ is not symmetrical about the y-axis. All along the graph, any two points with opposite x-values also have opposite y-values. So, $g(x) = x^3$ is an **odd function**. An **odd function** is a function such that two inputs *that are opposites* have outputs that are also opposites.



We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle.

Example 7.53 *Find* $\sin(\theta)$ *and* $\sin(-\theta)$ *and determine what this tells us about the sine function.*



Example 7.54 *Find the other five trigonometric functions of* θ *and* $-\theta$ *to determine if these functions are even or odd.*

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

Definition 7.4

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \& \quad \sec(\theta) = \frac{1}{\cos(\theta)} \quad (\text{for } \cos(\theta) \neq 0),$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \quad \& \quad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} \quad (\text{for } \sin(\theta) \neq 0 \text{ and } \tan(\theta) \neq 0).$$

Example 7.55 *Given* $\sin(45°) = \dfrac{1}{\sqrt{2}}$ *and* $\cos(45°) = \dfrac{1}{\sqrt{2}}$, *find* $\tan(45°)$.

Example 7.56 *Given* $\sin\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2}$ *and* $\cos\left(\dfrac{5\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$, *find* $\sec\left(\dfrac{5\pi}{6}\right)$.

Example 7.57 *If* $\cos(t) = \dfrac{12}{13}$ *and* t *is in quadrant IV, find the values of the other five trigonometric functions.*

Example 7.58 *If* $\sec(\theta) = -\dfrac{17}{8}$ *and* $0 < \theta < \pi$ *find the values of the other five trigonometric functions.*

Example 7.59 *Simplify* $\tan(\theta)\,\cos(\theta)$.

Example 7.60 *Simplify* $\dfrac{\sec(t)}{\tan(t)}$.

Example 7.61 *Simplify* $\csc(\theta)\cos(\theta)\tan(\theta)$.

Alternate Forms of the Pythagorean Identity

To derive alternate forms of the Pythagorean Identity, let's begin with the Pythagorean Identity:

Now, divide each term by $\sin^2(t)$ and use trigonometric identities to simplify:

For another form, begin with the Pythagorean Identity again, divide each term by $\cos^2(t)$, and then simplify:

Example 7.62 *Prove that* $\tan(\theta) + \cot(\theta) = \sec(\theta)\ \csc(\theta)\,.$

Example 7.63 *Using a calculator, evaluate the cosecant of* $\frac{5\pi}{7}$.

Example 7.64 *Using a calculator, evaluate the cotangent of* $-\frac{\pi}{8}$.

Section Recap

What are some take-away concepts from this section?

7.4 Exercises:

1. Find the exact values of all six trigonometric functions of $240°$.
2. If $\sec(\theta) = -\frac{13}{5}$ and θ is in Quadrant II, find the values of the other five trigonometric functions.
3. Simplify the expression
$$\frac{\sec(\theta)\sin(\theta)}{\tan(\theta)},$$
and explain why your result is true using fundamental trigonometric identities.

7.4 Scaffolding Questions:

1. Find the exact values of all six trigonometric functions of 240°.

   - Question Comprehension:
     - In which quadrant does 240° lie?
     - What is the reference angle for 240°?
     - What does the problem ask you to determine?
   - Content Knowledge:
     - What are the exact sine and cosine values associated with the reference angle?
     - How do the signs of the trigonometric functions depend on the quadrant?
     - How can secant, cosecant, and cotangent be found from sine, cosine, and tangent?
   - Solution:
     - Did you correctly find $\sin(240°)$ and $\cos(240°)$?
     - Did you correctly determine $\tan(240°)$?
     - Did you correctly use reciprocal relationships to find $\sec(240°)$, $\csc(240°)$, and $\cot(240°)$?
   - Reflect:
     - Why do some trigonometric functions have positive values while others are negative at 240°?
     - How would your answers change if the angle were 60° instead?

2. If $\sec(\theta) = -\frac{13}{5}$ and θ is in Quadrant II, find the values of the other five trigonometric functions.

   - Question Comprehension:
     - What information does the value of secant provide about cosine?
     - What signs should the trigonometric functions have in Quadrant II?
     - What quantities are you trying to find?
   - Content Knowledge:
     - How can you use the reciprocal relationship between secant and cosine?
     - How can the Pythagorean Identity be used to find sine?
     - How can tangent, cosecant, and cotangent be found once sine and cosine are known?
   - Solution:
     - Did you correctly find $\cos(\theta)$?
     - Did you correctly use the Pythagorean Identity to find $\sin(\theta)$?
     - Did you correctly determine the remaining trigonometric functions and their signs?
   - Reflect:
     - Why is it necessary to know the quadrant when using the Pythagorean Identity?
     - How would your answers differ if θ were in Quadrant III?

3. Simplify the expression

$$\frac{\sec(\theta)\sin(\theta)}{\tan(\theta)},$$

and explain why your result is true using fundamental trigonometric identities.

- Question Comprehension:
  - What does it mean to simplify a trigonometric expression?
  - Which trigonometric functions appear in the expression?
  - What identities might be useful?
- Content Knowledge:
  - How can secant be rewritten in terms of cosine?
  - How can tangent be rewritten in terms of sine and cosine?
  - What algebraic simplifications become possible after these substitutions?
- Solution:
  - Did you correctly rewrite secant and tangent using sine and cosine?
  - Did you simplify the resulting expression carefully?
  - Did you verify that your final expression is equivalent to the original expression?
- Reflect:
  - Why is rewriting everything in terms of sine and cosine often useful?
  - How do identities allow us to recognize different expressions as representing the same quantity?

8 Periodic Functions

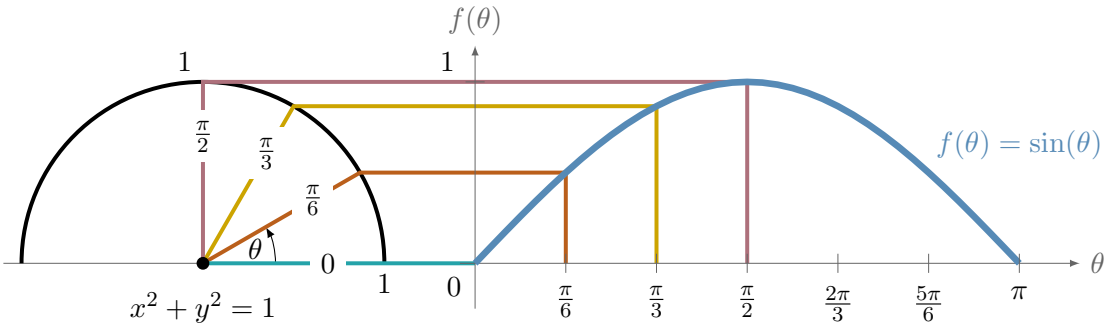
8.1 Graphs of the Sine and Cosine Functions

Learning Goals 8.1

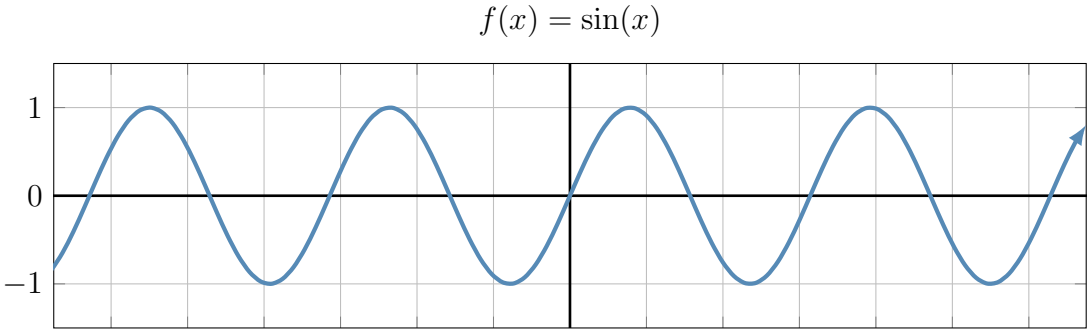
After this section, I should be able to:

1. Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
2. Use phase shifts of sine and cosine curves.

The sine function relates a real number t to the $y-$coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the $y-$value of the endpoint on the unit circle of an arc of length t. In the figure on the previous page, we see sine of t is equal to y. Like all functions, the sine function has an input and an output: its input is the measure of the angle, and its output is the $y-$coordinate of the corresponding point on the unit circle.

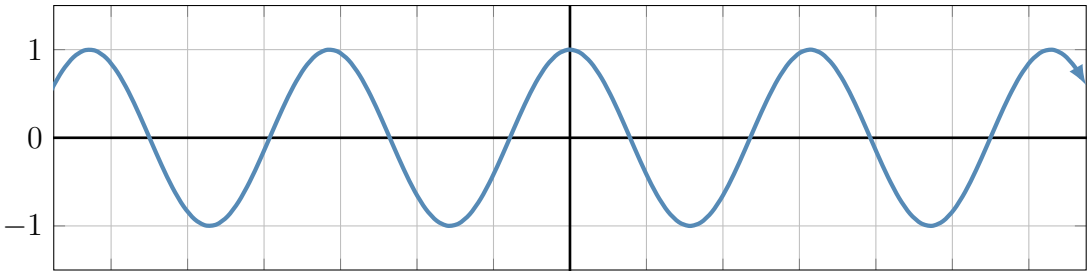


Example 8.1 *Plot points on the sine curve below.*



Example 8.2 *Plot points on the cosine curve below.*

$$g(x) = \cos(x)$$



In both graphs, the shape of the graph repeats after 2π which means the functions are **periodic** with a period of 2π. In general, a function $f(x)$ is **periodic** if there is a value $p > 0$ such that

$$f(x) = f(x + p)\,.$$

There could be many numbers p that satisfy the above requirements. If there is a smallest such number p, then we call that number the period of the function $f(x)$. How could represent this with the sine and cosine functions?

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. The sine function is **symmetric about the origin**. Earlier, we determined that $\sin(x)$ was an **odd function** since $\sin(x) = -\sin(-x)$. This can be seen in the graph.

The cosine function is **symmetric about the y-axis**. Earlier, we determined that $\cos(x)$ was an **even function** since $\cos(x) = \cos(-x)$. This can be seen in the graph as well.

Note: Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic with period 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin(x)$ is symmetric about the origin because it is an odd function.
- The graph of $y = \cos(x)$ is symmetric about the y-axis because it is an even function.

Investigating Sinusoidal Functions

A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D\,.$$

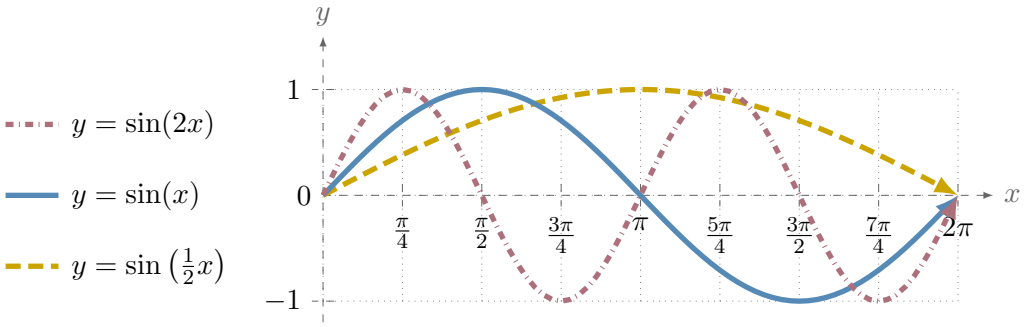
Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In general, B is related to the period by $P = \dfrac{2\pi}{|B|}$.

If $|B| > 1$, then the period is ___ 2π and the function is **compressed horizontally.**

If $|B| < 1$, then the period is ___ 2π and the function is **stretched horizontally.**

If $|B| = 1$, then the period is ___ 2π.



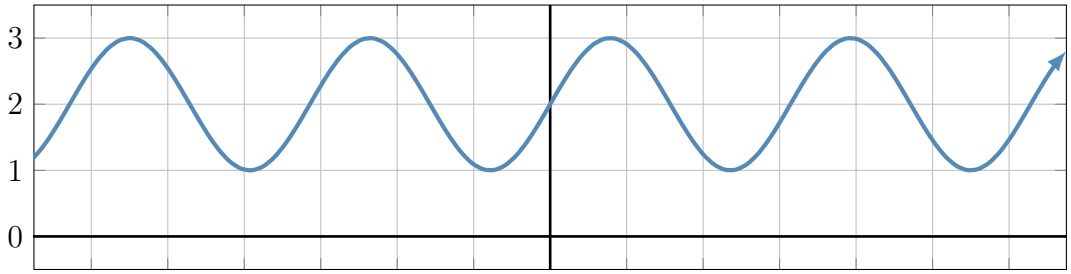
Example 8.3 *Determine the period of the function* $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$, *and then sketch the function.*

In the general form for a sinusoidal function, the B variable relates to the period. The A variable is related to the amplitude.

A represents the **vertical stretch factor** and its absolute value $|A|$ is the amplitude.

The local maxima will be a distance of $|A|$ units above the **horizontal midline** of the graph, which is the line $y = D$.

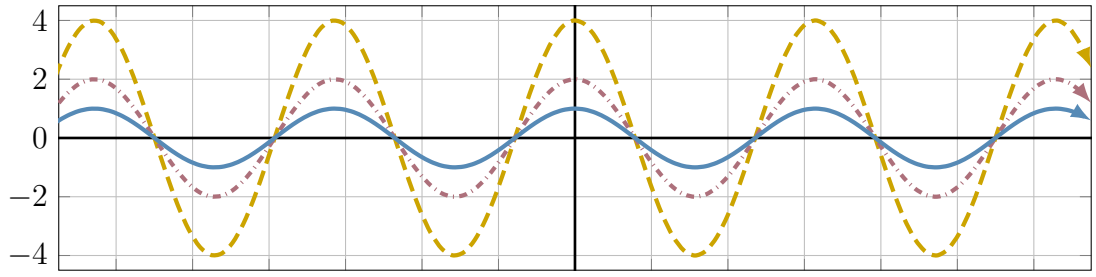
Example 8.4 *Sketch the horizontal midline in the graph below, and then determine the function's amplitude.*



If $|A| > 1$, then the function is **stretched vertically.**

If $|A| < 1$, then the function is **compressed vertically.**

Example 8.5 *Sketch the horizontal midline in the graph below, and then determine the amplitude of each function shown.*

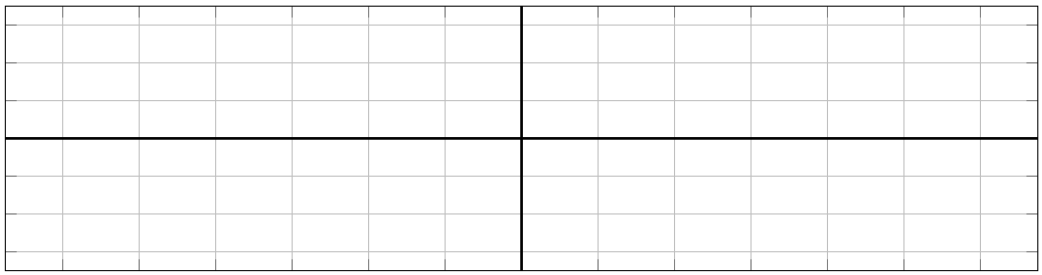


$y = f(x)$

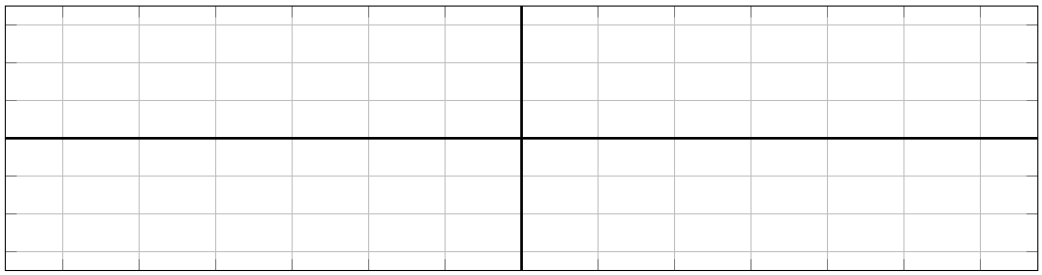
$y = g(x)$

$y = h(x)$

Example 8.6 *Determine the amplitude of the function* $f(x) = -4\sin(x)$*, then sketch the function. Is the function stretched or compressed vertically?*



Example 8.7 *Determine the period and amplitude of the function* $f(x) = 3\cos\left(\frac{\pi}{2}x\right)$*, then sketch the function.*



Now that we understand how A and B relate to the general form for the sine and cosine functions, we will explore the variables C and D. The general form is:

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \quad \text{and} \quad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D\,.$$

The value $\dfrac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function.

If $C > 0$, the graph shifts to the right.

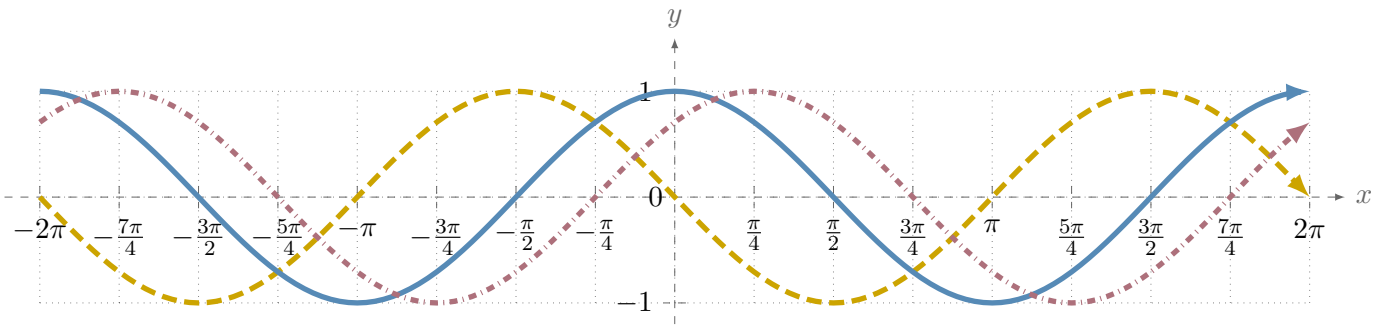
If $C < 0$, the graph shifts to the left.

Example 8.8 *Determine the equations for the cosine functions shown below.*

$y = f(x)$

$y = g(x)$

$y = h(x)$

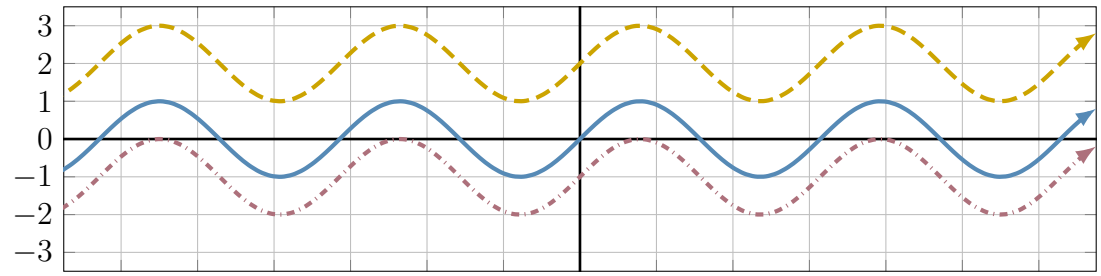


The value D represents the **vertical shift** in the graph. Any value of D other than zero shifts the graph up or down.

If $D > 0$, the graph shifts up.

If $D < 0$, the graph shifts down.

Example 8.9 *Determine the vertical shift of the sine functions below.*



$y = f(x)$

$y = g(x)$

$y = h(x)$

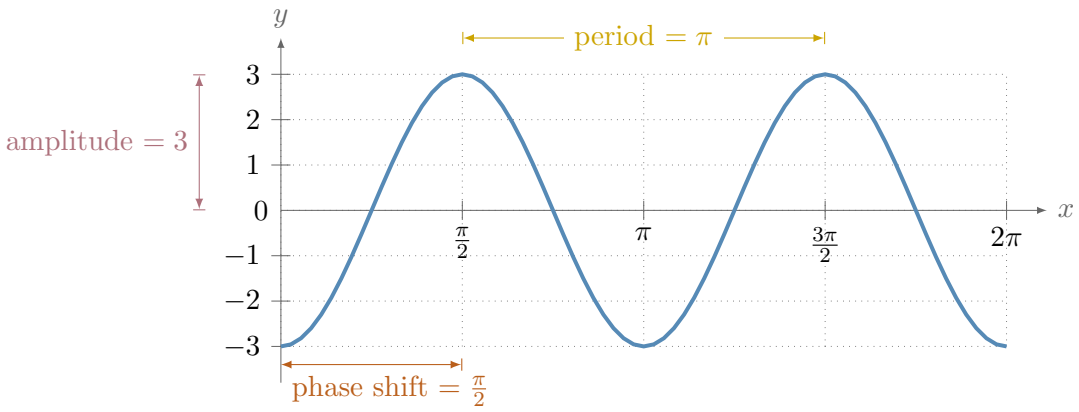
Note: Variations of the Sine and Cosine Function

Given the equation in the form

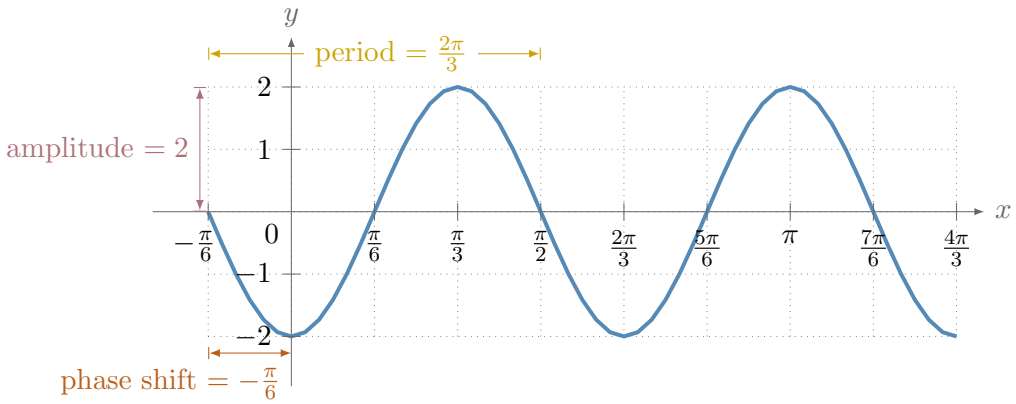
$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D\,,$$

- The A term is related to the amplitude– the **amplitude** is $|A|$,
- The B term is related to the **period**– the **period** is $P = \dfrac{2\pi}{|B|}$,
- The C term is related to the **horizontal shift**– the **phase shift** is $\dfrac{C}{B}$,
- The D term is the **vertical shift**– the **midline** is $y = D$.

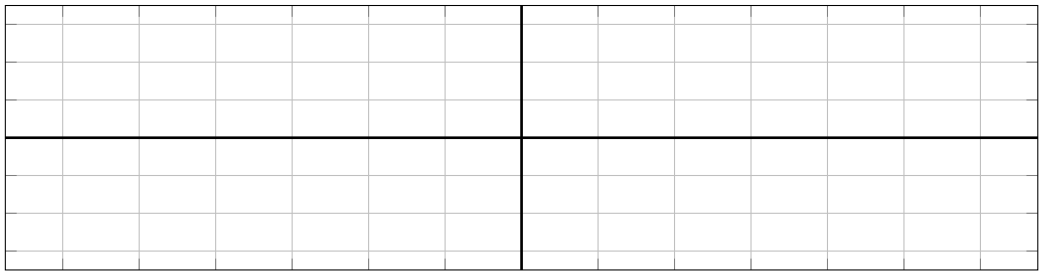
Example 8.10 *Given the amplitude, period, and phase shift for the* cosine *function below, determine the equation.*



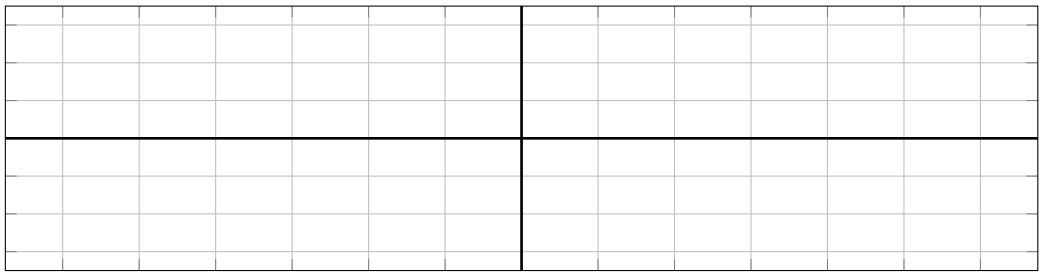
Example 8.11 *Given the amplitude, period, and phase shift for the* sine *function below, determine the equation of the function shown.*



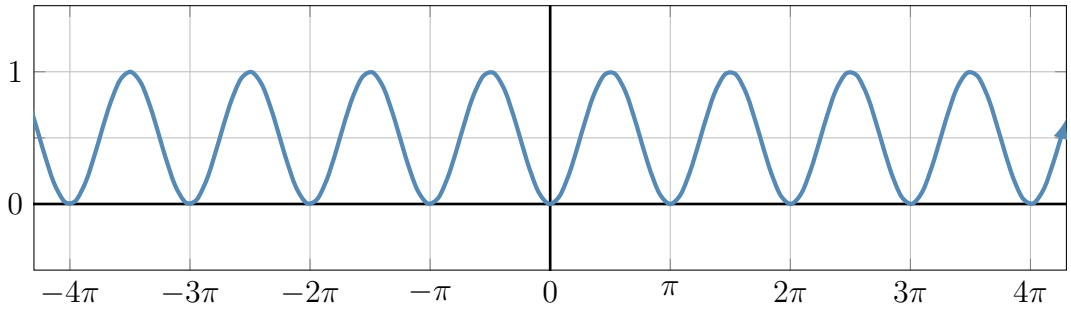
Example 8.12 *Determine the midline, amplitude, period, and phase shift of the function* $y = 2\sin\left(\frac{\pi}{2}\,x\right) + 1$. *Then, sketch the function.*



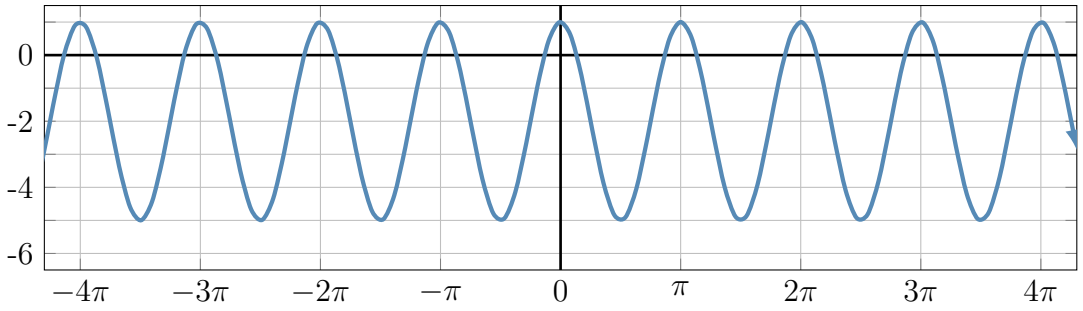
Example 8.13 *Determine the midline, amplitude, period, and phase shift of the function* $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$. *Then, sketch the function.*



Example 8.14 *Determine the formula for the* cosine *function below.*



Example 8.15 *Determine the formula for the* sinusoidal *function below.*

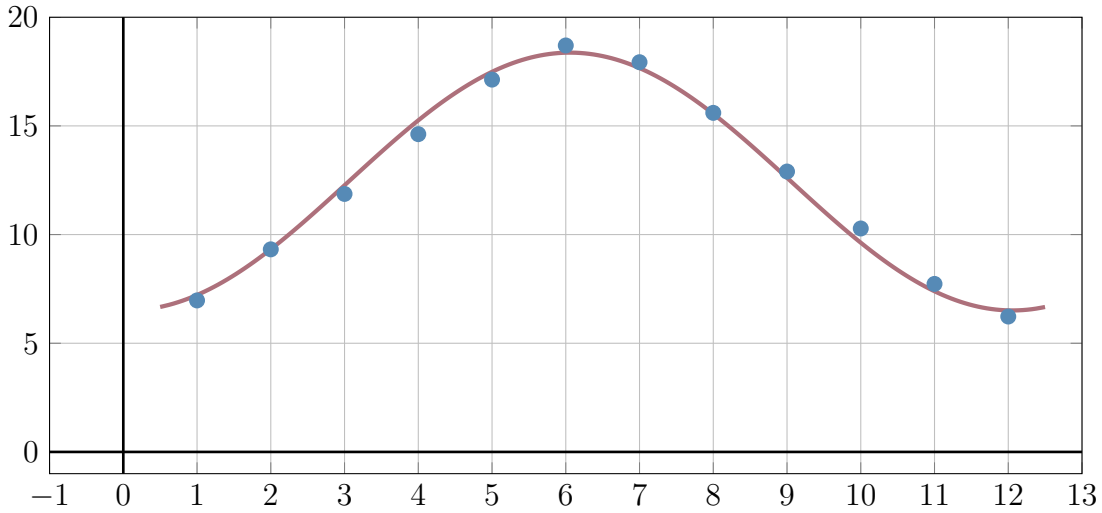


Example 8.16 *The times for sunrise and sunset in Sweden are significantly influenced by the high location in the northern hemisphere. Relatively high in the north, the days are long in summer (June to September) and short in winter (December to March).*

The number of hours $H(t)$ of daylight that Stockholm receives on average for the t^{th} month of the year is given below. Here, $t = 1$ represents January, $t = 2$ represents February, and so on.

Find a sinusoidal function which models these data points and use a graphing calculator to graph your answer along with the data.

t	**Hours of Daylight**	$H(t)$
1	6:58 h	6.97
2	9:19 h	9.32
3	11:52 h	11.87
4	14:37 h	14.62
5	17:08 h	17.13
6	18:42 h	18.7
7	17:56 h	17.93
8	15:36 h	15.6
9	12:54 h	12.9
10	10:17 h	10.28
11	7:44 h	7.73
12	6:14 h	6.23



Inspiration from: *A Little Night Music* by Sondheim and Wheeler.

Example 8.17 *Sketch the function* $y = -\sin\left(\frac{\pi}{2}x\right)$.



Section Recap

What are some take-away concepts from this section?

8.1 Exercises:

1. Determine the amplitude, period, phase shift, and midline of the function

$$y = 2\sin\left(4x + \frac{\pi}{3}\right),$$

and then sketch one complete period of the graph, labeling all key points.

2. Determine the amplitude, period, phase shift, and midline of the function

$$y = -3\cos\left(\frac{x}{2} - \pi\right),$$

and then sketch one complete period of the graph, labeling all key points.

3. Determine the amplitude, period, phase shift, and midline of the function

$$y = \sin\left(3x - \frac{\pi}{4}\right),$$

and then sketch one complete period of the graph, labeling the intercepts, maximum point, and minimum point.

8.1 Scaffolding Questions:

1. Determine the amplitude, period, phase shift, and midline of the function

$$y = 2\sin\left(4x + \frac{\pi}{3}\right),$$

and then sketch one complete period of the graph, labeling all key points.

- Question Comprehension
  - What quantities are you asked to determine before sketching the graph?
  - Which values in the equation correspond to the parameters A, B, C, and D in the form $y = A\sin(Bx - C) + D$?
  - What information is needed to sketch one complete period?
- Content Knowledge
  - How is the amplitude determined from the value of A?
  - How is the period determined from the value of B?
  - How is the phase shift determined from the values of B and C?
  - How does the value of D determine the midline?
  - How can the period be divided into four equal intervals to locate key points?
- Solution
  - Did you correctly determine the amplitude, period, phase shift, and midline?
  - Did you identify the five key points of one period?
  - Does your sketch reflect the horizontal shift and vertical scaling of the function?
  - Are all key points clearly labeled?
- Reflect
  - How would the graph change if the amplitude were doubled?
  - How would the graph change if the coefficient of x were smaller?
  - What effect does the phase shift have on the location of the graph?

2. Determine the amplitude, period, phase shift, and midline of the function

$$y = -3\cos\left(\frac{x}{2} - \pi\right),$$

and then sketch one complete period of the graph, labeling all key points.

- Question Comprehension
  - What quantities must be identified from the equation before sketching?
  - Which values correspond to A, B, C, and D in the form $y = A\cos(Bx - C) + D$?
  - What effect does the negative value of A have on the graph?
- Content Knowledge
  - How is the amplitude determined from A?
  - How is the period determined from B?
  - How is the phase shift calculated?
  - How is the midline determined?
  - How does a negative amplitude affect the usual cosine pattern?
- Solution
  - Did you correctly determine the amplitude, period, phase shift, and midline?
  - Did you correctly account for the reflection caused by the negative coefficient?
  - Did you identify and label the maximum and minimum values?
  - Does your graph show one complete period?
- Reflect
  - How would the graph differ if the coefficient were $+3$ instead of -3?
  - How would changing the period affect the spacing of the key points?
  - Why does the amplitude remain positive even when the coefficient is negative?

3. Determine the amplitude, period, phase shift, and midline of the function

$$y = \sin\left(3x - \frac{\pi}{4}\right),$$

and then sketch one complete period of the graph, labeling the intercepts, maximum point, and minimum point.

- Question Comprehension
  - What characteristics of the sinusoidal function must be determined?
  - Which values correspond to A, B, C, and D?
  - What points should be labeled on the completed graph?
- Content Knowledge
  - How is the amplitude determined?
  - How is the period calculated using B?
  - How is the phase shift calculated using C and B?
  - What is the midline of the graph?
  - How can the intercepts, maximum, and minimum be found from the key points of a sine curve?
- Solution
  - Did you correctly determine the amplitude, period, phase shift, and midline?
  - Did you correctly locate one complete period of the graph?
  - Did you label the intercepts, maximum point, and minimum point?
  - Does the graph reflect the correct horizontal shift?
- Reflect
  - How would the graph change if the coefficient of x were increased?
  - How would the graph change if the phase shift were in the opposite direction?
  - How can you verify that your graph has the correct period?

8.2 Graphs of the Other Trigonometric Functions

Learning Goals 8.2

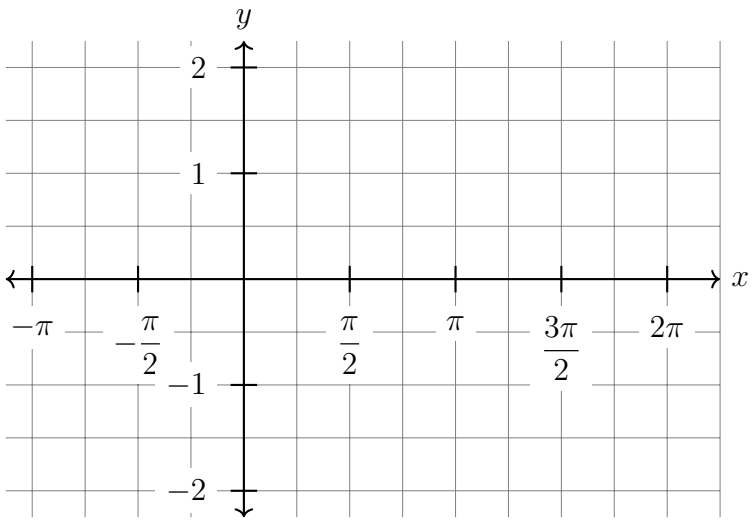
After this section, I should be able to:

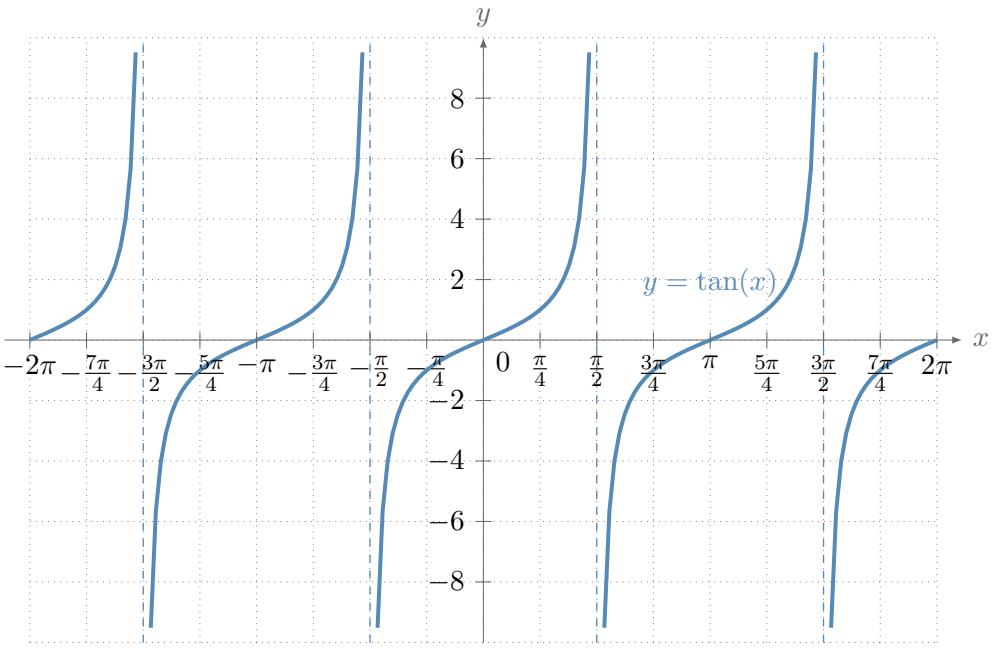
1. Analyze the graphs of $y = \tan(x)$, $y = \csc(x)$, $y = \sec(x)$, and $y = \cot(x)$.
2. Graph vertical and horizontal shifts with graphs of $y = \tan(x)$, $y = \csc(x)$, $y = \sec(x)$, and $y = \cot(x)$.

Example 8.18 *Fill out the table below with the exact* $\sin(x)$ *and* $\cos(x)$ *values, use that information to determine the* $\tan(x)$ *values, and then sketch the tangent curve on the interval* $[0, \pi]$.

Further, $\tan(x)$ *has a period of* π. *Use this information to sketch* $\tan(x)$ *on the remaining parts of the graph.*

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$									
$\cos(x)$									
$\tan(x)$									





Recall that the tangent is positive for angles in QI and QIII, and is negative in QII and QIV.

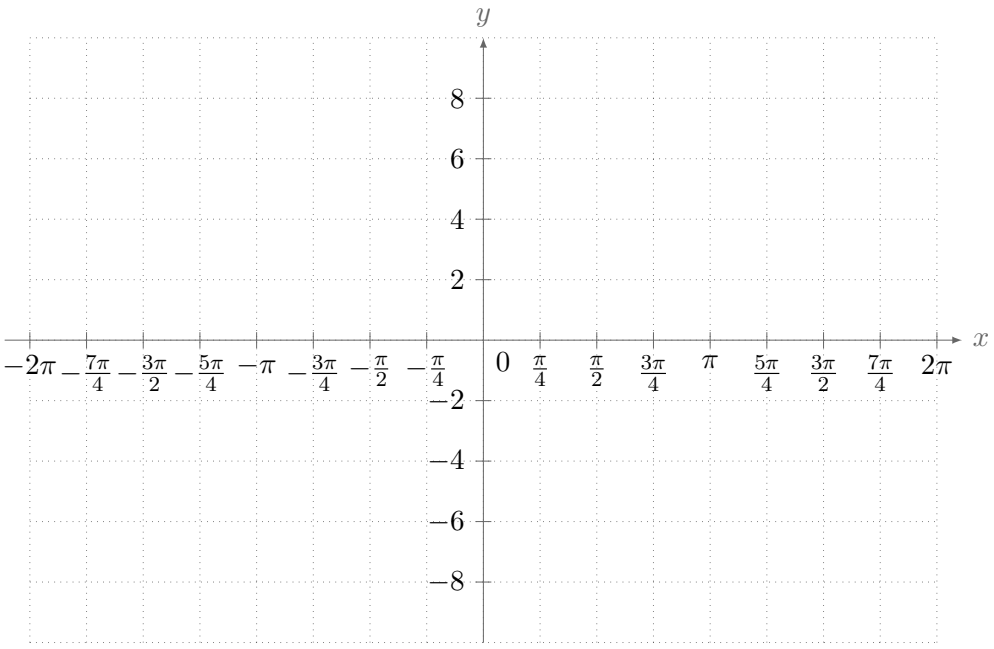
We know that $\tan(x)$ is not defined when $\cos(x) = 0$, i.e. at odd multiples of $\frac{\pi}{2}$: $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Although $\tan(x)$ is not defined at those values, we can figure out what happens *near* those angles by looking at the sine and cosine functions. For example, for x in QI a little less than $\frac{\pi}{2}$, fill in the $\sin(x)$ and $\cos(x)$ to find the $\tan(x)$ values as x approaches $\frac{\pi}{2}$:

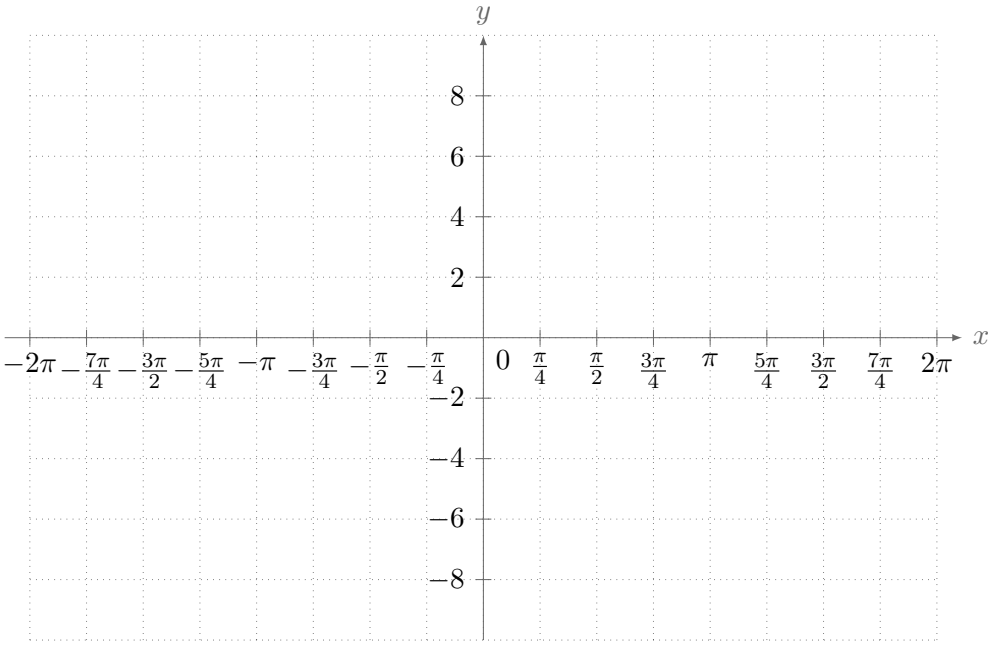
x	1.0	1.5	1.57	$\to \pi/2$
$\sin(x)$				
$\cos(x)$				
$\tan(x)$				

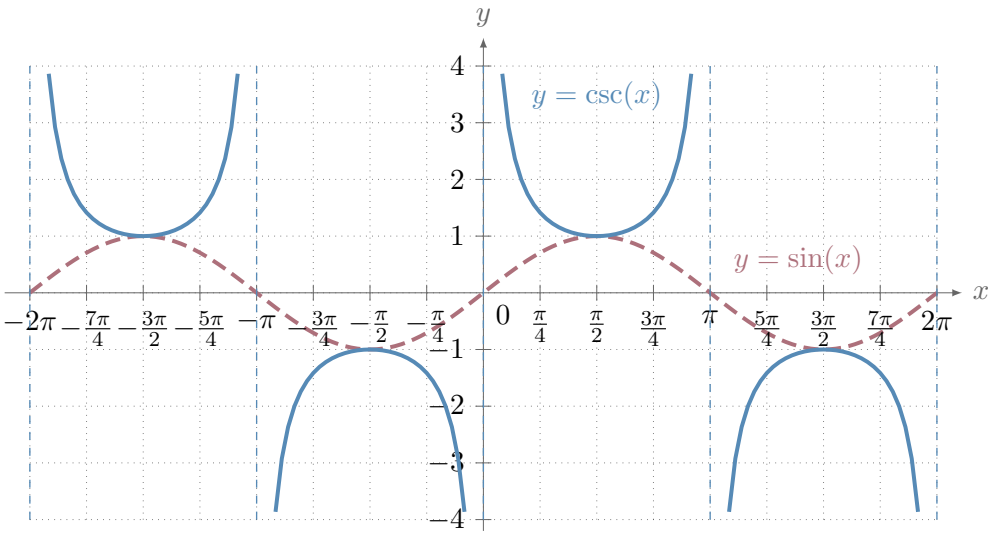
Likewise, for x in QII very close to $\frac{\pi}{2}$ (but this time, greater than $\frac{\pi}{2}$), $\sin(x)$ is very close to 1 and $\cos(x)$ is negative and very close to 0; so, the quotient $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is a negative number that is very large, and it gets larger in the negative direction the closer x gets to $\frac{\pi}{2}$.

Example 8.19 *Sketch the function* $f(x) = 2\tan(x) + 1$. *Does this variation have the same vertical asymptotes as* $y = \tan(x)$*?*



Example 8.20 *Sketch the function* $f(x) = -\tan\left(x - \dfrac{\pi}{2}\right)$. *Does this variation have the same vertical asymptotes as* $y = \tan(x)$*?*





With $\csc(x)$ defined as $\dfrac{1}{\sin(x)}$, cosecant will be undefined when the denominator is 0.

This will happen at the roots of $\sin(x)$: $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

What does this tell us about vertical asymptotes?

We can figure out what happens *near* those values by looking at the sine function. For example, for x in QII a little less than π, fill in the $\sin(x)$ values to find what $\csc(x)$ approaches as x gets closer to π:

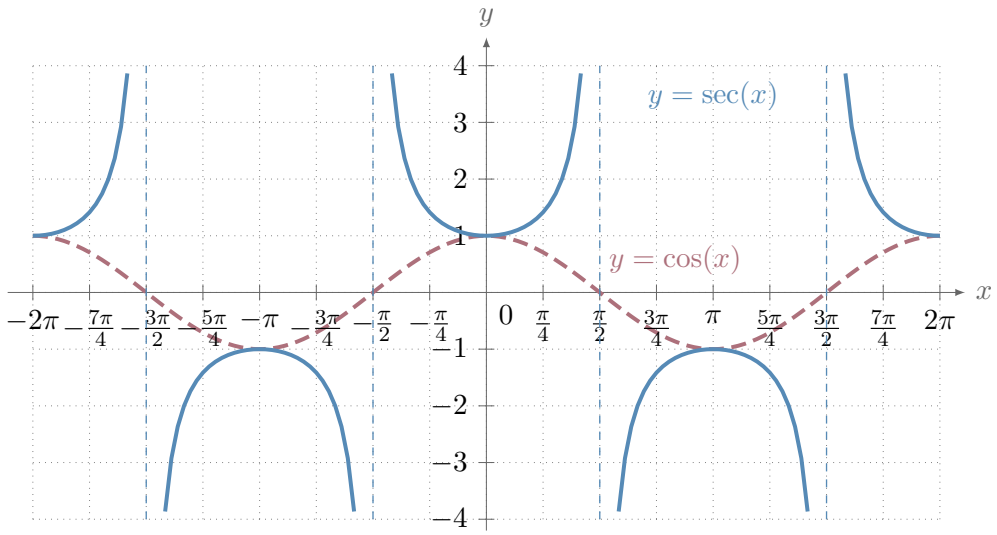
x	3.0	3.1	3.14	$\to \pi$
$\sin(x)$				
$\csc(x)$				

Now, approach $x = \pi$ from QIII, where values are a little *greater than* π:

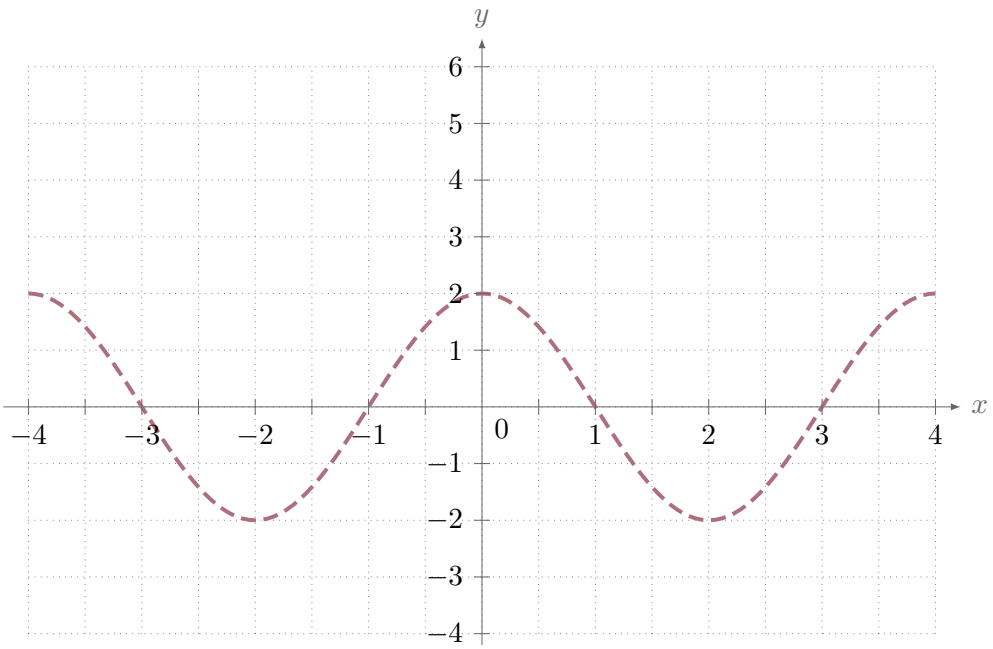
x	$\pi \leftarrow$	3.15	3.2	3.5
$\sin(x)$				
$\csc(x)$				

Likewise, we have the the graph of $\sec(x) = \dfrac{1}{\cos(x)}$ along with the graph of $y = \cos(x)$ for reference. Note the vertical asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ (where $\cos(x) = 0$).

Notice also that the secant function is just cosecant function shifted to the left by $\frac{\pi}{2}$ radians, just as cosine is the sine function shifted to the left by $\frac{\pi}{2}$ radians.



Example 8.21 *Given the graph of* $y = 2\cos\left(\frac{\pi}{2}x\right)$*, sketch* $y = 2\cos\left(\frac{\pi}{2}x\right) + 1$*. Then, sketch the secant curve* $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$*.*



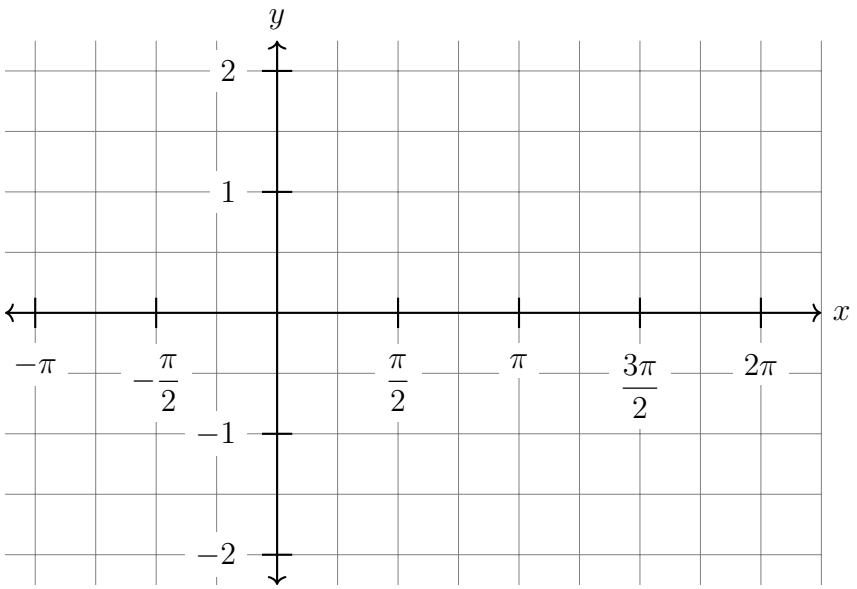
The cotangent function is defined by the reciprocal identity $\cot(x) = \dfrac{1}{\tan(x)}$.

Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at $x = 0, \pm\pi, \pm 2\pi, \ldots$

Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

Example 8.22 *Fill out the table below with the exact* $\tan(x)$ *values. Then, use that information to determine the* $\cot(x)$ *values and sketch the tangent curve on the interval* $[0, \pi]$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan(x)$									
$\cot(x)$									



Example 8.23 *Standing on the shore of the ocean, a fisherman sights a submarine far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. The submarine floats from due west to due east and, ignoring the curvature of the Earth, the distance in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.*

(a) Label variables x and $d(x)$ on the figure below.

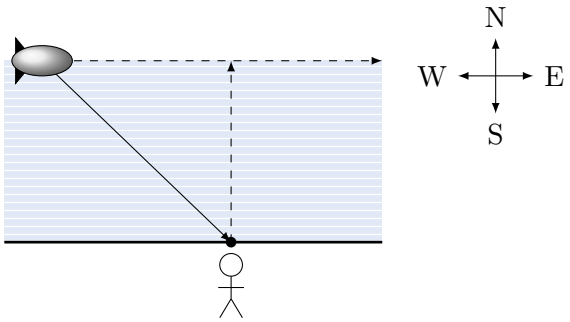
(b) What is a reasonable domain for $d(x)$?

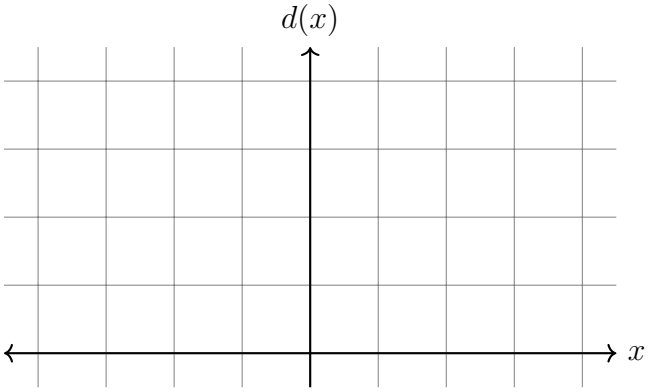
(c) Graph $d(x)$ on this domain.

(d) Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.

(e) Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.

(f) What is the minimum distance between the fisherman and the boat? When does this occur?





Example 8.24 *A laser rangefinder is locked on a comet approaching Earth! The distance $g(x)$ (measured in kilometers) of the comet after x days, for x in the interval of 0 to 30 days, is given by*

$$g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right) .$$

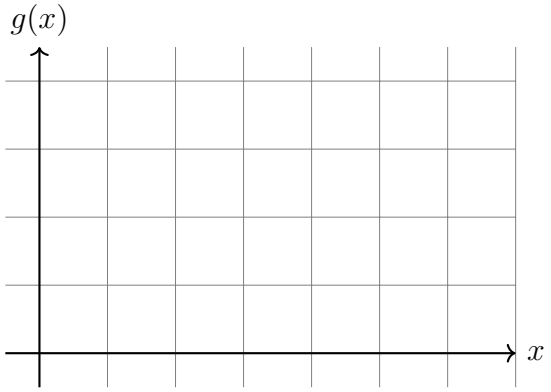
(a) Sketch out the situation. Label any constants and variables in your diagram.

(b) Graph $g(x)$ on the interval $[0, 30]$.

(c) Evaluate $g(5)$ and interpret the information.

(d) What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond.

(e) Find and explain the meaning of any vertical asymptotes.



Example 8.25 *A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.*

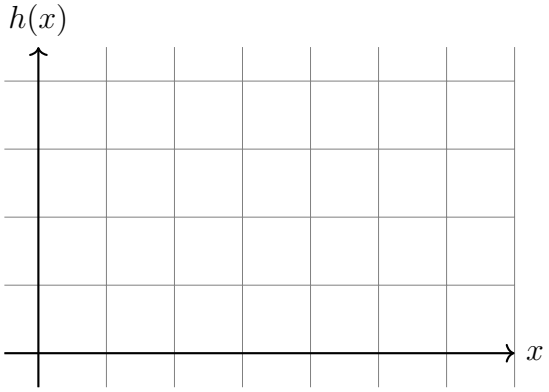
(a) Sketch out the situation. Label any constants and variables in your diagram.

(b) Write a function expressing the altitude $h(x)$, measured in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.

(c) Graph $h(x)$ on the interval $[0, 60)$.

(d) Evaluate and interpret the values $h(0)$ and $h(30)$.

(e) What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.



Section Recap

What are some take-away concepts from this section?

8.2 Exercises:

1. Sketch the graph of
$$y = 2\tan\left(\frac{x}{3}\right).$$
Identify the period, vertical asymptotes, and x-intercepts. Explain how the graph differs from the graph of $y = \tan(x)$.

2. Sketch the graph of
$$y = 4\sec\left(x - \frac{\pi}{2}\right).$$
Identify the period, phase shift, vertical asymptotes, and the coordinates of the vertices of the secant branches on one period.

3. Sketch the graph of
$$y = \cot\left(\frac{x}{4}\right).$$
Identify the period, vertical asymptotes, and x-intercepts. Explain how the graph differs from the graph of $y = \cot(x)$.

8.2 Scaffolding Questions:

1. Sketch the graph of
$$y = 2\tan\left(\frac{x}{3}\right).$$
Identify the period, vertical asymptotes, and x-intercepts. Explain how the graph differs from the graph of $y = \tan(x)$.

   - Question Comprehension:
     - What transformations have been applied to the parent function $y = \tan(x)$?
     - Which graph features must be identified before sketching the curve?
     - How are the period and vertical asymptotes related for tangent functions?
   - Content Knowledge:
     - What is the period of $y = \tan(Bx)$?
     - Where do the vertical asymptotes of $y = \tan(Bx)$ occur?
     - How does multiplying by 2 affect the outputs of the tangent function?
     - How can you find the x-intercepts of a tangent graph?
   - Solution:
     - Did you determine the period correctly?
     - Did you locate at least two consecutive vertical asymptotes?
     - Did you identify the x-intercepts?
     - Does your sketch show the correct shape and orientation between asymptotes?
   - Reflect:
     - Which transformation changes the period?
     - Which transformation changes the steepness of the graph without moving the asymptotes?

2. Sketch the graph of
$$y = 4\sec\left(x - \frac{\pi}{2}\right).$$
Identify the period, phase shift, vertical asymptotes, and the coordinates of the vertices of the secant branches on one period.

   - Question Comprehension:
     - How is the graph of secant related to the graph of cosine?
     - What effect does the expression $x - \frac{\pi}{2}$ have on the graph?
     - What information is needed to locate the secant branches?
   - Content Knowledge:
     - What is the period of the secant function?
     - How do you determine the phase shift?
     - Where do the vertical asymptotes occur for a secant graph?
     - How can the corresponding cosine graph help locate the vertices of the secant branches?
   - Solution:
     - Did you identify the phase shift correctly?
     - Did you locate the vertical asymptotes?
     - Did you determine the coordinates of the vertices of the secant branches?
     - Does your sketch accurately show the secant branches approaching the asymptotes?
   - Reflect:
     - How would the graph change if the coefficient 4 were replaced by 2?
     - How does shifting the cosine graph help graph the secant function?

3. Sketch the graph of
$$y = \cot\left(\frac{x}{4}\right).$$
Identify the period, vertical asymptotes, and x-intercepts. Explain how the graph differs from the graph of $y = \cot(x)$.

   - Question Comprehension:
     - What transformations have been applied to the parent function $y = \cot(x)$?
     - Which key features are needed to sketch a cotangent graph?
     - How does the graph of cotangent differ from the graph of tangent?
   - Content Knowledge:
     - What is the period of $y = \cot(Bx)$?
     - Where do the vertical asymptotes of a cotangent graph occur?
     - How can you find the x-intercepts of a cotangent graph?
     - Why is the cotangent graph decreasing on each interval between asymptotes?
   - Solution:
     - Did you calculate the period correctly?
     - Did you identify the vertical asymptotes?
     - Did you locate the x-intercepts?
     - Does your sketch show the correct decreasing shape between asymptotes?
   - Reflect:
     - How does the horizontal stretch affect the period?
     - Which features remain unchanged when the graph is stretched horizontally?

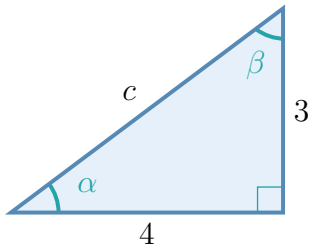
8.3 Inverse Trigonometric Functions

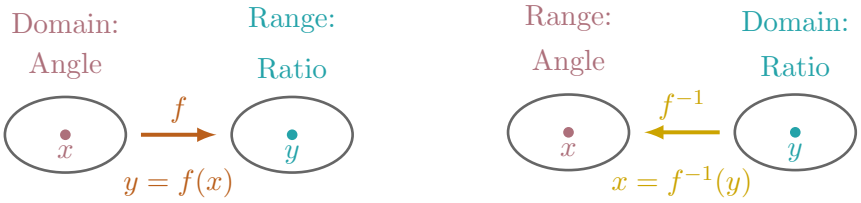
Learning Goals 8.3

After this section, I should be able to:

1. Understand and use the inverse sine, cosine, and tangent functions.
2. Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
3. Use a calculator to evaluate inverse trigonometric functions.
4. Find exact values of composite functions with inverse trigonometric functions.

Example 8.26 *Solve for the missing side length and angles in the right triangle below.*





An **inverse trigonometric function** "undoes" what the original trigonometric function "does." For example:

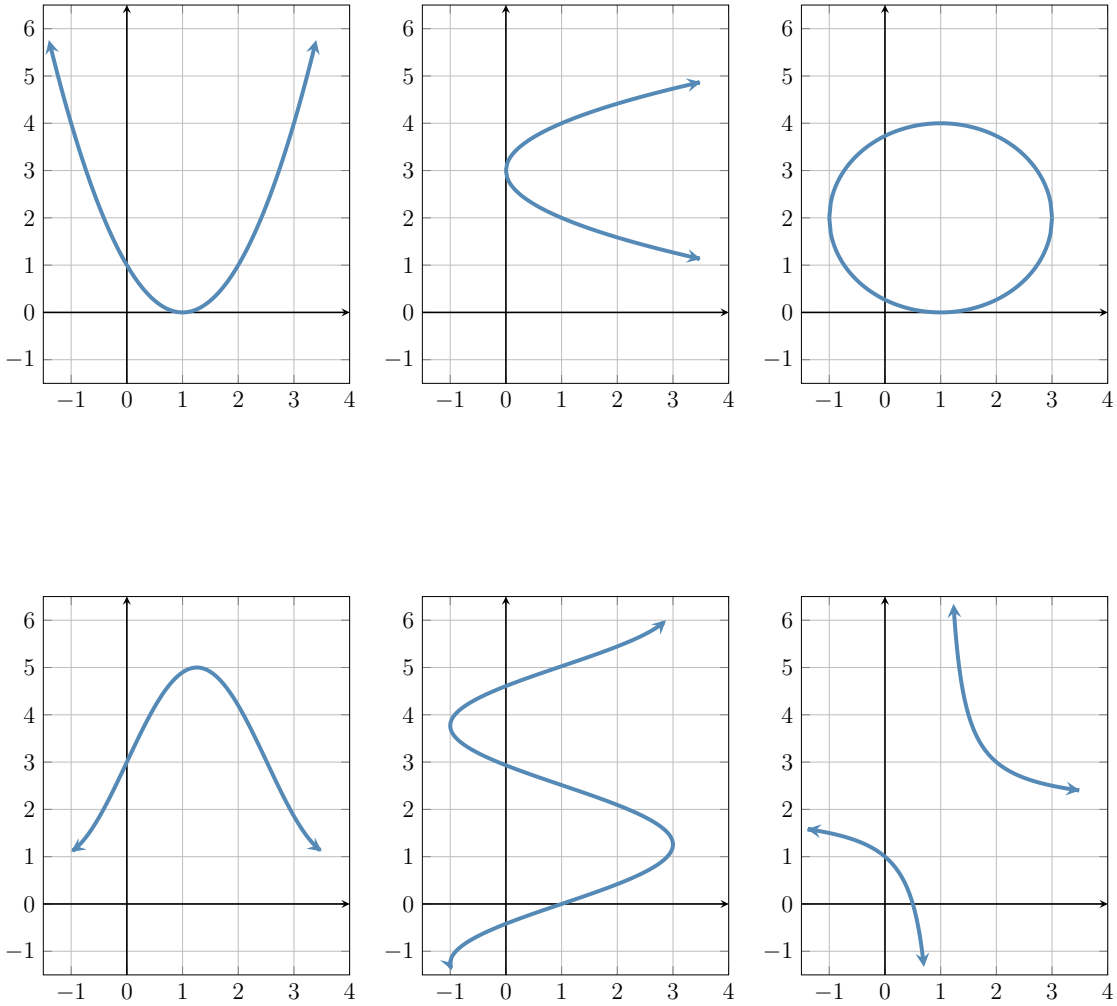
- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then
- Since $\cos(\pi) = -1$, then
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then

The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph *does not* define a function because a function has only one output value for each input value.

How To: Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would interest the curve more than once.
2. If there is any such line, determine that the graph is not a function.

Example 8.27 *Identify which of the following graphs are functions.*

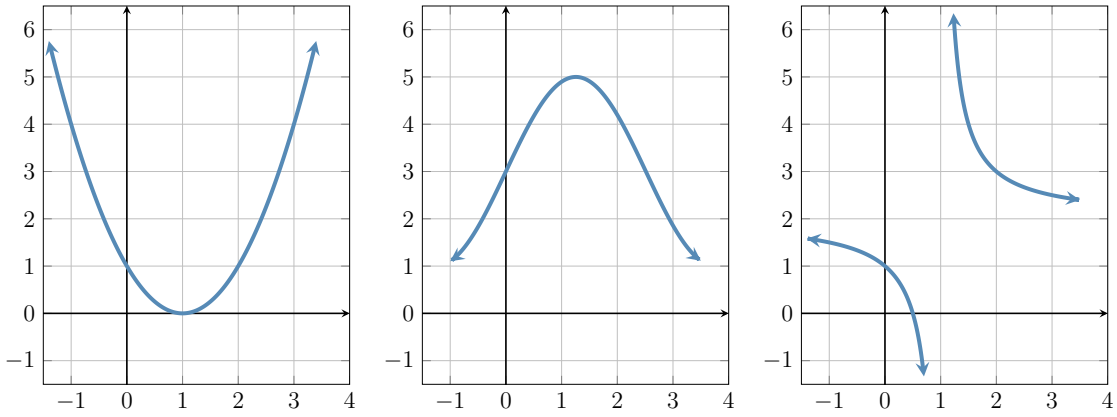


A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

How To: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

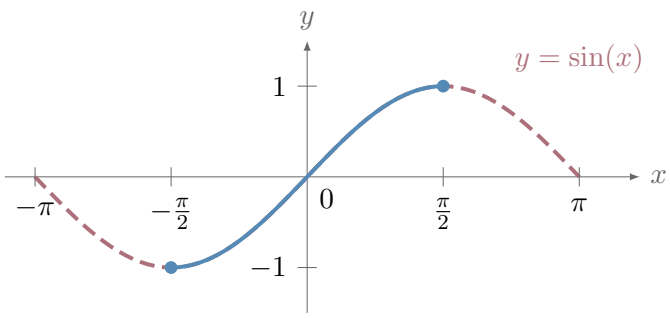
1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is *not* one-to-one.

Example 8.28 *Identify which of the following functions are one-to-one.*



If a function f is one-to-one on its domain, then f has an **inverse function**, denoted by f^{-1}, such that $y = f(x)$ if and only if $f^{-1}(y) = x$. The domain of f^{-1} is the range of f.

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *parts* of their domains where they *are* one-to-one. For example, $y = \sin(x)$ is one-to-one over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we see in the graph below:



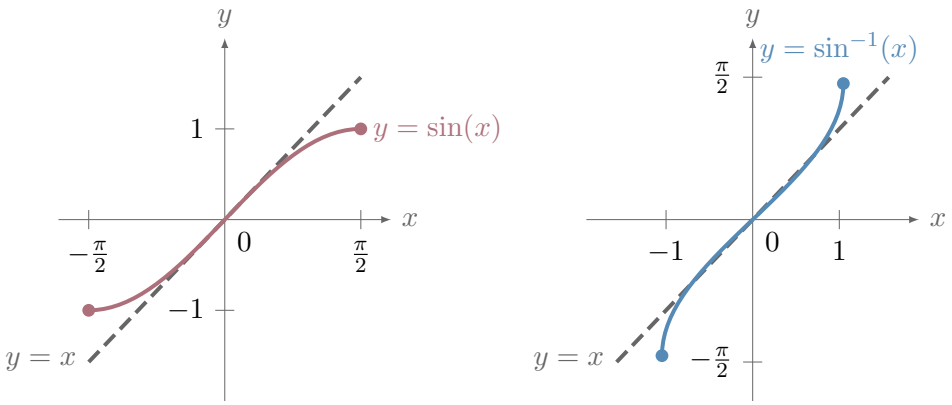
Find the domain and range of $\sin(x)$ on the *restricted domain*.

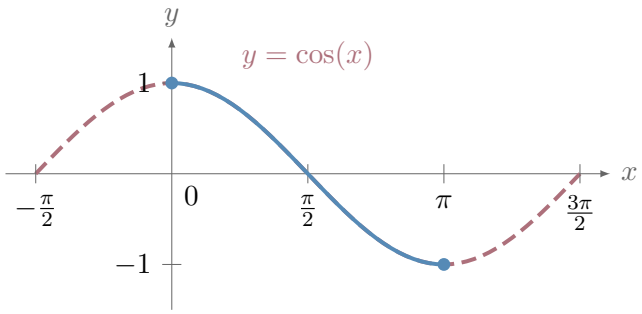
We can define the **inverse sine** function $y = \sin^{-1}(x)$, sometimes called the **arc sine** and denoted by $y = \arcsin(x)$. Now, find the domain and range of $y = \arcsin(x)$.

In other words, for angles y in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin(y) = x$, then $\sin^{-1}(x) = y$.

Example 8.29 *Given* $\sin\left(\frac{5\pi}{12}\right) \approx 0.97$, *write a relation involving the inverse sine function.*

Example 8.30 *Find* $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$.





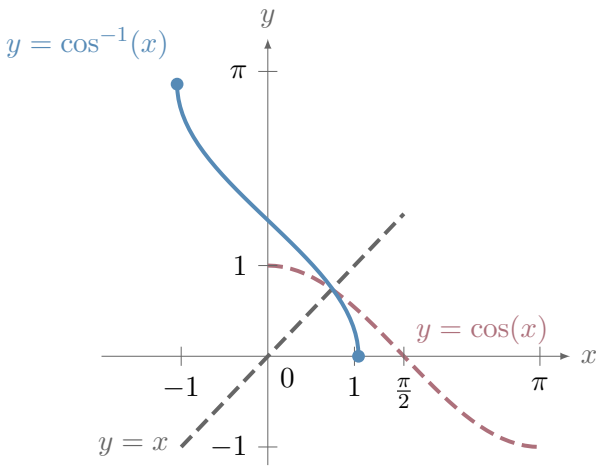
Find the domain and range of $\cos(x)$ on the *restricted domain.*

We can define the **inverse cosine** function $y = \cos^{-1}(x)$, sometimes called the **arc cosine** and denoted by $y = \arccos(x)$. Now, find the domain and range of $y = \arccos(x)$.

In other words, for angles y in the interval $[0, \pi]$, if $\cos(y) = x$, then $\cos^{-1}(x) = y$.

Example 8.31 *Given* $\cos(.5) \approx 0.88$, *write a relation involving the inverse cosine function.*

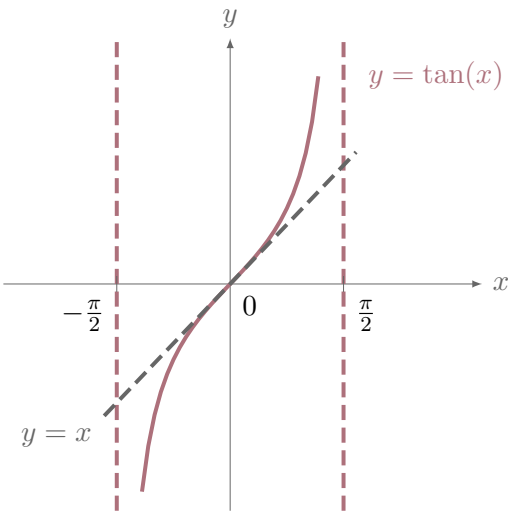
Example 8.32 *Find* $\cos^{-1}\left(\cos\left(\frac{\pi}{2}\right)\right)$.

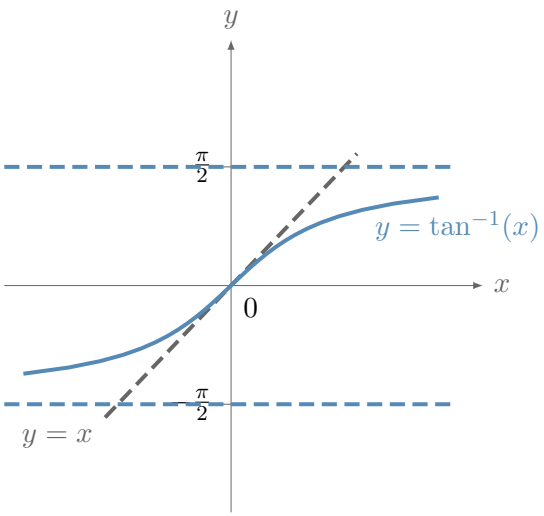


Find the domain and range of $\tan(x)$ on the *restricted domain*.

We can define the **inverse tangent** function $y = \tan^{-1}(x)$, sometimes called the **arc tangent** and denoted by $y = \arctan(x)$. Now, find the domain and range of $y = \arctan(x)$.

In other words, for angles y in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan(y) = x$, then $\tan^{-1}(x) = y$.





Example 8.33 *Evaluate* $\tan^{-1}(1)$.

Example 8.34 *Determine* $\cos\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$.

Example 8.35 *Find an exact value for* $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

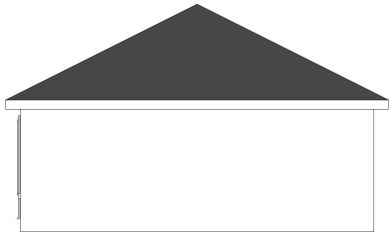
Example 8.36 *Find an exact value for* $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Example 8.37 *Find an exact value for* $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Example 8.38 *The roof on the house below has a '* $^{6}/_{12}$ *pitch'. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.*



Front View



Side View

Example 8.39 *A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it touches level ground 360 feet from the base of the tower. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.*

Example 8.40 *The top of Crown Mountain is 1556 feet above sea level. From atop the mountain, a ship is seen 4 miles away (four miles of purely horizontal distance). What angle of depression is made by the line of sight from the mountain top to the ship? Express your answer using degree measure rounded to one decimal place.*

Example 8.41 *Shelving is being built at the library, and the shelves are to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?*

Example 8.42 *A parasailor is being pulled by a boat at Lindbergh Bay Beach. The cable is 300 feet long and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.*

Example 8.43 *A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?*

Example 8.44 *Without using a calculator, approximate the value of* $\arctan(10,000)$. *Explain why your answer is reasonable.*

Section Recap

What are some take-away concepts from this section?

8.3 Exercises:

1. A wheelchair ramp rises 2 feet for every 12 feet of horizontal run. Find the angle of inclination of the ramp with the ground. Express your answer in degrees rounded to the nearest tenth.

2. A kite is flying 120 feet above the ground. The string attached to the kite is 200 feet long and is pulled taut. Find the angle of elevation from the person holding the string to the kite. Express your answer in degrees rounded to the nearest tenth.

3. A surveyor stands 500 feet from the base of a radio tower and measures an angle of elevation of 38° to the top of the tower. Determine the height of the tower. Then, find the exact value of

$$\cos\left(\tan^{-1}\left(\frac{\text{height of tower}}{500}\right)\right).$$

8.3 Scaffolding Questions:

1. A wheelchair ramp rises 2 feet for every 12 feet of horizontal run. Find the angle of inclination of the ramp with the ground. Express your answer in degrees rounded to the nearest tenth.

    - Question Comprehension:
        - What quantity is being asked for in this problem?
        - Which side of the right triangle represents the rise?
        - Which side represents the horizontal run?
        - Which trigonometric ratio relates the opposite side and adjacent side?
    - Content Knowledge:
        - How can the tangent ratio be written using the given measurements?
        - Which inverse trigonometric function can be used to find the angle?
        - How do you use a calculator to evaluate an inverse tangent?
    - Solution:
        - Did you write the tangent ratio correctly?
        - Did you use the inverse tangent function to solve for the angle?
        - Did you round your answer to the nearest tenth of a degree?
    - Reflect:
        - Why is inverse tangent the appropriate inverse trigonometric function for this problem?
        - How would the angle change if the ramp became steeper?

2. A kite is flying 120 feet above the ground. The string attached to the kite is 200 feet long and is pulled taut. Find the angle of elevation from the person holding the string to the kite. Express your answer in degrees rounded to the nearest tenth.

   - Question Comprehension:
     - What quantity is being asked for?
     - Which side of the triangle is the hypotenuse?
     - Which side is opposite the angle of elevation?
     - Which trigonometric ratio involves the opposite side and hypotenuse?
   - Content Knowledge:
     - How can you write a sine equation using the given measurements?
     - Which inverse trigonometric function should be used to solve for the angle?
     - How do you evaluate the inverse sine function on a calculator?
   - Solution:
     - Did you identify the correct trigonometric ratio?
     - Did you set up the equation correctly?
     - Did you use inverse sine to determine the angle?
     - Did you round your answer appropriately?
   - Reflect:
     - Why would inverse cosine not be the most direct method here?
     - How would the angle change if the kite remained at the same height but the string became longer?

3. A surveyor stands 500 feet from the base of a radio tower and measures an angle of elevation of 38° to the top of the tower. Determine the height of the tower. Then, find the exact value of

$$\cos\left(\tan^{-1}\left(\frac{\text{height of tower}}{500}\right)\right).$$

- Question Comprehension:
  - What information is given about the right triangle formed by the tower, ground, and line of sight?
  - What quantity must be found first?
  - What quantity must be evaluated after finding the tower's height?
  - Why does this problem involve both trigonometric and inverse trigonometric functions?
- Content Knowledge:
  - Which trigonometric ratio relates the tower's height and the horizontal distance?
  - How can you determine the tower's height?
  - If
$$\theta = \tan^{-1}\left(\frac{\text{height}}{500}\right),$$
what does this tell you about $\tan(\theta)$?
  - How can a right triangle be used to evaluate $\cos(\theta)$ exactly?
- Solution:
  - Did you correctly calculate the height of the tower?
  - Did you rewrite the inverse tangent expression as an angle?
  - Did you construct an appropriate right triangle?
  - Did you simplify the exact value correctly?
- Reflect:
  - How does the expression
$$\cos(\tan^{-1}(x))$$
connect inverse trigonometric functions to right-triangle geometry?
  - Why is it often easier to use a triangle rather than a calculator when finding exact values of composite trigonometric expressions?

8.4 Graphs of the Sine and Cosine Functions, Revisited*

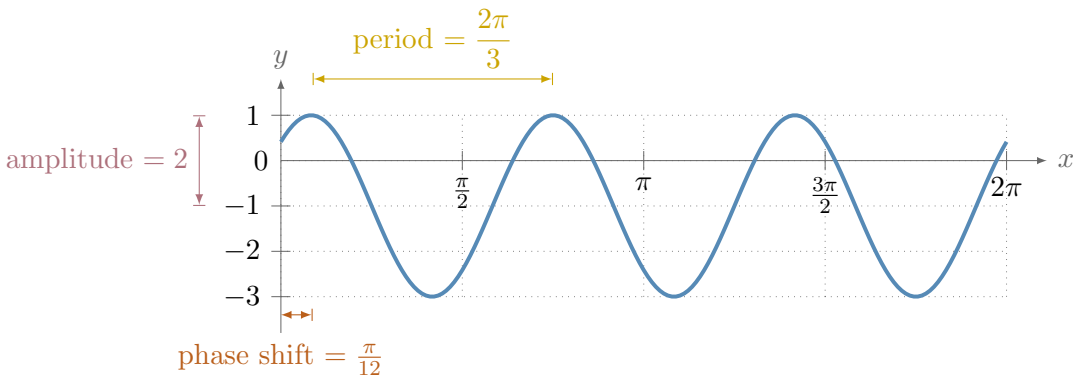
Note: Variations of the Sine and Cosine Function

Given the equation in the form

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D\,,$$

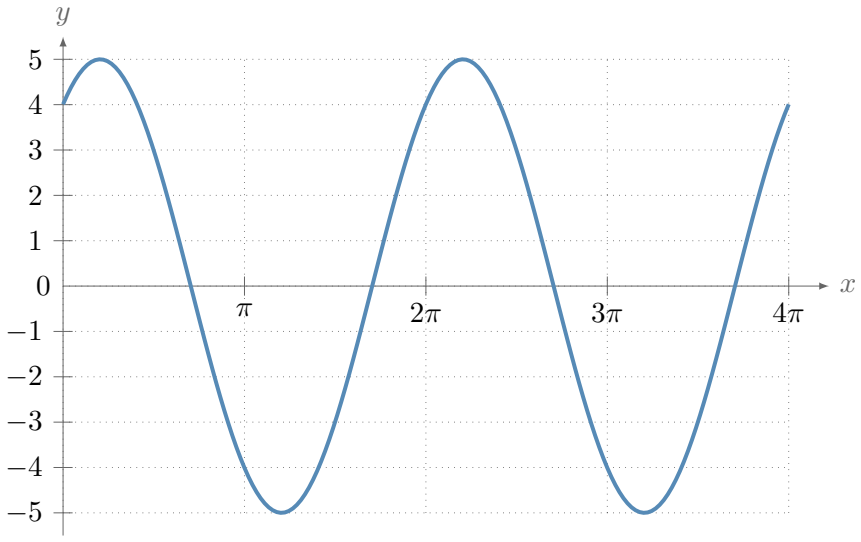
- The A term is related to the amplitude– the **amplitude** is $|A|$,
- The B term is related to the **period**– the **period** is $P = \dfrac{2\pi}{|B|}$,
- The C term is related to the **horizontal shift**– the **phase shift** is $\dfrac{C}{B}$,
- The D term is the **vertical shift**– the **midline** is $y = D$.

Example 8.45 *Given the amplitude, period, and phase shift for the* cosine *function below, determine the equation.*

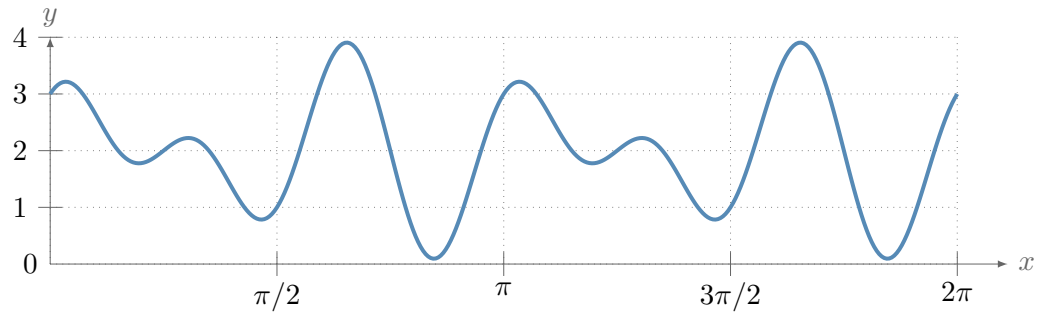


Example 8.46 *Find the amplitude and period of* $y = 3\sin(x) + 4\cos(x)$.

Then, write the combination sinusoidal curve *as a single sine equation. Plot your solution on Desmos to verify it works!*

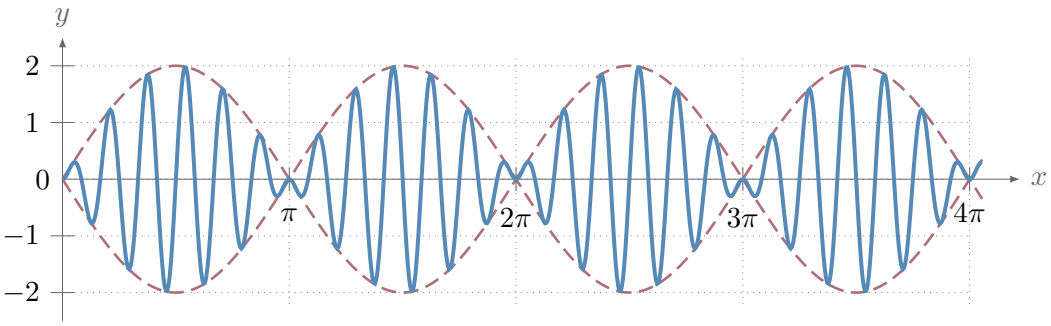


Example 8.47 *Find the amplitude and period of* $y = \sin(6x) + \cos(4x) + 2$.



Example 8.48 *Find the amplitude and period of* $y = 2\sin(x)\sin(12x)$.

The function below $y = 2\sin(x)\sin(12x)$ *is known as a* modulated wave *and the functions* $y = 2\sin(x)$ *and* $y = -2\sin(x)$ *form an* amplitude envelope *for the wave (i.e., they enclose the wave).*



Example 8.49 *With the* product-to-sum identity:

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right],$$

rewrite the modulated wave $y = 2\sin(x)\sin(12x)$ *as a sum of sinusoidal curves.*

Plot your solution on Desmos to verify it works!

Section Recap

What are some take-away concepts from this section?

8.4 Exercises:

1. A Ferris wheel rotates smoothly so that a rider moves 20 feet above and below the wheel's center. The wheel completes one revolution every 30 seconds, and the rider starts at the highest point when $t = 0$.

   Write a cosine function that models the rider's height relative to the center of the wheel. State the amplitude, period, and phase shift.

2. A tidal monitoring station records the water level (in feet relative to average sea level) as

   $$h(t) = 3\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) + 1.$$

   Determine the amplitude, period, phase shift, and midline. Then, determine the maximum and minimum water levels and explain what they mean in context.

3. Two speakers emit sound waves that can be modeled by

   $$y = 5\sin(x) + 12\cos(x).$$

   Find the amplitude and period of the resulting wave. Then, rewrite the function as a single sinusoidal function of the form

   $$y = A\sin(x + \phi).$$

   Verify your result by graphing both functions with technology.

8.4 Scaffolding Questions:

1. A Ferris wheel rotates smoothly so that a rider moves 20 feet above and below the wheel's center. The wheel completes one revolution every 30 seconds, and the rider starts at the highest point when $t = 0$.

   Write a cosine function that models the rider's height relative to the center of the wheel. State the amplitude, period, and phase shift.

   - Question Comprehension:
     - What quantity is changing periodically?
     - What does the amplitude represent in this context?
     - What does one complete revolution correspond to on the graph?
     - Why might cosine be more convenient than sine for this situation?
   - Content Knowledge:
     - How is amplitude determined from the description?
     - How can the period be found from the wheel's rotation time?
     - How is the coefficient B related to the period?
     - What does a phase shift represent?
   - Solution:
     - Did you correctly identify the amplitude?
     - Did you calculate the period correctly?
     - Did your model begin at a maximum height when $t = 0$?
     - Did you state the amplitude, period, and phase shift?
   - Reflect:
     - Why is a sinusoidal function appropriate for modeling circular motion?
     - How would the equation change if the rider started at the lowest point instead?

2. A tidal monitoring station records the water level (in feet relative to average sea level) as

$$h(t) = 3\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) + 1.$$

Determine the amplitude, period, phase shift, and midline. Then, determine the maximum and minimum water levels and explain what they mean in context.

- Question Comprehension:
  - Which quantities correspond to amplitude, period, phase shift, and vertical shift?
  - What does the variable t represent?
  - What does the constant $+1$ represent in the context of water level?
- Content Knowledge:
  - How is amplitude determined from the equation?
  - How is period calculated from the coefficient of t?
  - How is phase shift calculated?
  - How can the maximum and minimum values be found from the amplitude and midline?
- Solution:
  - Did you correctly identify all four transformations?
  - Did you calculate the maximum water level?
  - Did you calculate the minimum water level?
  - Did you interpret the results in the context of tides?
- Reflect:
  - What does the phase shift tell you about when high tide occurs?
  - How would increasing the amplitude affect the tide levels?

3. Two speakers emit sound waves that can be modeled by

$$y = 5\sin(x) + 12\cos(x).$$

Find the amplitude and period of the resulting wave. Then, rewrite the function as a single sinusoidal function of the form

$$y = A\sin(x + \phi).$$

Verify your result by graphing both functions with technology.

- Question Comprehension:
  - What is being asked besides finding amplitude and period?
  - What does it mean to rewrite a combination of sine and cosine as a single sinusoid?
  - Why might two different-looking equations represent the same graph?
- Content Knowledge:
  - What relationship can be used to write

$$A\sin(x + \phi)$$

    in terms of sine and cosine?
  - How can the coefficients of $\sin(x)$ and $\cos(x)$ be matched?
  - How is the amplitude found from the coefficients 5 and 12?
  - What is the period of the resulting function?
- Solution:
  - Did you correctly determine the amplitude?
  - Did you determine the period?
  - Did you find an appropriate phase angle ϕ?
  - Did your rewritten function match the original when graphed?
- Reflect:
  - Why is it useful in science and engineering to combine multiple sinusoidal terms into a single sinusoid?
  - How is this problem connected to the example from the notes involving

$$3\sin(x) + 4\cos(x)?$$

9 Trigonometric Identities and Equations

9.1 Verifying Trig. Identities & Using Trig. Identities to Simplify Trig. Expressions

Learning Goals 9.1

After this section, I should be able to:

1. Verify the fundamental trigonometric identities.
2. Simplify trigonometric expressions using algebra and the identities.

"In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports– there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions."

Verifying the Fundamental Trig. Identities

Identities enable us to simplify complicated expressions.

The **Pythagorean identities** are equations involving trigonometric functions based on the properties of a right triangle. The first Pythagorean identity is:

$$\sin^2(\theta) + \cos^2(\theta) = 1\,.$$

For the second form, begin with the Pythagorean Identity, divide each term by $\sin^2(\theta)$, and simplify:

For another form, begin with the Pythagorean Identity again, divide each term by $\cos^2(\theta)$ this time, and then simplify:

The next tool we'll be using is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle:

$$\tan(-\theta) = -\tan(\theta)\,, \qquad \sin(-\theta) = -\sin(\theta)\,, \qquad \cos(-\theta) = \cos(\theta)\,,$$
$$\cot(-\theta) = -\cot(\theta)\,, \qquad \csc(-\theta) = -\csc(\theta)\,, \qquad \sec(-\theta) = \sec(\theta)\,.$$

Another set of fundamental identities we'll use is the set of **reciprocal identities** which relate trigonometric functions that are reciprocals of each other:

$$\csc(t) = \frac{1}{\sin(t)}\,, \qquad \sec(t) = \frac{1}{\cos(t)}\,, \qquad \cot(t) = \frac{1}{\tan(t)}\,,$$
$$\sin(t) = \frac{1}{\csc(t)}\,, \qquad \cos(t) = \frac{1}{\sec(t)}\,, \qquad \tan(t) = \frac{1}{\cot(t)}\,.$$

The final set of identities is the set of **quotient identities**:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}\,, \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}\,.$$

Example 9.1 *Verify* $\tan(\theta)\cos(\theta) = \sin(\theta)$.

Example 9.2 *Simplify* $5\sin^2(u) + 4\cos^2(u)$ *by rewriting strictly in terms of sine.*

Example 9.3 *Verify the identity* $\dfrac{\sec^2(\theta) - 1}{\sec^2(\theta)} = \sin^2(\theta)$.

Example 9.4 *Create an identity for the expression* $2\tan(x)\sec(x)$ *by rewriting strictly in terms of sine.*

Example 9.5 *Simplify* $\cos^2(t)\tan^2(t)$ *by rewriting strictly in terms of cosine.*

Example 9.6 *Prove the following statement:* $\dfrac{1-\sin(\theta)}{1+\sin(\theta)} = [\sec(\theta)-\tan(\theta)]^2$

Using Algebra to Simplify Trigonometric Expressions

Example 9.7 *Solve the equation* $2x^2 + x - 1 = 0$.

Example 9.8 *Solve the equation* $2\cos^2(\theta) + \cos(\theta) - 1 = 0$ *for* θ *in the interval* $[0, 2\pi)$.

Example 9.9 *Rewrite the trigonometric expression using the difference of squares:* $4\cos^2(\theta) - 1$.

Section Recap

What are some take-away concepts from this section?

9.1 Exercises:

1. A surveyor measures the angle of elevation to the top of a building and records both $\tan(\theta)$ and the horizontal distance. To simplify a calculation, the surveyor wants to replace $\tan(\theta)\cos(\theta)$ with an equivalent expression.

   Verify the identity
   $$\tan(\theta)\cos(\theta) = \sin(\theta)\,.$$

2. A signal-processing program models a wave using the expression
   $$5\sin^2(\theta) + 4\cos^2(\theta)\,.$$
   Rewrite the expression using only $\sin(\theta)$ and simplify as much as possible.

3. A solar panel is tilted at an angle θ. The panel's efficiency model leads to the equation
   $$2\cos^2(\theta) + \cos(\theta) - 1 = 0\,.$$
   Find all values of θ in the interval $[0, 2\pi)$ that satisfy the equation.

9.1 Scaffolding Questions:

1. A surveyor measures the angle of elevation to the top of a building and records both $\tan(\theta)$ and the horizontal distance. To simplify a calculation, the surveyor wants to replace $\tan(\theta)\cos(\theta)$ with an equivalent expression.

   Verify the identity
   $$\tan(\theta)\cos(\theta) = \sin(\theta)\,.$$

   - Question Comprehension:
     - What does it mean to verify a trigonometric identity?
     - Which side of the identity is easier to simplify?
     - What identities involving tangent might be useful?
   - Content Knowledge:
     - How can $\tan(\theta)$ be rewritten using sine and cosine?
     - What happens when you multiply that expression by $\cos(\theta)$?
     - Can any factors be cancelled?
   - Solution:
     - Did you replace tangent using the quotient identity?
     - Did you simplify correctly after multiplying by $\cos(\theta)$?
     - Did your final result match the other side of the identity?
   - Reflect:
     - Why is it often helpful to rewrite tangent in terms of sine and cosine?
     - How could this identity simplify calculations involving right triangles?

2. A signal-processing program models a wave using the expression

$$5\sin^2(\theta) + 4\cos^2(\theta).$$

Rewrite the expression using only $\sin(\theta)$ and simplify as much as possible.

- Question Comprehension:
  - What does it mean to rewrite an expression using only $\sin(\theta)$?
  - Which part of the expression is not written in terms of sine?
  - Which trigonometric identity relates $\sin^2(\theta)$ and $\cos^2(\theta)$?
- Content Knowledge:
  - How can you rewrite $\cos^2(\theta)$ using the Pythagorean identity?
  - What expression do you obtain after substituting for $\cos^2(\theta)$?
  - How can you combine like terms to simplify the result?
- Solution:
  - Did you correctly substitute $1 - \sin^2(\theta)$ for $\cos^2(\theta)$?
  - Did you distribute the coefficient 4 correctly?
  - Did you combine like terms and simplify completely?
- Reflect:
  - Why is it useful to rewrite an expression in terms of a single trigonometric function?
  - Could you instead rewrite the expression using only $\cos(\theta)$? How would that process compare?

3. A solar panel is tilted at an angle θ. The panel's efficiency model leads to the equation

$$2\cos^2(\theta) + \cos(\theta) - 1 = 0\,.$$

Find all values of θ in the interval $[0, 2\pi)$ that satisfy the equation.

- Question Comprehension:
  - What quantity are you solving for?
  - What interval must all solutions belong to?
  - How is this equation similar to a quadratic equation?
- Content Knowledge:
  - What substitution can turn this trigonometric equation into a quadratic equation?
  - How can you factor the resulting quadratic?
  - What cosine values do you obtain after solving?
  - At which angles in $[0, 2\pi)$ do those cosine values occur?
- Solution:
  - Did you rewrite the equation as a quadratic in $\cos(\theta)$?
  - Did you solve for all possible cosine values?
  - Did you find every angle in the specified interval?
  - Did you check that each solution satisfies the original equation?
- Reflect:
  - How did algebra help you solve a trigonometric equation?
  - Why is it important to find all angles corresponding to a trigonometric value?

9.2 Sum and Difference Identities

Learning Goals 9.2

After this section, I should be able to:

1. Use sum and difference formulas for sine, cosine, and tangent.
2. Use sum and difference formulas to verify identities.

Definition: Sum Identities

For any two angles α and β, we have the **addition formulas**, or the **sum identities**:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\,,$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\,.$$

Example 9.10 *Given the angles α and β such that:*

$$\sin(\alpha) = \frac{4}{5}\,, \quad \sin(\beta) = \frac{12}{13}\,, \quad \cos(\alpha) = \frac{3}{5}\,, \quad \cos(\beta) = \frac{5}{13}\,,$$

find the exact values of $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$.

Example 9.11 *Find the exact value of* $\cos(75°)$. *(Hint: use* $45° + 30° = 75°$.*)*

Definition: Difference Identities

For any two angles α and β, we have the **difference formulas**, or the **difference identities**:

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)\,,$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\,.$$

Example 9.12 *Given the angles α and β such that:*

$$\sin(\alpha) = \frac{4}{5}\,, \quad \sin(\beta) = \frac{12}{13}\,, \quad \cos(\alpha) = \frac{3}{5}\,, \quad \cos(\beta) = \frac{5}{13}\,,$$

find the exact values of $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$.

Example 9.13 *Find the exact value of $\cos(15°)$. (Hint: use $15° = 45° - 30°$.)*

Example 9.14 *Find the exact value of $\sin(15°)$. (Hint: use $15° = 135° - 120°$.)*

Definition: Sum and Difference Identities for Tangent

For any two angles α and β, we have the **sum and difference formulas** for tangent:

$$\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)},$$

$$\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}.$$

Example 9.15 *Given the angles α and β such that:*

$$\sin(\alpha)=\frac{4}{5},\quad \sin(\beta)=\frac{12}{13},\quad \cos(\alpha)=\frac{3}{5},\quad \cos(\beta)=\frac{5}{13},$$

find the exact values of $\tan(\alpha+\beta)$ *and* $\tan(\alpha-\beta)$.

Example 9.16 *Find the exact value of* $\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$.

Example 9.17 *Find the exact value of* $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right)$.

Example 9.18 *Given the following information:* $\sin(\alpha)=\frac{3}{5}$ *with* α *in Quadrant I and* $\cos(\beta)=-\frac{5}{13}$ *with* β *in Quadrant III, determine:*

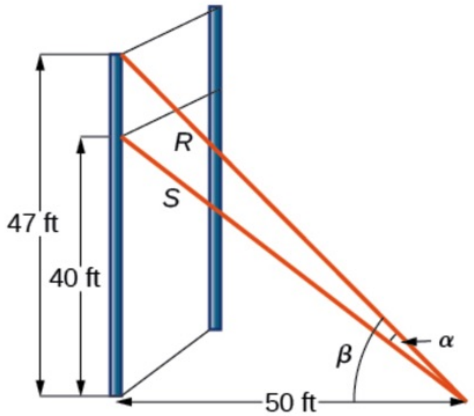
$$\sin(\alpha+\beta)\,,\quad \cos(\alpha+\beta)\,,\quad \textit{and}\quad \tan(\alpha+\beta)\,.$$

Example 9.19 *Verify the identity* $\tan(\pi - \theta) = -\tan(\theta)$.

Example 9.20 *Verify the identity* $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$.

Example 9.21 *Verify the identity* $\dfrac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) - \tan(\beta)\,.$

Example 9.22 *For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle between the wires.*



Section Recap

What are some take-away concepts from this section?

9.2 Exercises:

1. A surveyor measures two angles of elevation, 45° and 30°, from the same observation point. To combine the measurements, she needs the exact value of

$$\cos(45^\circ + 30^\circ).$$

   Find the exact value.

2. Two forces act on an object at angles whose tangents are

$$\tan(\alpha) = \frac{3}{4} \qquad \text{and} \qquad \tan(\beta) = \frac{5}{12}.$$

   Determine the exact value of

$$\tan(\alpha + \beta).$$

3. A navigation system reports that

$$\sin(\alpha) = \frac{3}{5},$$

   where α is in Quadrant I, and

$$\cos(\beta) = -\frac{5}{13},$$

   where β is in Quadrant III.

   Determine the exact values of

$$\sin(\alpha + \beta), \qquad \cos(\alpha + \beta), \qquad \tan(\alpha + \beta).$$

9.2 Scaffolding Questions:

1. A surveyor measures two angles of elevation, $45°$ and $30°$, from the same observation point. To combine the measurements, she needs the exact value of

$$\cos(45° + 30°).$$

Find the exact value.

- Question Comprehension:
  - What quantity are you being asked to find?
  - Why is $75°$ not typically listed among the special angles you memorize?
  - What information do you already know about $45°$ and $30°$?
- Content Knowledge:
  - Which trigonometric identity involves the cosine of a sum?
  - What are the exact values of $\cos(45°)$, $\sin(45°)$, $\cos(30°)$, and $\sin(30°)$?
  - How can these values be substituted into the identity?
- Solution:
  - Did you correctly substitute the special-angle values?
  - Did you simplify all radicals and fractions completely?
  - Is your final answer an exact value rather than a decimal approximation?
- Reflect:
  - Why are sum identities useful for finding exact values of angles such as $75°$?
  - Could the same strategy be used to find exact values of other non-special angles?

2. Two forces act on an object at angles whose tangents are

$$\tan(\alpha) = \frac{3}{4} \qquad \text{and} \qquad \tan(\beta) = \frac{5}{12}.$$

Determine the exact value of

$$\tan(\alpha + \beta).$$

- Question Comprehension:
  - What quantity are you trying to determine?
  - What information is given directly?
  - Are the angles themselves known, or only certain trigonometric values?
- Content Knowledge:
  - Which identity relates $\tan(\alpha + \beta)$ to $\tan(\alpha)$ and $\tan(\beta)$?
  - What values should be substituted into the identity?
  - What restrictions must be considered when simplifying a rational expression?
- Solution:
  - Did you correctly substitute both tangent values?
  - Did you simplify the numerator and denominator carefully?
  - Is the final answer expressed as a simplified fraction?
- Reflect:
  - Why is the tangent sum identity more efficient than first finding α and β?
  - How would your work change if you were asked to find $\tan(\alpha - \beta)$ instead?

3. A navigation system reports that

$$\sin(\alpha) = \frac{3}{5},$$

where α is in Quadrant I, and

$$\cos(\beta) = -\frac{5}{13},$$

where β is in Quadrant III.

Determine the exact values of

$$\sin(\alpha + \beta), \qquad \cos(\alpha + \beta), \qquad \tan(\alpha + \beta).$$

- Question Comprehension:
  - What quantities must be found?
  - What information is provided about each angle?
  - Why is the quadrant information important?
- Content Knowledge:
  - How can a right triangle be used to find missing trigonometric values?
  - Which trigonometric functions must be determined before applying the sum identities?
  - Which sum identities will be needed to find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$?
  - How can $\tan(\alpha + \beta)$ be obtained once sine and cosine are known?
- Solution:
  - Did you correctly determine all missing trigonometric values and signs?
  - Did you apply the sum identities accurately?
  - Did you simplify each final answer completely?
- Reflect:
  - How did the quadrant information affect your answers?
  - Which prerequisite topics from earlier sections were needed to solve this problem?
  - Why is this problem more involved than simply evaluating a special angle?

9.3 Double-Angle, Half-Angle, and Reduction Formulas

Learning Goals 9.3

After this section, I should be able to:

1. Use double-angle formulas to find exact values.
2. Use double-angle formulas to verify identities.
3. Use reduction formulas to simplify an expression.
4. Use half-angle formulas to find exact values.

Bicycle and skateboard ramps made for competition must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be such that $\tan(\theta) = 5/3$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

Example 9.23 *Using the sum identity* $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, *calculate* $\sin(2\theta)$.

Example 9.24 *Using the sum identity* $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$, *calculate* $\cos(2\theta)$.

Definition: Double Angle Formulas

The **double angle formulas** are:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ \downarrow \quad &= 1 - 2\sin^2(\theta)\\ \cos(2\theta) &= 2\cos^2(\theta) - 1\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

Example 9.25 *Given that* $\tan(\theta) = -\frac{3}{4}$ *and* θ *is in quadrant II, find the following:*

(a) $\sin(2\theta)$

(b) $\cos(2\theta)$

(c) $\tan(2\theta)$

Example 9.26 *Rewrite* $\cos(6x)$ *in terms of* $\cos(3x)$.

Example 9.27 *Verify the identity* $1 + \sin(2\theta) = (\sin(\theta) + \cos(\theta))^2$.

Example 9.28 *Verify the identity* $\tan(2\theta) = \dfrac{2}{\cot(\theta) - \tan(\theta)}$.

Ideal projectile motion

In physics, a projectile launched with specified conditions (initial velocity of projectile, angle of elevation, gravity, etc.) will have a set distance it can travel. If the projectile is launched on flat ground, the range of the projectile is

$$d = \frac{v_0^2 \sin(2\theta)}{g},$$

where

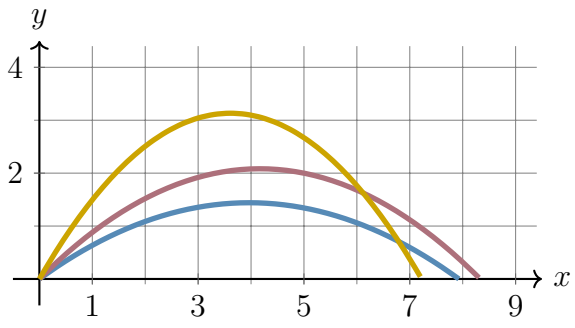
- d is the total horizontal distance travelled by the projectile,
- v_0 is the initial velocity of the projectile,
- g is the gravitational acceleration constant (usually 9.81m/s^2 near the Earth's surface),
- θ is the angle of elevation at which the projectile is launched.

Example 9.29 *With $g = 3$ and $v_0 = 5$, determine the range of a projectile with angles of elevation of:*

(a) $\theta = \pi/5$

(b) $\theta = \pi/4$

(c) $\theta = \pi/3$



[link to Desmos graph]

The horizontal position of the projectile is $x(t) = v_0\, t\, \cos(\theta)$.

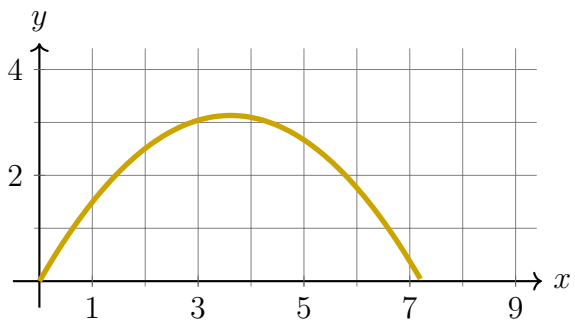
The vertical position of the projectile is $y(t) = v_0\, t\, \sin(\theta) - \frac{1}{2} g\, t^2$.

The position at time t can be determined by

$$(x(t), y(t)) = \left(v_0\, t\, \cos(\theta), v_0\, t\, \sin(\theta) - \frac{1}{2} g\, t^2 \right).$$

This is an example of a **parametric equation**, where the x- and y-components are given in terms of some **parameter**, usually t or θ.

Example 9.30 *Using the constants $g = 3$, $v_0 = 5$, and $\theta = \pi/3$, find the x- and y-coordinates of the projectile at $t = 1$.*



Example 9.31 *Determine the t-value(s) when the projectile will be at a vertical height of 0. That is, find the times where the projectile will be on the ground.*

Example 9.32 *At the time value found in the previous example, what is the horizontal distance of the projectile then? That is, find the distance travelled by the projectile.*

Example 9.33 *For what angle of elevation does this distance attain its maximum value?*

Example 9.34 *Use* $\cos(2\theta) = 1 - 2\sin^2(\theta)$ *to find an expression for* $\sin^2(\theta)$.

Example 9.35 *Use* $\cos(2\theta) = 2\cos^2(\theta) - 1$ *to find an expression for* $\cos^2(\theta)$.

Example 9.36 *Use* $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$ *to find an expression for* $\tan^2(\theta)$.

Definition: Reduction Formulas

The **reduction formulas** are:

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$
$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + 2\cos(2\theta)}$$

Example 9.37 *Write an expression for* $\cos^4(x)$ *that does not involve any powers of sine or cosine greater than 1.*

Example 9.38 *Use the power-reducing formulas to prove* $\sin^3(2x) = \frac{1}{2}\sin(2x)\left[1 - \cos(4x)\right]$.

Definition: Half Angle Formulas

The **half angle formulas** are:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

$$\downarrow \quad = \frac{\sin(\theta)}{1+\cos(\theta)}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{\sin(\theta)}$$

Example 9.39 *Find* $\sin(15°)$ *using a half-angle formula.*

Example 9.40 *Find* $\cos(15°)$ *using a half-angle formula.*

Example 9.41 *A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan(\theta) = 5/3$ for higher-level competition, what is the measurement of the angle for novice competition?*

Section Recap

What are some take-away concepts from this section?

9.3 Exercises:

1. A projectile is launched with an angle of elevation θ, where

$$\sin(\theta) = \frac{3}{5} \quad \text{and} \quad \cos(\theta) = \frac{4}{5}.$$

   The range formula involves $\sin(2\theta)$.

   Find the exact value of $\sin(2\theta)$ and explain what this value represents in the projectile's range formula.

2. A bicycle ramp has an angle of inclination θ satisfying

$$\cos(\theta) = \frac{\sqrt{3}}{2},$$

   where $0 < \theta < \pi$.

   A beginner ramp is constructed with half the angle of inclination.

   Use a half-angle formula to find the exact value of

$$\sin\left(\frac{\theta}{2}\right).$$

3. A stunt rider is testing two launch ramps. The first ramp launches at an angle θ with

$$\tan(\theta) = \frac{5}{3}.$$

   The second ramp launches at twice that angle.

   (a) Find the exact value of $\tan(2\theta)$.

   (b) Determine whether the doubled angle is steeper or less steep than a 60° ramp.

   (c) Explain why the double-angle formula is useful in this situation instead of first finding θ with a calculator.

9.3 Scaffolding Questions:

1. A projectile is launched with an angle of elevation θ, where

$$\sin(\theta) = \frac{3}{5} \quad \text{and} \quad \cos(\theta) = \frac{4}{5}.$$

The range formula involves $\sin(2\theta)$.

Find the exact value of $\sin(2\theta)$ and explain what this value represents in the projectile's range formula.

- Question Comprehension:
  - What quantity are you asked to find?
  - Which double-angle formula involves $\sin(2\theta)$?
  - What information about $\sin(\theta)$ and $\cos(\theta)$ is provided?
- Content Knowledge:
  - What is the double-angle identity for sine?
  - How can you use the given values directly in the formula?
  - In the projectile range formula, what role does $\sin(2\theta)$ play?
- Solution:
  - Did you substitute correctly into the double-angle formula?
  - Did you simplify the result completely?
  - Did you connect your answer to the projectile's range?
- Reflect:
  - Why is knowing $\sin(2\theta)$ useful in projectile motion?
  - How would the calculation change if only $\tan(\theta)$ were given?

2. A bicycle ramp has an angle of inclination θ satisfying

$$\cos(\theta) = \frac{\sqrt{3}}{2},$$

where $0 < \theta < \pi$.

A beginner ramp is constructed with half the angle of inclination.

Use a half-angle formula to find the exact value of

$$\sin\left(\frac{\theta}{2}\right).$$

- Question Comprehension:
  - What quantity are you trying to find?
  - Which type of identity is designed for angles such as $\frac{\theta}{2}$?
  - What information about θ is given?
- Content Knowledge:
  - What is the half-angle formula for sine?
  - Which sign should be used in the formula?
  - How does the interval $0 < \theta < \pi$ help determine the sign?
- Solution:
  - Did you correctly substitute $\cos(\theta) = \frac{\sqrt{3}}{2}$ into the formula?
  - Did you simplify the radical correctly?
  - Did you choose the appropriate sign?
- Reflect:
  - Why do half-angle formulas involve square roots?
  - How could you verify your answer using a known special angle?

3. A stunt rider is testing two launch ramps. The first ramp launches at an angle θ with
$$\tan(\theta) = \frac{5}{3}.$$
The second ramp launches at twice that angle.

   (a) Find the exact value of $\tan(2\theta)$.

   (b) Determine whether the doubled angle is steeper or less steep than a 60° ramp.

   (c) Explain why the double-angle formula is useful in this situation instead of first finding θ with a calculator.

   - Question Comprehension:
     - What information is given about the original ramp?
     - What quantity must be found in part (a)?
     - How can comparing tangent values help determine whether an angle is steeper than 60°?
   - Content Knowledge:
     - What is the double-angle formula for tangent?
     - What is the exact value of $\tan(60^\circ)$?
     - Why can tangent values be compared directly for angles between 0° and 90°?
   - Solution:
     - Did you correctly substitute $\tan(\theta) = \frac{5}{3}$ into the formula?
     - Did you simplify the resulting fraction completely?
     - Did you use the result to compare the doubled angle with 60°?
   - Reflect:
     - Why is the double-angle formula more efficient than first finding θ?
     - In what other situations might doubling an angle naturally occur in engineering or physics?

9.4 Sum-to-Product and Product-to-Sum Formulas

Learning Goals 9.4

After this section, I should be able to:

1. Express products as sums.
2. Express sums as products.

Definition: Product-to-Sum Formulas

The **product-to-sum formulas** are:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin(\alpha)\sin(\beta) = -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]$$

Example 9.42 *Verify the identity* $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$ *using the sum and difference formulas.*

Prosthaphaeresis (from the Greek προσθαφαίρεσιζ) was an algorithm used in the late 16th century and early 17th century for approximate multiplication and division using formulas from trigonometry.

First, let's sound out the Greek word:

π ρ o σ θ α φ αί ρ ε σ ι ζ

Using $\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$, here is how multiplication of two numbers would work. We'll show each step with the example of multiplying 309 and 78.8:

(1) **Scale down:** By shifting the decimal point to the left or right, scale both numbers to values between -1 and 1. The first number will be denoted $\cos(\alpha)$ and the second will be $\cos(\beta)$.

(2) **Inverse Cosine:** Using an inverse cosine table, find two angles α and β whose cosines are our two values.

(3) **Sum and difference:** Find the sum and difference of the two angles.

(4) **Average the cosines:** Find the cosines of the sum and difference angles using a cosine table and average them, giving the product $\cos(\alpha)\cos(\beta)$.

(5) **Scale up:** Shift the decimal place in the answer the combined number of places we have shifted the decimal in the first step for each input, but in the opposite direction.

Example 9.43 *Use the Prosthaphaeresis approach to find the product* $386 \cdot 555$.

Example 9.44 *Use the Prosthaphaeresis approach to find the quotient* $7125/619$. *Here, we'll need to alter the approach slightly.*

Definition: Sum-to-Product Formulas

The **sum-to-product formulas** are:

$$\sin(\alpha)+\sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin(\alpha)-\sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Example 9.45 *Write* $\sin(3x)+\sin(5x)$ *as a product.*

Example 9.46 *Write* $\sin(3x)\sin(5x)$ *as a sum.*

Section Recap

What are some take-away concepts from this section?

9.4 Exercises:

1. Two sound waves are modeled by

$$y = \sin(3t) + \sin(5t).$$

   Rewrite the expression as a product using a sum-to-product formula.

2. In signal processing, products of trigonometric functions are often rewritten as sums. Rewrite

$$\cos(4t)\cos(2t)$$

   as a sum using a product-to-sum formula.

3. A vibrating system is modeled by

$$y = \cos(7t) - \cos(3t).$$

   (a) Rewrite the expression as a product.

   (b) Use your result to identify the slower oscillation and the faster oscillation in the model.

   (c) Explain why the product form might be more useful than the original expression when studying the motion.

9.4 Scaffolding Questions:

1. Two sound waves are modeled by

$$y = \sin(3t) + \sin(5t).$$

Rewrite the expression as a product using a sum-to-product formula.

- Question Comprehension:
  - What form is the expression currently written in?
  - Are you being asked to convert a sum into a product or a product into a sum?
  - Which sum-to-product identity involves the sum of two sine functions?
- Content Knowledge:
  - What is the sum-to-product formula for $\sin(\alpha) + \sin(\beta)$?
  - What values correspond to α and β in this problem?
  - How do you compute $\frac{\alpha+\beta}{2}$ and $\frac{\alpha-\beta}{2}$?
- Solution:
  - Did you correctly identify $\alpha = 3t$ and $\beta = 5t$?
  - Did you simplify the resulting angles completely?
  - Is your final answer written as a product?
- Reflect:
  - Why might a product form reveal features of the wave that are harder to see in the original sum?
  - How would the process change if the expression involved cosines instead of sines?

2. In signal processing, products of trigonometric functions are often rewritten as sums. Rewrite

$$\cos(4t)\cos(2t)$$

as a sum using a product-to-sum formula.

- Question Comprehension:
  - What form is the expression currently written in?
  - Are you converting a product into a sum or a sum into a product?
  - Which product-to-sum identity applies to a product of two cosine functions?
- Content Knowledge:
  - What is the product-to-sum formula for $\cos(\alpha)\cos(\beta)$?
  - What values correspond to α and β?
  - How do you find $\alpha + \beta$ and $\alpha - \beta$?
- Solution:
  - Did you correctly substitute into the formula?
  - Did you simplify the resulting angles?
  - Is your final answer expressed as a sum of cosine functions?
- Reflect:
  - Why might engineers prefer the sum form instead of the product form?
  - What patterns do you notice in the frequencies that appear after rewriting the expression?

3. A vibrating system is modeled by

$$y = \cos(7t) - \cos(3t).$$

(a) Rewrite the expression as a product.

(b) Use your result to identify the slower oscillation and the faster oscillation in the model.

(c) Explain why the product form might be more useful than the original expression when studying the motion.

- Question Comprehension:
  - What type of expression are you given?
  - Which sum-to-product formula applies to $\cos(\alpha) - \cos(\beta)$?
  - What additional interpretation is requested after the algebraic simplification?
- Content Knowledge:
  - What is the sum-to-product formula for $\cos(\alpha) - \cos(\beta)$?
  - What values correspond to α and β?
  - After rewriting the expression, which factor oscillates more rapidly?
- Solution:
  - Did you correctly apply the identity?
  - Did you simplify the resulting angles completely?
  - Did you identify the faster and slower oscillations from the product form?
- Reflect:
  - How does rewriting a sum or difference as a product reveal information about the behavior of a wave?
  - Why might scientists and engineers prefer one form of an expression over another depending on the application?

9.5 Solving Trigonometric Equations

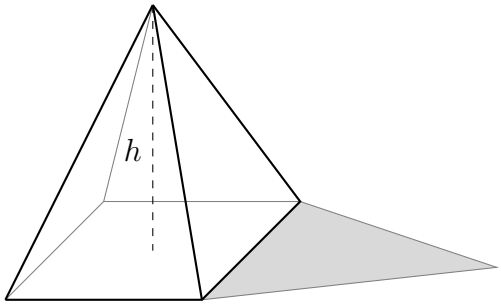
Learning Goals 9.5

After this section, I should be able to:

1. Solve linear trigonometric equations in sine and cosine.
2. Solve equations involving a single trigonometric function.
3. Solve trigonometric equations using a calculator.
4. Solve trigonometric equations that are quadratic in form.
5. Solve trigonometric equations using fundamental identities.
6. Solve trigonometric equations with multiple angles.
7. Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is often credited as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. He reasoned that when the height of his staff's shadow was exactly equal to the actual height of the staff, then the height of the nearby pyramid's shadow must also be equal to the height of the actual pyramid. Since the structures and their shadows were creating a right triangle with two equal sides, they were similar triangles. By measuring the length of the pyramid's shadow at that moment, he could obtain the height of the pyramid.

Example 9.47 *Thales found that the base of the pyramid of Cheops has sides that are 756 feet long. If his 3 ft. staff casts a shadow that was 6 ft. long at the same time that the shadow of the pyramid was 586 ft. long, determine the height of the pyramid.*



Example 9.48 *Find all possible exact solutions for the equation* $\cos(\theta) = \frac{1}{2}$.

Example 9.49 *Find all possible exact solutions for the equation* $\sin(t) = \frac{1}{2}$.

Example 9.50 *Find all possible exact solutions for the equation* $2\sin(x) - \sqrt{3} = 0$.

Example 9.51 *Solve the equation* $2\cos(\theta) - 3 = -5$ *for* $0 \leq \theta < 2\pi$.

Example 9.52 *Solve the equation* $2\sin^2(t) - 1 = 0$ *for* $0 \leq t < 2\pi$.

Example 9.53 *Solve the equation* $2\sec(\theta) - 1 = 0$ *for* $0 \leq \theta < 4\pi$.

Example 9.54 *Solve the equation* $\tan\left(\theta - \frac{\pi}{2}\right) = 1$ *for* $0 \leq \theta < 2\pi$.

Example 9.55 *Solve the equation* $2\left[\tan(x) + 3\right] = 5 + \tan(x)$ *for* $0 \leq x < 2\pi$.

Example 9.56 *Use a calculator to solve the equation* $5\sin(\theta)-4=0$ *for* $0 \leq \theta < 2\pi$.

Example 9.57 *Use a calculator to solve the equation* $\sec(x)+4=0$ *for* $0 \leq x < 2\pi$.

Example 9.58 *Use a calculator to solve* $\cos(\theta) = \tan(\theta)$ *for* $0 \leq \theta < 2\pi$.

Example 9.59 *Use a calculator to solve the equation* $\sin(\theta) = \tan(\theta)$ *for* $0 \leq x < 2\pi$.

Example 9.60 *Solve the equation:* $\cos^2(\theta) + 3\cos(\theta) - 1 = 0$ *for* $0 \leq \theta < 2\pi$.

Example 9.61 *Solve the equation exactly:* $2\sin^2(\theta) - 5\sin(\theta) + 3 = 0$ *for* $0 \leq \theta \leq 2\pi$.

Example 9.62 *Solve the equation exactly:* $2\sin^2(\theta) + \sin(\theta) = 0$ *for* $0 \leq \theta \leq 2\pi$.

Example 9.63 *Solve the equation exactly:* $2\sin^2(\theta) - 3\sin(\theta) + 1 = 0$ *for* $0 \leq \theta < 2\pi$.

Example 9.64 *Solve the equation exactly for* $0 \leq x < 2\pi$*:*

$$\cos(x)\cos(2x) + \sin(x)\sin(2x) = \frac{\sqrt{3}}{2}\,.$$

Example 9.65 *Solve the equation exactly:* $\cos(2\theta) = \cos(\theta)$ *for* $0 \leq \theta < 2\pi\,.$

Example 9.66 *Solve the equation exactly for* $0 \leq \theta < 2\pi$*:*

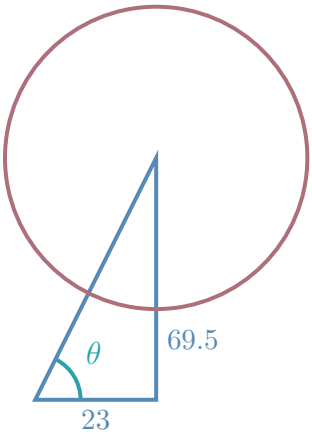
$$3\cos(\theta) + 3 = 2\sin^2(\theta)\,.$$

Example 9.67 *Solve the equation exactly:* $\cos(2x) = \dfrac{1}{2}$ *for* $0 \leq x < 2\pi\,.$

Example 9.68 *Use the Pythagorean Theorem, and the properties of right triangles to model an equation that fits the problem.*

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation?

The angle of elevation is the angle θ shown, from ground up to the center of the Ferris wheel.



Example 9.69 *OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.*

Section Recap

What are some take-away concepts from this section?

9.5 Exercises:

1. A Ferris wheel ride operator wants a passenger car to reach a height where the vertical position satisfies
$$\sin(\theta) = \frac{1}{2}\,.$$
Find all angles θ in the interval $0 \leq \theta < 2\pi$ at which this occurs.

2. A rotating radar antenna produces a signal that is strongest when
$$2\cos^2(\theta) - 3\cos(\theta) + 1 = 0\,.$$
Find all solutions in the interval $0 \leq \theta < 2\pi$.

3. A surveyor stands 50 meters from the base of a radio tower. The angle of elevation from the surveyor to the top of the tower is $60°$.
Find the height of the tower and explain how the trigonometric equation used in your solution models the situation.

9.5 Scaffolding Questions:

1. A Ferris wheel ride operator wants a passenger car to reach a height where the vertical position satisfies
$$\sin(\theta) = \frac{1}{2}\,.$$
Find all angles θ in the interval $0 \leq \theta < 2\pi$ at which this occurs.

   - Question Comprehension:
     - What quantity are you solving for?
     - What interval must contain all solutions?
     - What does the equation tell you about the sine value of the angle?
   - Content Knowledge:
     - Which special angle has reference angle $\pi/6$ and sine value $\frac{1}{2}$?
     - In which quadrants is sine positive?
     - What angles in $[0, 2\pi)$ have sine equal to $\frac{1}{2}$?
   - Solution:
     - Did you identify all angles with the correct reference angle?
     - Did you check that each solution lies in the required interval?
     - Did you verify that substituting each solution gives $\frac{1}{2}$?
   - Reflect:
     - How would your answers change if the sine value were negative?
     - How does the unit circle help organize possible solutions?

2. A rotating radar antenna produces a signal that is strongest when

$$2\cos^2(\theta) - 3\cos(\theta) + 1 = 0\,.$$

Find all solutions in the interval $0 \leq \theta < 2\pi$.

- Question Comprehension:
  - Which trigonometric function appears in the equation?
  - Why might this equation be considered quadratic in form?
  - What interval must contain all solutions?
- Content Knowledge:
  - What substitution could simplify the equation into a quadratic equation?
  - Can the quadratic be factored?
  - After solving for $\cos(\theta)$, which values are actually possible for cosine?
  - For each valid cosine value, what angles satisfy the equation on the unit circle?
- Solution:
  - Did you solve the quadratic correctly?
  - Did you reject any impossible trigonometric values?
  - Did you find every solution in $[0, 2\pi)$?
- Reflect:
  - Why is it important to check whether a quadratic solution is within the range of cosine?
  - How would this process change if the equation involved $\sin(\theta)$ instead?

3. A surveyor stands 50 meters from the base of a radio tower. The angle of elevation from the surveyor to the top of the tower is $60°$.

   Find the height of the tower and explain how the trigonometric equation used in your solution models the situation.

   - Question Comprehension:
     - What information is given about the triangle?
     - Which side length represents the tower's height?
     - Which trigonometric ratio relates the known side and the unknown height?
   - Content Knowledge:
     - Relative to the $60°$ angle, which side is adjacent and which side is opposite?
     - What equation can be written using the tangent ratio?
     - What is the exact value of $\tan(60°)$?
   - Solution:
     - Did you write a correct trigonometric equation?
     - Did you solve the equation correctly for the tower's height?
     - Did you include appropriate units in your answer?
   - Reflect:
     - Why is tangent the most convenient ratio in this problem?
     - How would the solution change if the distance from the surveyor to the tower increased?

10 Further Applications of Trigonometry

10.1 Non-right Triangles: Law of Sines

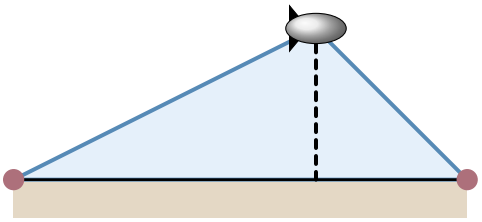
Learning Goals 10.1

After this section, I should be able to:

1. Use the Law of Sines to solve oblique triangles.
2. Solve applied problems using the Law of Sines.

Example 10.1 *Suppose two radar stations located 20 miles apart each detect a blimp between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees.*

What other information would we need to determine the height of the blimp? Why can we not determine this with the tools we have so far?



Earlier in the course, we saw how to solve a right triangle. Given two sides, or one side and one acute angle, we could find the remaining sides and angles. In both of those cases, we were actually given *three pieces* of information, since we already knew one angle was 90°.

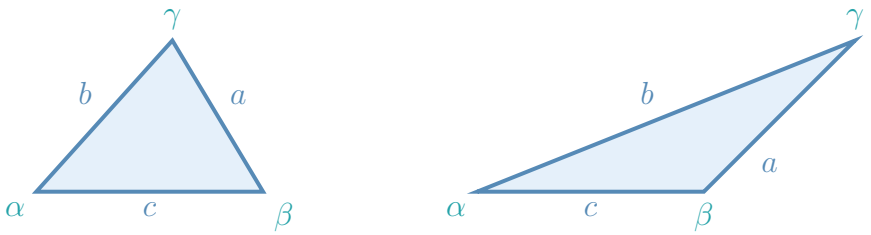
For a general triangle (which may or may not have a right angle!), we will again need *three pieces of information*. The four cases are:

Case 1: One side and two angles
Case 2: Two sides and one opposite angle
Case 3: Two sides and the angle between them
Case 4: Three sides

Note that if we were given all three angles, we cannot determine the sides uniquely. All of the triangles we find in this case would be *similar triangles*.

In this chapter, we will learn how to solve a general triangle in all four of the above cases. Though the methods described will work for right triangles, they are mostly used to solve **oblique triangles**– that is, triangles which do not have a right angle.

There are two types of oblique triangles: an **acute triangle** has all acute angles, and an **obtuse triangle** has one obtuse angle.



This section of notes will focus on cases 1 and 2, which can be solved with the **Law of Sines**. Cases 3 and 4 will be solved in later sections with the **Law of Cosines** and **Law of Tangents**.

Definition

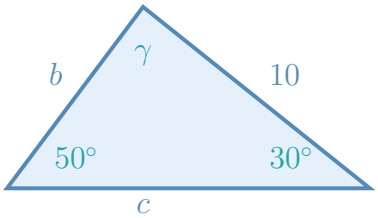
Given a triangle with angles and opposite sides (like labeled in the acute triangle above), the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal.

The **Law of Sines** is based on proportions and is presented symbolically two ways:

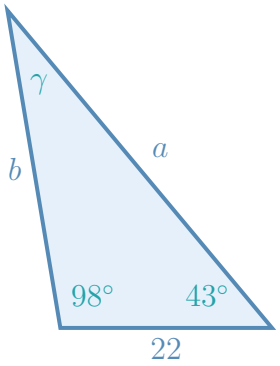
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \quad \text{or} \quad \frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}.$$

To solve an oblique triangle, use any pair of applicable ratios.

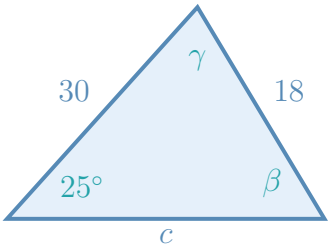
Example 10.2 *Solve for the remaining sides and angles for the triangle shown below to the nearest tenth.*



Example 10.3 *Solve for the remaining sides and angles for the triangle shown below to the nearest tenth.*

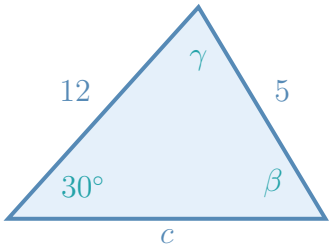


Example 10.4 *Solve for the remaining sides and angles for the triangle shown below to the nearest tenth.*

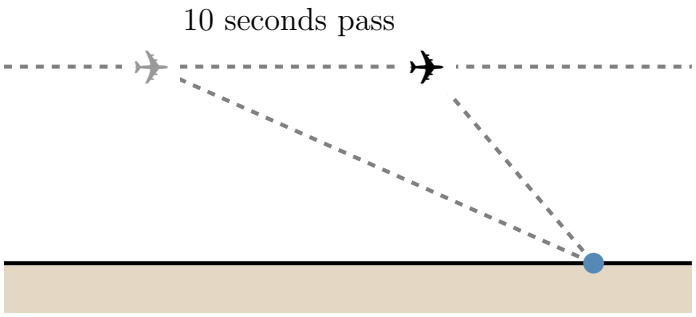


The last example is an example of an **ambiguous case**. That is, there may be more than one solution. It is also possible for there to be exactly one solution or no solution at all!

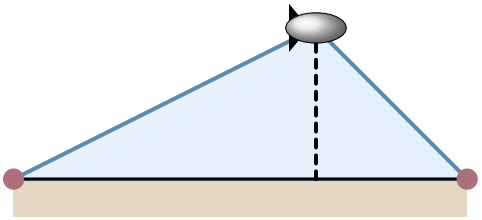
Example 10.5 *Solve for the remaining sides and angles for the triangle shown below to the nearest tenth.*



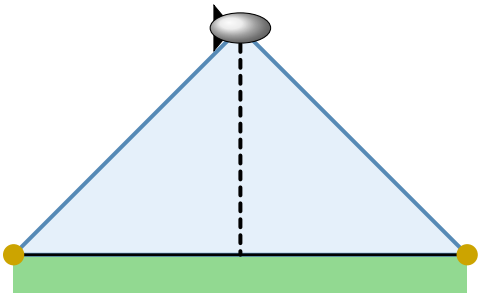
Example 10.6 *An observer on the ground measures an angle of inclination of* 30° *to an approaching airplane, and* 10 *seconds later measures an angle of inclination of* 55°. *If the airplane is flying at a constant speed and at a steady altitude of* 6000 *ft in a straight line directly over the observer, find the speed of the airplane in miles per hour.*



Example 10.7 *Suppose two radar stations located 20 miles apart each detect a blimp between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. Find the height of the blimp.*



Example 10.8 *A blimp is flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point is 62°, and the distance between the viewing points of the two end zones is 145 yards.*



Section Recap

What are some take-away concepts from this section?

10.1 Exercises:

1. Solve the triangle using the Law of Sines. Given $A = 42°$, $B = 68°$, and side $a = 15$, find angle C and the lengths of sides b and c. Round side lengths to the nearest tenth and show all work.

2. Solve the triangle using the Law of Sines. Given $A = 35°$, side $a = 12$, and side $b = 18$, determine all possible triangles that satisfy the given information. If more than one triangle exists, find all missing sides and angles for each triangle. Round answers to the nearest tenth.

3. A surveyor is measuring the distance across a river. Two points A and B on the same bank are 120 meters apart. The angle from A to a tree on the opposite bank is $48°$, and the angle from B to the same tree is $72°$. Find the distances from the tree to points A and B. Round your answers to the nearest tenth of a meter.

10.1 Scaffolding Questions:

1. Solve the triangle using the Law of Sines. Given $A = 42°$, $B = 68°$, and side $a = 15$, find angle C and the lengths of sides b and c. Round side lengths to the nearest tenth and show all work.

    - Question Comprehension:
        - What information is given, and what quantities must be found?
        - Why is the Law of Sines an appropriate method for this problem?
        - What must be determined before solving for the unknown side lengths?
    - Content Knowledge:
        - How can you find the third angle of a triangle when two angles are known?
        - What is the Law of Sines?
        - How do you set up a proportion using a known side-angle pair and an unknown side-angle pair?
    - Solution:
        - Did you correctly find angle C using the angle sum of a triangle?
        - Did you correctly apply the Law of Sines to find side b?
        - Did you correctly apply the Law of Sines to find side c and round appropriately?
    - Reflect:
        - Why does knowing two angles guarantee a unique triangle?
        - How would the solution process change if only one angle were given?

2. Solve the triangle using the Law of Sines. Given $A = 35°$, side $a = 12$, and side $b = 18$, determine all possible triangles that satisfy the given information. If more than one triangle exists, find all missing sides and angles for each triangle. Round answers to the nearest tenth.

   - Question Comprehension:
     - What type of triangle-solving case is represented by the given information?
     - Why is this known as the ambiguous SSA case?
     - Is it possible for more than one triangle to satisfy the given measurements?
   - Content Knowledge:
     - How can the Law of Sines be used to find angle B?
     - Why can $\sin(B) = k$ correspond to two different angles?
     - How do you determine whether a second triangle is possible?
   - Solution:
     - Did you use the Law of Sines to calculate a value for $\sin(B)$?
     - Did you find both possible values of angle B?
     - For each possible triangle, did you find angle C and the remaining side correctly?
   - Reflect:
     - Under what circumstances does an SSA problem have two solutions, one solution, or no solution?
     - How can a sketch help identify whether multiple triangles are possible?

3. A surveyor is measuring the distance across a river. Two points A and B on the same bank are 120 meters apart. The angle from A to a tree on the opposite bank is $48°$, and the angle from B to the same tree is $72°$. Find the distances from the tree to points A and B. Round your answers to the nearest tenth of a meter.

   - Question Comprehension:
     - How can this situation be represented as a triangle?
     - Which side length is known?
     - What information is needed before applying the Law of Sines?
   - Content Knowledge:
     - How do you find the third angle of the triangle?
     - Which side is opposite each angle?
     - How do you use the Law of Sines to find unknown side lengths?
   - Solution:
     - Did you correctly determine the third angle?
     - Did you correctly identify the side-angle pairs needed for the Law of Sines?
     - Did you find both unknown distances and round appropriately?
   - Reflect:
     - Why is the Law of Sines particularly useful in surveying applications?
     - How would the problem change if only one angle were known?

10.2 Non-right Triangles: Law of Cosines

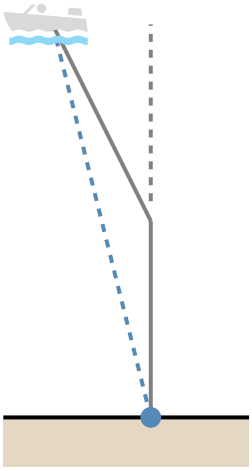
Learning Goals 10.2

After this section, I should be able to:

1. Use the Law of Cosines to solve oblique triangles.
2. Solve applied problems using the Law of Cosines.

Example 10.9 *Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles.*

What other information would we need to determine the distance from the boat to the port? Why can we not determine this with the tools we have so far?



If we have 'one side & two angles' or 'two sides & one opposite angle,' we can use the **Law of Sines** to help determine the other information.

If we have 'two sides & the angle between them' like in the problem above, the Law of Sines cannot help us...

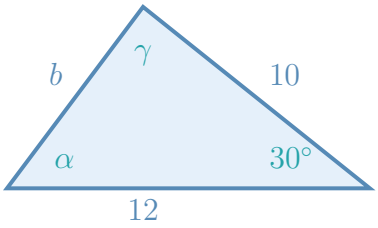
Definition: Law of Cosines

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

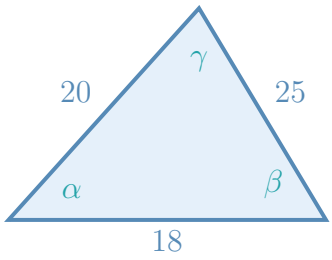
That is, if we know two sides x and y and the angle between them θ, we can find the other side length, z, with

$$z^2 = x^2 + y^2 - 2xy\cos(\theta)\,.$$

Example 10.10 *Solve for the remaining sides and angles for the triangle shown below to the nearest tenth.*

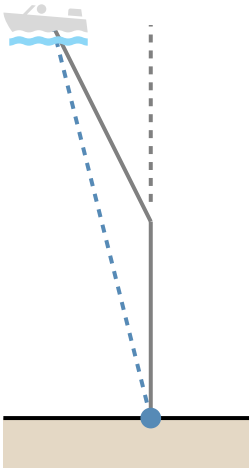


Example 10.11 *Solve for the angles of the triangle shown below to the nearest tenth.*

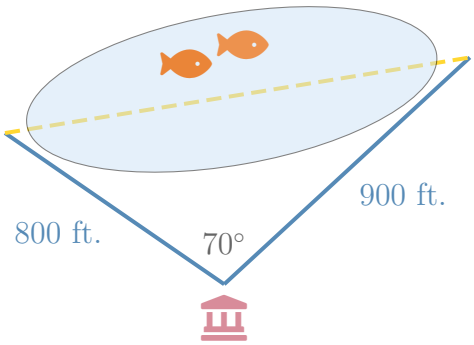


Example 10.12 *Solve for the angles of the triangle with sides* 5, 12, *and* 13.

Example 10.13 *Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. Determine the distance from the boat to the port.*



Example 10.14 *From the landmark, a surveyor measures it is 800 feet to one end of a lake. She also measures it is 900 feet from the landmark to the opposite end of the lake, and there is an angle of* 70° *between the two lines measured. Find the distance across the lake.*



Example 10.15 *On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points.*

Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5,050 feet from the first tower and 2,420 feet from the second tower. Determine how far the phone is from the highway.



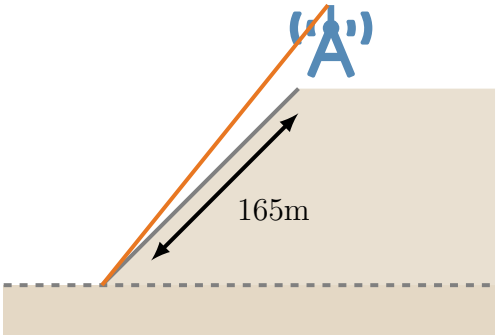
Example 10.16 *The figure below shows satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 70 miles apart. When the satellite is on one side of the two stations, the angles of elevation are measured to be $83.9°$ and $86.2°$, respectively. How far is the satellite above the ground? Round answers to the nearest whole mile.*



Example 10.17 *Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.*

Example 10.18 *A pilot flies in a straight path for 1 hour 30 min. She then makes a slight turn, heading* 10° *to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?*

Example 10.19 *A communications tower is located at the top of a steep hill. The angle of inclination of the hill is* 67°*. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is* 16°*. Find the length of the cable required for the guy wire, rounded up to the nearest whole meter.*



Section Recap

What are some take-away concepts from this section?

10.2 Exercises:

1. A triangle has sides of length 9 and 12, and the included angle between them is 70°. Use the Law of Cosines to find the third side. Then find the remaining angles of the triangle. Round all answers to the nearest tenth.

2. A triangular park has sides measuring 120 meters, 175 meters, and 210 meters. Determine all three angles of the triangle. Round answers to the nearest tenth of a degree.

3. Two hiking trails leave the same ranger station. One trail extends 4.5 miles and the other extends 6.2 miles. The angle between the trails at the ranger station is 48°. How far apart are the endpoints of the two trails? Then, determine the remaining angles of the triangle formed by the trails and the segment connecting their endpoints. Round answers to the nearest tenth.

10.2 Scaffolding Questions:

1. A triangle has sides of length 9 and 12, and the included angle between them is 70°. Use the Law of Cosines to find the third side. Then find the remaining angles of the triangle. Round all answers to the nearest tenth.

    - Question Comprehension:
        - What information is given about the triangle?
        - Why is the Law of Cosines an appropriate method to begin solving this problem?
        - Which side can be found directly using the given information?
    - Content Knowledge:
        - What is the Law of Cosines formula for finding a side when two sides and the included angle are known?
        - After finding the third side, what methods can be used to find the remaining angles?
        - How can the angle sum of a triangle be used to check your work?
    - Solution:
        - Did you correctly substitute the two side lengths and included angle into the Law of Cosines?
        - Did you solve for the third side before attempting to find the remaining angles?
        - Did you find all three angles and round answers appropriately?
    - Reflect:
        - Why would the Law of Sines be difficult to use as the first step in this problem?
        - How can you verify that your angle measurements are reasonable?

2. A triangular park has sides measuring 120 meters, 175 meters, and 210 meters. Determine all three angles of the triangle. Round answers to the nearest tenth of a degree.

   - Question Comprehension:
     - What information is given about the triangle?
     - Which side is opposite the largest angle?
     - Why can the Law of Cosines be used in this situation?
   - Content Knowledge:
     - What form of the Law of Cosines is used to find an angle when all three sides are known?
     - Which angle would be easiest to find first?
     - Once one angle is known, how can the remaining angles be found?
   - Solution:
     - Did you correctly identify the side opposite the angle you solved for first?
     - Did you use the Law of Cosines correctly to find an angle?
     - Did you determine all three angles and verify that they add to $180°$?
   - Reflect:
     - Why is it often helpful to begin with the largest side in an SSS problem?
     - How would a small computational error affect the remaining angle calculations?

3. Two hiking trails leave the same ranger station. One trail extends 4.5 miles and the other extends 6.2 miles. The angle between the trails at the ranger station is 48°. How far apart are the endpoints of the two trails? Then, determine the remaining angles of the triangle formed by the trails and the segment connecting their endpoints. Round answers to the nearest tenth.

   - Question Comprehension:
     - What geometric figure is formed by the two trails and the segment connecting their endpoints?
     - What information is known about this triangle?
     - What quantity is represented by the distance between the endpoints?
   - Content Knowledge:
     - Which theorem allows you to find the distance between the trail endpoints?
     - Once the third side is known, how can the remaining angles be found?
     - What relationship among the angles can be used to check the final answers?
   - Solution:
     - Did you use the Law of Cosines correctly to find the distance between the endpoints?
     - Did you determine the remaining angles using either the Law of Cosines or the Law of Sines?
     - Did you include appropriate units and rounding?
   - Reflect:
     - How does this problem illustrate a real-world use of the Law of Cosines?
     - If the included angle increased, what would happen to the distance between the trail endpoints?

10.3 Polar Coordinates

Learning Goals 10.3

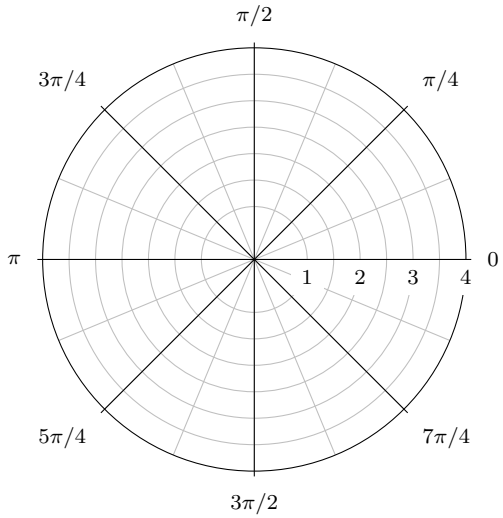
After this section, I should be able to:

1. Plot points using polar coordinates.
2. Convert from polar coordinates to rectangular coordinates.
3. Convert from rectangular coordinates to polar coordinates.
4. Transform equations between polar and rectangular coordinates.
5. Identify and graph polar equations by converting to rectangular coordinates.

When we think of plotting points, we typically use **rectangular coordinates** on the **Cartesian plane**. However, there are other ways to represent coordinates, like **polar coordinates**, which are plotted on a the polar grid. A polar grid is made up of concentric circles extending outward from the **pole**, which is the origin of this coordinate system.

In the polar system, the positive x-axis becomes the **polar axis**, and the origin is called the **pole**. Polar coordinates consist of two parts (r, θ). The **radius**, r, representing the distance from the pole. The **angle**, θ, measured in **radians** to indicate direction. To plot a point, move **counterclockwise** by θ, then go out r units from the pole. For instance, to plot $(2, \frac{\pi}{4})$, rotate $\frac{\pi}{4}$ units counterclockwise, and then extend 2 units outward from the pole.

Example 10.20 *Plot the points in polar coordinates:* $\left(3, \frac{\pi}{4}\right)$, $\left(3, -\frac{\pi}{4}\right)$, $\left(-3, \frac{\pi}{4}\right)$, $\left(2, \frac{9\pi}{4}\right)$.



Sometimes, we'll be given a set of polar coordinates and will need to convert them to rectangular coordinates.

How To: Polar to Rectangular coordinates

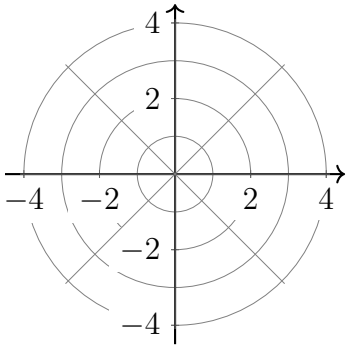
To convert polar coordinates to rectangular coordinates, let:

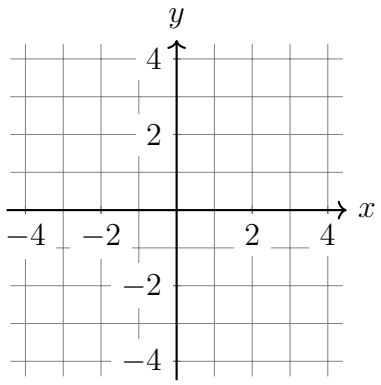
$$\frac{x}{r} = \cos(\theta) \quad \longrightarrow \quad x = r\cos(\theta)\,,$$

$$\frac{y}{r} = \sin(\theta) \quad \longrightarrow \quad y = r\sin(\theta)\,.$$

Example 10.21 *Write the polar coordinates* $\left(3, \frac{\pi}{2}\right)$ *in rectangular coordinates. Then, plot the points on their respective graphs below.*

Example 10.22 *Write the polar coordinates* $(-2, 0)$ *in rectangular coordinates. Then, plot the points on their respective graphs below.*





Other times, we may be given a set of rectangular coordinates and will need to convert them to polar.

How To: Rectangular to Polar coordinates

To convert rectangular coordinates to polar coordinates, let:

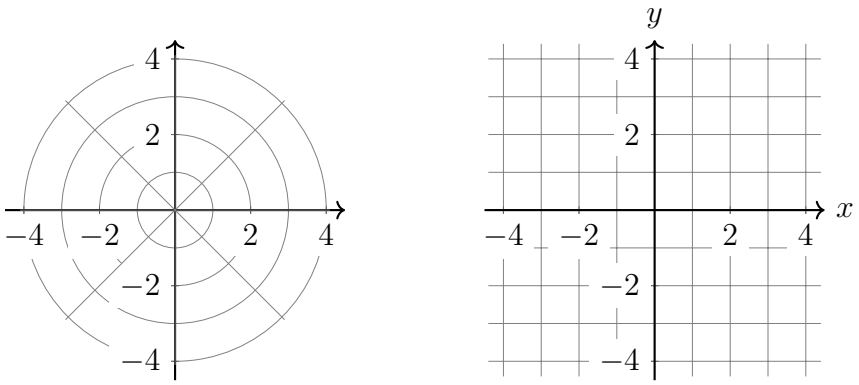
$$\frac{x}{r} = \cos(\theta) \quad \longrightarrow \quad x = r\cos(\theta)\,,$$
$$\frac{y}{r} = \sin(\theta) \quad \longrightarrow \quad y = r\sin(\theta)\,.$$

Then,

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan(\theta) = \frac{y}{x}\,.$$

Example 10.23 *Write the rectangular coordinates* $(3, 3)$ *in polar coordinates. Then, plot the points on their respective graphs below.*

Example 10.24 *Write the rectangular coordinates* $(-3, -4)$ *in polar coordinates. Then, plot the points on their respective graphs below.*



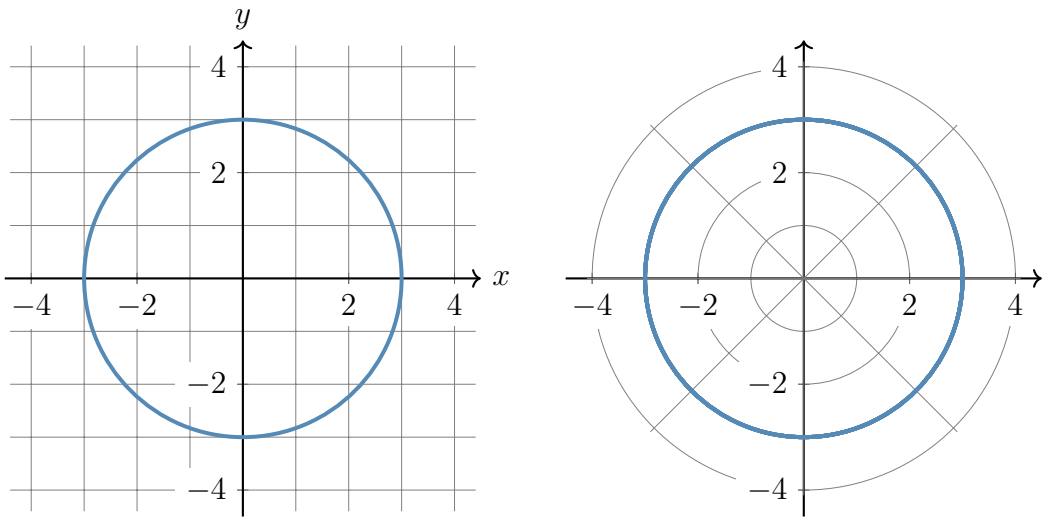
Now that we can convert coordinates between polar and rectangular forms, we can also apply this to equations.

Although converting equations is more complex, it's useful to know how to switch between these forms. Since some polar equations do not easily translate to Cartesian form (and vice versa!) we use the same procedures we used with converting points.

Example 10.25 *The equation for a circle centered at the origin in Cartesian coordinates is $x^2 + y^2 = r^2$. To write this as a function, it needs to be split into*

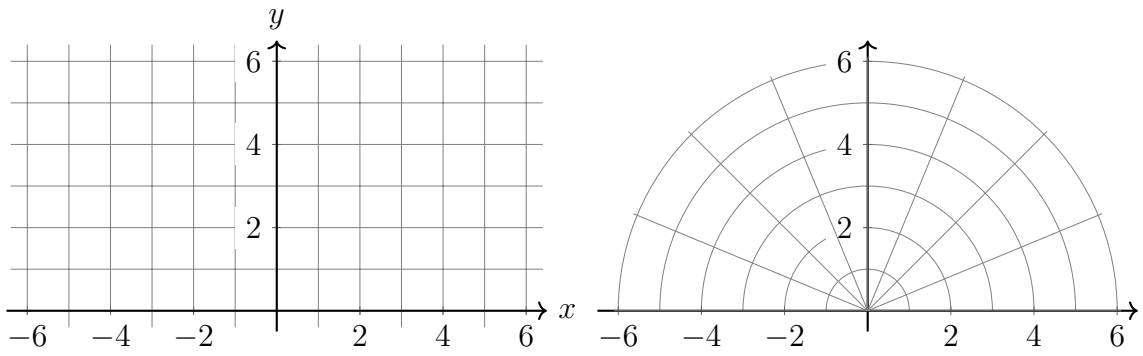
$$y = \sqrt{9 - x^2} \quad \textit{and} \quad y = -\sqrt{9 - x^2}\,.$$

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.



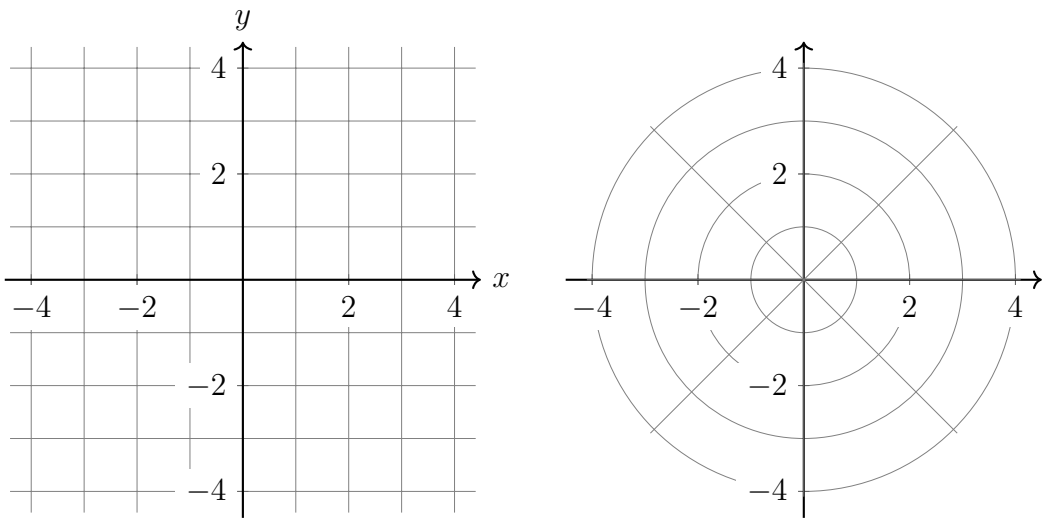
Switching between polar and rectangular forms is important because certain equations or shapes are simpler to express and understand in one form over the other. By choosing the appropriate form, we can make complex calculations easier, visualize graphs more effectively, and solve problems more efficiently, depending on the context.

Example 10.26 *Write the Cartesian equation $x^2 + y^2 = 6y$ in polar form. Then, plot each in their respective graphs below.*



Example 10.27 *Write the Cartesian equation $y = 3x + 2$ in polar form.*

Example 10.28 *Write the polar equation* $r = 2\sec(\theta)$ *in rectangular form. Then, plot each in their respective graphs below.*



Example 10.29 *Write the polar equation* $r = \sin(2\theta)$ *in rectangular form.*

Example 10.30 *Write the polar equation* $r = \dfrac{3}{1 - 2\cos(\theta)}$ *in rectangular form.*

Section Recap

What are some take-away concepts from this section?

10.3 Exercises:

1. Plot the polar points $\left(3, \frac{\pi}{3}\right)$ and $\left(-3, \frac{\pi}{6}\right)$ on a polar grid. Then convert each point to rectangular coordinates. Show all work.

2. Convert the rectangular equation

$$x^2 + y^2 = 8y$$

to polar form. Then, describe the graph and identify its center and radius.

3. Convert the polar equation

$$r = 4\cos(\theta)$$

to rectangular form. Use the rectangular equation to identify and sketch the graph.

10.3 Scaffolding Questions:

1. Plot the polar points $\left(3, \frac{\pi}{3}\right)$ and $\left(-3, \frac{\pi}{6}\right)$ on a polar grid. Then convert each point to rectangular coordinates. Show all work.

   - Question Comprehension:
     - What do the values r and θ represent in a polar coordinate?
     - How does a negative value of r affect the location of a point?
     - What is the goal of this problem?
   - Content Knowledge:
     - What formulas convert polar coordinates to rectangular coordinates?
     - How can you determine where a polar point should be plotted?
     - How do special-angle values help evaluate sine and cosine?
   - Solution:
     - Did you correctly plot $\left(3, \frac{\pi}{3}\right)$?
     - Did you correctly account for the negative radius in $\left(-3, \frac{\pi}{6}\right)$?
     - Did you use
       $$x = r\cos(\theta), \qquad y = r\sin(\theta)$$
       for each point?
     - Are your final rectangular coordinates simplified exactly?
   - Reflect:
     - Why can different polar coordinates represent the same point?
     - Which coordinate system makes the location of these points easier to describe?

2. Convert the rectangular equation

$$x^2 + y^2 = 8y$$

to polar form. Then, describe the graph and identify its center and radius.

- Question Comprehension:
  - What substitutions connect rectangular coordinates and polar coordinates?
  - What information are you asked to determine after converting the equation?
- Content Knowledge:
  - What polar expression can replace $x^2 + y^2$?
  - What polar expression can replace y?
  - How can you rewrite the resulting equation in polar form?
  - How can completing the square help identify the circle in rectangular form?
- Solution:
  - Did you substitute
$$x^2 + y^2 = r^2$$
and
$$y = r\sin(\theta)?$$
  - Did you simplify the equation correctly?
  - Did you identify the center and radius of the circle?
  - Does your polar equation describe the same graph as the original rectangular equation?
- Reflect:
  - Why is the polar form of this circle often simpler than the rectangular form?
  - How could you verify that both equations represent the same graph?

3. Convert the polar equation
$$r = 4\cos(\theta)$$
to rectangular form. Use the rectangular equation to identify and sketch the graph.

   - Question Comprehension:
     - What information is given in polar form?
     - What are you being asked to determine about the graph?
   - Content Knowledge:
     - How are $\cos(\theta)$ and x related?
     - What substitution can be made using
$$x = r\cos(\theta)?$$
     - How can you eliminate r from the equation?
     - What algebraic techniques help identify the resulting graph?
   - Solution:
     - Did you multiply both sides by r when needed?
     - Did you replace $r\cos(\theta)$ with x?
     - Did you use
$$r^2 = x^2 + y^2$$
to obtain a rectangular equation?
     - Did you identify the graph correctly and sketch its key features?
   - Reflect:
     - Why might this graph be easier to recognize in rectangular form?
     - What advantages does polar form provide before conversion?

10.4 Polar Coordinates: Graphs

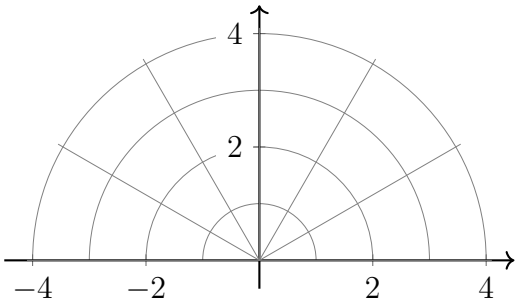
Learning Goals 10.4

After this section, I should be able to:

1. Graph polar equations by plotting points.

Example 10.31 *Plot the equation* $r = 4\sin(\theta)$ *by: filling out the rows on the table below, and then plotting the points.*

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\sin(\theta)$							
$4\sin(\theta)$							



Why did we only use half of a rotation and not the *full* rotation? Use the table below to determine what those points would contribute to the graph!

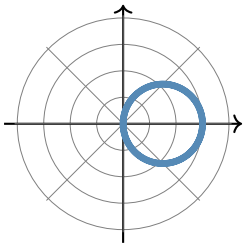
θ	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin(\theta)$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$4\sin(\theta)$	0	-2	$-2\sqrt{3}$	-4	$-2\sqrt{3}$	-2	0

Circles are some of the more simple polar curves, both in terms of their graphs and their representations. Some of the formulas that produce the graph of a circle in polar coordinates are given by

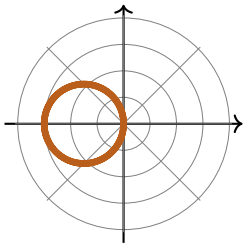
$$r = a\cos(\theta) \quad \text{and} \quad r = a\sin(\theta)\,.$$

With these circles:

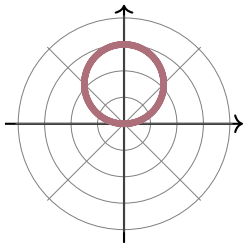
- the curves go through the origin (but are not centered there),
- their diameter is $|a|$,
- their radius is $\dfrac{|a|}{2}$,
- the center is along the x-axis for cosine curves and the y-axis for sine curves.



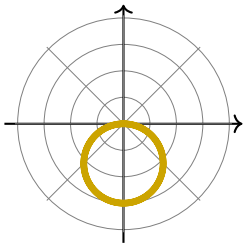
$r = a\cos(\theta)$
$a > 0$



$r = a\cos(\theta)$
$a < 0$

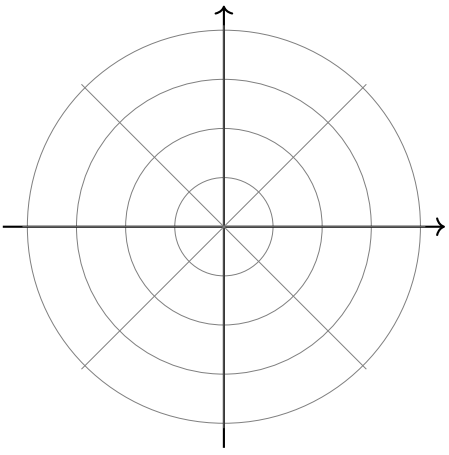


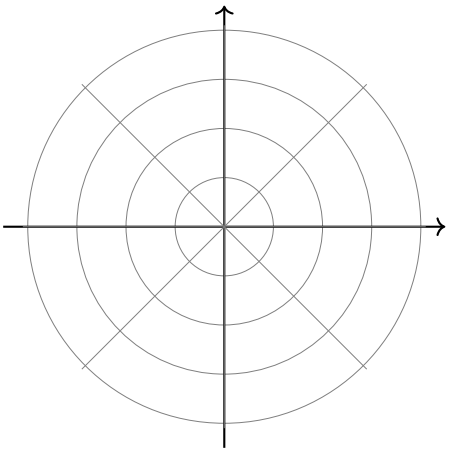
$r = a\sin(\theta)$
$a > 0$



$r = a\sin(\theta)$
$a < 0$

Example 10.32 *Sketch the circles* $r = -2\sin(\theta)$ *and* $r = 2\cos(\theta)$ *on the graphs below.*





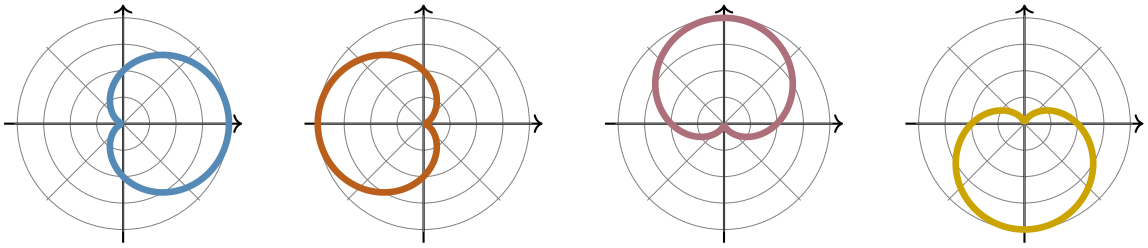
A **cardioid** is a heart-shaped curve in mathematics, known for its distinctive, rounded shape resembling an apple or heart. It emerged as a concept in the 17th century, introduced by mathematicians such as Étienne Pascal and Jacob Bernoulli, who studied its properties while exploring reflected light paths. The cardioid shape is generated by tracing a point on a circle as it rolls around another circle of the same size.

Over time, it became relevant in fields like physics and engineering, especially in acoustics and radio design. For example, cardioid-shaped microphones capture sound in a specific directional pattern, reducing unwanted noise. Similarly, cardioid antenna patterns in radio transmitters help direct signals in desired directions, improving efficiency and reception.

Cardioids are found with

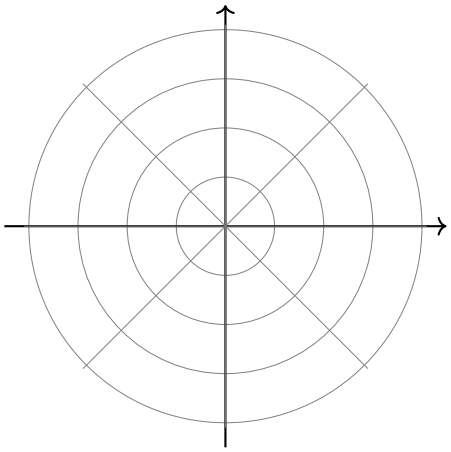
$$r = a \pm b\cos(\theta)\,, \quad \text{and} \quad r = a \pm b\sin(\theta)$$

where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. Cardioids pass through the origin.



$r = a + b\cos(\theta)$ $r = a - b\cos(\theta)$ $r = a + b\sin(\theta)$ $r = a - b\sin(\theta)$

Example 10.33 *Sketch the cardioid* $r = 2 + 2\cos(\theta)$.

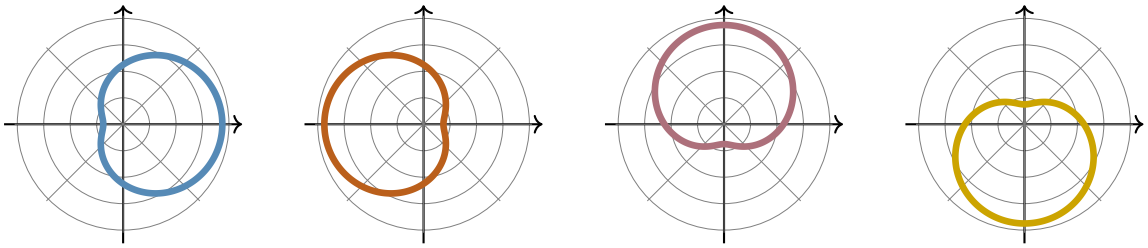


The **limaçon**, named after the (old) French word for 'snail,' is a family of mathematical curves that emerged in the 17th century, studied by Étienne Pascal and Gilles de Roberval. Like cardioids, limaçons have applications in physics and engineering, particularly in optics, where their reflective properties are used in certain mirror designs. They also find use in antenna design, where limaçon shapes help direct electromagnetic waves efficiently, improving signal focus and range.

One-loop limaçons, or **dimpled limaçons**, are found with

$$r = a \pm b\cos(\theta)\,, \quad \text{and} \quad r = a \pm b\sin(\theta)$$

where $a > 0$, $b > 0$, and $1 < \frac{a}{b} < 2$.

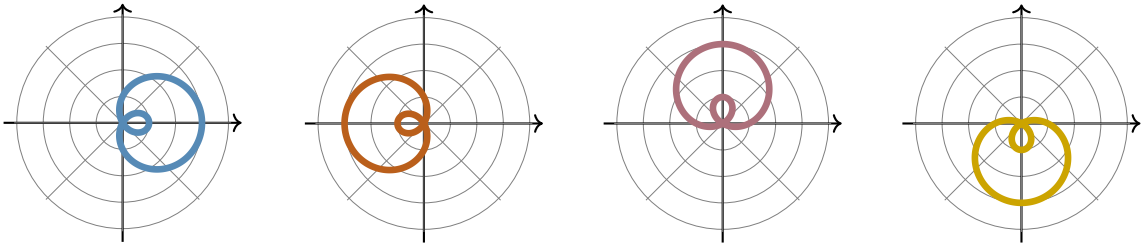


$r = a + b\cos(\theta)$ $r = a - b\cos(\theta)$ $r = a + b\sin(\theta)$ $r = a - b\sin(\theta)$

Inner-loop limaçons are found with

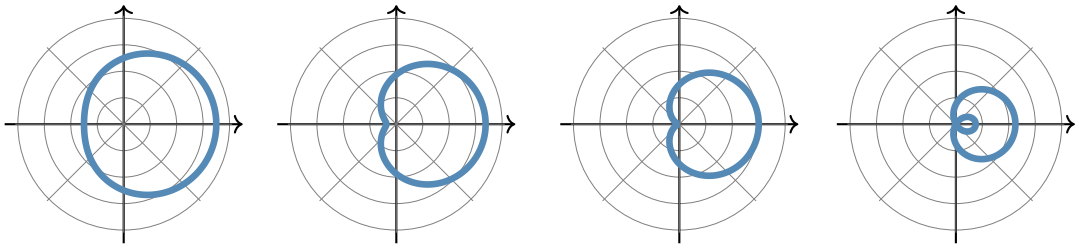
$$r = a \pm b\cos(\theta)\,, \quad \text{and} \quad r = a \pm b\sin(\theta)$$

where $a > 0$, $b > 0$, and $0 < \frac{a}{b} < 1$.



$r^2 = a + b\cos(\theta)$ $r = a - b\cos(\theta)$ $r = a + b\sin(\theta)$ $r = a - b\sin(\theta)$

Here, we see that cardioids are a special case of limaçons:



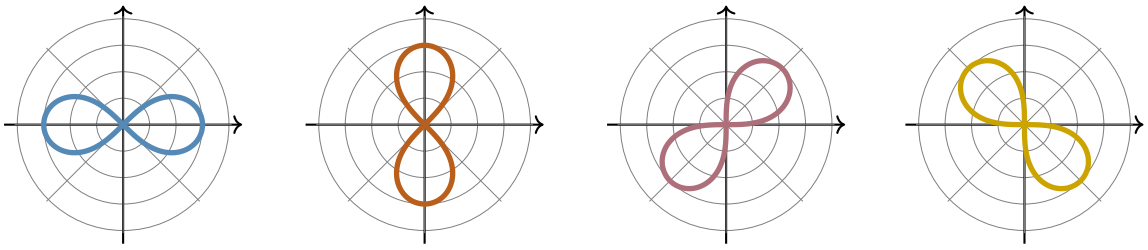
$r = 2.5 + \cos(\theta)$ $r = 1.25 + \cos(\theta)$ $r = 1 + \cos(\theta)$ $r = 0.5 + \cos(\theta)$

Lemniscates are figure-eight-shaped curves with fascinating mathematical properties, first studied in detail by Jacob Bernoulli in the late 17th century. The term 'lemniscate' comes from the Latin word for 'ribbon,' referencing their looped shapes.

Lemniscates are found by

$$r^2 = a^2\cos(2\theta) \quad \text{and} \quad r^2 = a^2\sin(2\theta)$$

where $a \neq 0$.



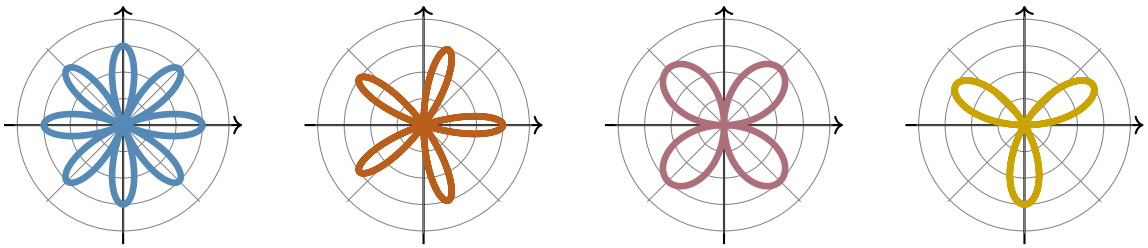
$r = a^2\cos(2\theta)$ $\quad$ $r^2 = -a^2\cos(2\theta)$ $\quad$ $r^2 = a^2\sin(2\theta)$ $\quad$ $r = -a^2\sin(2\theta)$

Rose curves have been studied for their aesthetic appeal and mathematical properties. Historically, they appeared in the work of 18th century mathematicians exploring polar equations and periodicity. These curves find applications in art, design, and signal analysis, where their periodic and symmetric properties are used to model oscillatory systems or generate patterns. In engineering, rose curve principles inspire designs for certain mechanical gears and waveforms. Their visual symmetry also makes them a favorite in graphical representations and mathematical art.

Rose curves are given by

$$r = a\cos(n\theta) \quad \text{and} \quad r = a\sin(n\theta)$$

where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curves has n petals.



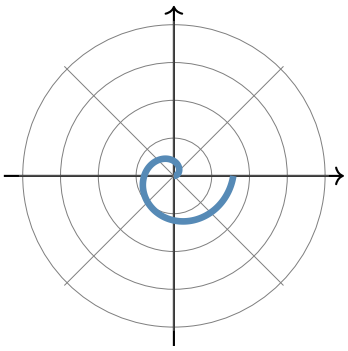
$r = a\cos(n\theta)$, n even $\quad$ $r = a\cos(n\theta)$, n odd $\quad$ $r = a\sin(n\theta)$, n even $\quad$ $r = a\sin(n\theta)$, n odd

Example 10.34 *Determine the polar equations used for the rose curve graphs above.*

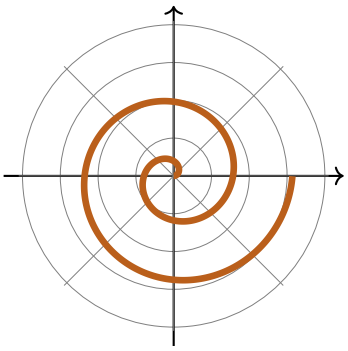
Archimedes' spiral describes a curve that starts at the origin and spirals outward, with the distance between successive turns remaining constant. First studied by the ancient Greek mathematician Archimedes, this spiral has numerous applications. In engineering, it is used in the design of scroll compressors and spiral antennas, where its uniform spacing helps optimize performance. In physics, it models phenomena like the paths of charged particles in uniform magnetic fields. Additionally, it appears in computer graphics and data visualization to represent uniform growth or to arrange data points in a visually appealing way.

Archimedes' spiral is given by

$$r = \theta \quad \text{for } \theta \geq 0\,.$$



$r = \theta$
$\theta \in [0, 2\pi]$



$r = \theta$
$\theta \in [0, 4\pi]$

Section Recap

What are some take-away concepts from this section?

10.4 Exercises:

1. Sketch the graph of the polar equation $r = 3 + 2\cos(\theta)$. Identify and label key features of the graph, such as intercepts and symmetry. Provide a step-by-step explanation of the graphing process.

2. Graph the polar equation $r = 4 - 3\sin(\theta)$. Label the key features of the graph, including intercepts and symmetries. Show all steps involved in sketching the graph.

3. Sketch the graph of the polar equation $r = 1 + 2\sin(\theta)$. Identify key features of the graph, such as the maximum radius, intercepts, and symmetry. Provide a detailed explanation of each step in the graphing process.

10.4 Scaffolding Questions:

1. Sketch the graph of the polar equation $r = 3 + 2\cos(\theta)$. Identify and label key features of the graph, such as intercepts and symmetry. Provide a step-by-step explanation of the graphing process.

    - Question Comprehension:
        - What is the general form of a polar equation, and how does the equation $r = 3 + 2\cos(\theta)$ suggest specific features for the graph?
        - What key features should we look for when graphing a polar equation, and how do we identify intercepts and symmetries?
    - Content Knowledge:
        - How do you determine the intercepts of the polar graph, and what role do the values of r and θ play in these intercepts?
        - How do you use symmetry to simplify the graphing process for polar equations?
    - Solution:
        - Did you first consider the range of r as θ varies, particularly the maximum and minimum values of r?
        - Did you calculate key points for specific values of θ, such as $0°, 90°, 180°, 270°$?
        - Did you check for symmetry (polar symmetry or symmetry about the horizontal axis) based on the equation's form, particularly the presence of cosine or sine functions?
    - Reflect:
        - How would the graph change if we modified the equation by changing the coefficients of r or $\cos(\theta)$?
        - What is the significance of the "offset" value in the equation, such as the 3 in this case?

2. Graph the polar equation $r = 4 - 3\sin(\theta)$. Label the key features of the graph, including intercepts and symmetries. Show all steps involved in sketching the graph.

   - Question Comprehension:
     - How does the presence of the sine function affect the graph of the polar equation, and what can we infer from the equation about its features?
     - What are the intercepts of a polar graph, and how do we find them for an equation like $r = 4 - 3\sin(\theta)$?
   - Content Knowledge:
     - How do we determine the maximum and minimum values of r by considering the equation's form and using specific angles like $0°, 90°, 180°$?
     - How do we identify symmetries in the graph based on the sine function's properties and the equation's structure?
   - Solution:
     - Did you plot key points for values of θ, such as $0°, 90°, 180°, 270°$, and check their corresponding r values?
     - Did you identify and label the intercepts (where $r = 0$) and verify the symmetry of the graph?
     - Did you account for any shifts or transformations caused by the $-3\sin(\theta)$ term, such as a shift in the graph's center or changes in symmetry?
   - Reflect:
     - How would the graph change if the negative coefficient of the sine term were positive instead?
     - What effect would changing the constant 4 in the equation have on the graph?

3. Sketch the graph of the polar equation $r = 1 + 2\sin(\theta)$. Identify key features of the graph, such as the maximum radius, intercepts, and symmetry. Provide a detailed explanation of each step in the graphing process.

   - Question Comprehension:
     - How does the structure of the equation $r = 1 + 2\sin(\theta)$ suggest a shifted graph, and how does the sine function impact its shape?
     - What are the key features to identify when graphing this equation, and how do we determine their positions?
   - Content Knowledge:
     - How do we find the maximum and minimum values of r by considering the sine function's behavior?
     - How do we calculate the intercepts and other key points for specific values of θ, such as $0°, 90°, 180°, 270°$?
   - Solution:
     - Did you first calculate the maximum and minimum values of r by evaluating the equation for specific θ values?
     - Did you correctly identify the symmetry of the graph, which is likely to be symmetric about the vertical axis?
     - Did you sketch the graph, labeling key points and the maximum radius, and indicating the symmetry and intercepts?
   - Reflect:
     - How would the graph differ if the coefficients of r or $\sin(\theta)$ were changed?
     - How do the different forms of polar equations (with cosine or sine terms) affect the graph's symmetry and features?

10.6 Parametric Equations

Learning Goals 10.6

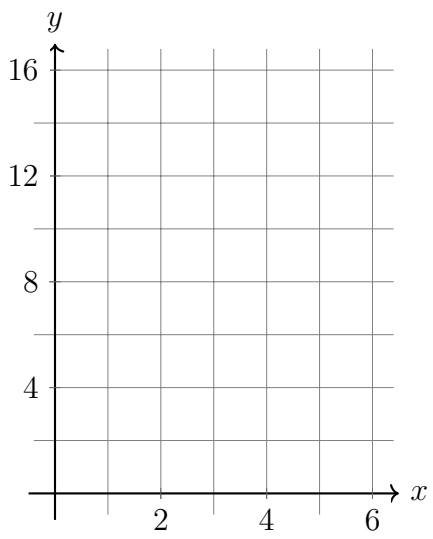
After this section, I should be able to:

1. Parameterize a curve.
2. Eliminate the parameter.
3. Find a rectangular equation for a curve defined parametrically.
4. Find parametric equation equations for curves defined by rectangular equations.

A set of equations that are defined using a single independent variable are called **parametric equations**. Typically, we use t, called the **parameter**, as the independent variable to define the functions $x(t)$ and $y(t)$. These equations define a **parametric curve**, C, in the plane such that the points on C are the set of points given by $(x(t), y(t))$.

Example 10.35 *Sketch and identify the curve defined by the parametric equations:*

$$x = t + 1\,, \quad y = t^2 - 2t\,. \qquad (*)$$



The equation (†) in terms of x and y describes where the particle has been, but it doesn't tell us when the particle was at a particular point. The parametric equations $(*)$ have an advantage– they tell us when the particle was at a point and they also indicate the direction of the motion.

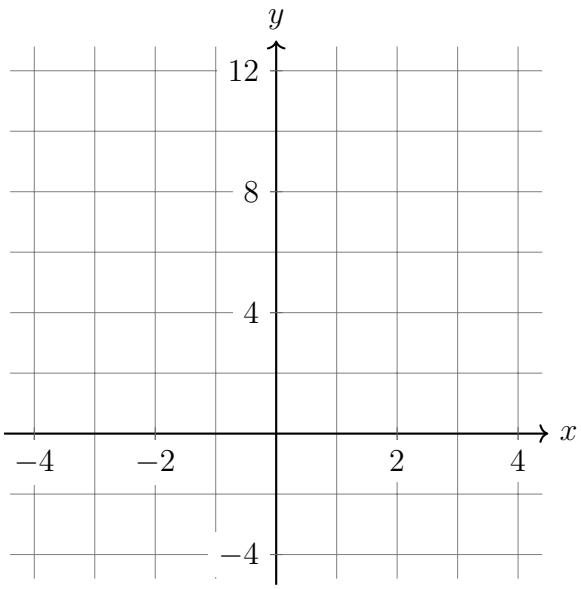
Example 10.36 *Define the line given by* $y = x^2 - 4$ *using parametric equations with* t *as the parameter with* two different *parameterizations. Sketch the curve for* $0 \leq t \leq 4$.

Option 1:

t	x	y

Option 2:

t	x	y

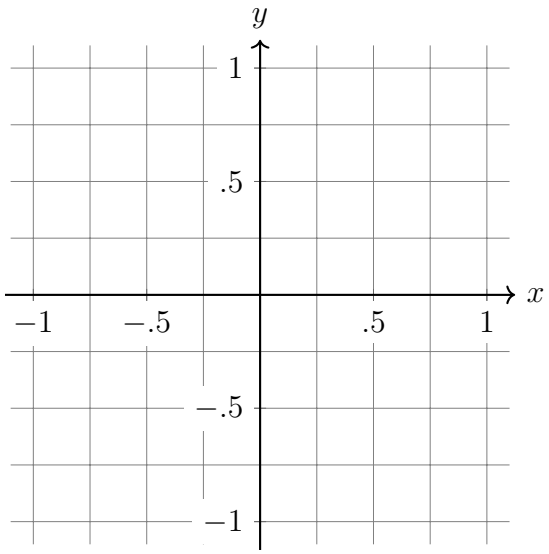


Example 10.37 *Determine the curve represented by the following parametric equations:*

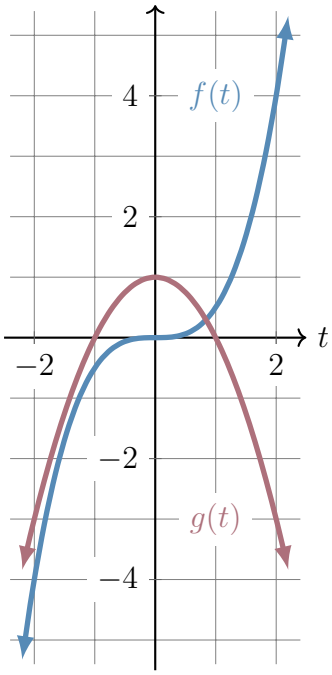
$$x = \cos(t)\,, \quad y = \sin(t)\,, \qquad 0 \le t \le 2\pi\,.$$

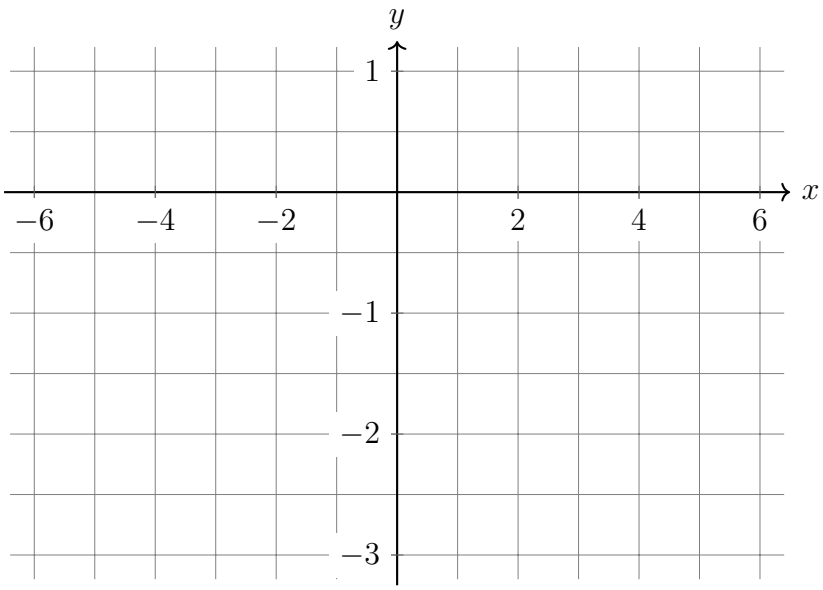
Then, eliminate the parameter to get an equation in terms of x and y.

t	$x = \cos(t)$	$y = \sin(t)$
0		
$\pi/4$		
$\pi/2$		
$3\pi/4$		
π		
$5\pi/4$		
$3\pi/2$		
$7\pi/4$		
2π		



Example 10.38 *Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $(x, y) = (f(t), g(t))$. Use an arrow to indicate the direction of the curve as t increases.*

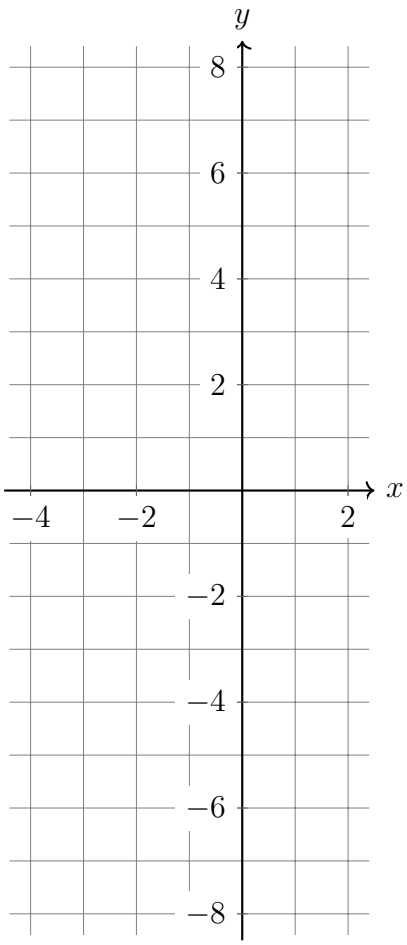




Example 10.39 *Determine the curve represented by the following parametric equations:*

$$x = 1 - t^2\,, \quad y = t^3 + 2\,, \qquad -2 \le t \le 2\,.$$

Then, eliminate the parameter to get an equation in terms of x and y.



Section Recap

What are some take-away concepts from this section?

10.6 Exercises:

1. Given the parametric equations $x = 3t + 2$ and $y = 2t^2 - 1$, eliminate the parameter t to find the rectangular equation. Then, sketch the graph of the equation. Show all steps involved in the solution.

2. Consider the parametric equations $x = \cos(t)$ and $y = \sin(t)$. Eliminate the parameter t to obtain the rectangular equation. Then, sketch the graph of the equation. Provide all necessary steps and explanations.

3. For the parametric equations $x = 2t-3$ and $y = 4-t^2$, eliminate the parameter t to express y as a function of x. Then, sketch the graph of the resulting equation. Show all work involved in the process.

10.6 Scaffolding Questions:

1. Given the parametric equations $x = 3t + 2$ and $y = 2t^2 - 1$, eliminate the parameter t to find the rectangular equation. Then, sketch the graph of the equation. Show all steps involved in the solution.

    - Question Comprehension:
        - What is the goal of eliminating the parameter t, and why do we need to find the rectangular equation?
        - How can we express the relationship between x and y without the parameter t?
    - Content Knowledge:
        - How can we isolate t from the equation $x = 3t + 2$ to substitute into the second equation for y?
        - After substituting, what form should the equation for y take in terms of x?
    - Solution:
        - Did you solve for t from $x = 3t + 2$ and substitute into the equation for $y = 2t^2 - 1$?
        - Did you simplify the resulting equation to express y in terms of x?
        - Did you sketch the graph based on the rectangular equation, considering the behavior of the equation?
    - Reflect:
        - How does the graph of the equation behave based on the form of the equation you derived?
        - If the parametric equations were altered, how would the graph of the resulting equation change?

2. Consider the parametric equations $x = \cos(t)$ and $y = \sin(t)$. Eliminate the parameter t to obtain the rectangular equation. Then, sketch the graph of the equation. Provide all necessary steps and explanations.

   - Question Comprehension:
     - What is the relationship between $x = \cos(t)$ and $y = \sin(t)$, and how can we eliminate t?
     - What geometric shape should we expect for the graph of this equation?
   - Content Knowledge:
     - How do we use the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$ to eliminate t and express a relationship between x and y?
   - Solution:
     - Did you substitute $x = \cos(t)$ and $y = \sin(t)$ into the identity $\cos^2(t) + \sin^2(t) = 1$?
     - Did you simplify to obtain the rectangular equation $x^2 + y^2 = 1$?
     - Did you sketch the graph of the equation, recognizing that it represents a circle with radius 1 centered at the origin?
   - Reflect:
     - How would the graph change if the parametric equations involved different functions of t?
     - What other types of graphs can be obtained by eliminating parameters in different parametric equations?

3. For the parametric equations $x = 2t-3$ and $y = 4-t^2$, eliminate the parameter t to express y as a function of x. Then, sketch the graph of the resulting equation. Show all work involved in the process.

   - Question Comprehension:
     - What is the goal of eliminating t, and how can we express y as a function of x?
     - What steps should we take to solve for t and then substitute into the equation for y?
   - Content Knowledge:
     - How do we isolate t from $x = 2t - 3$ to substitute into $y = 4 - t^2$?
     - What form should the equation for y take after substitution, and how do we simplify it?
   - Solution:
     - Did you solve for t from $x = 2t - 3$ and substitute into $y = 4 - t^2$?
     - Did you simplify the resulting equation to express y in terms of x?
     - Did you sketch the graph of the equation based on the relationship between x and y?
   - Reflect:
     - How would the graph change if the equation for y were different, such as if the coefficient of t^2 were changed?
     - What insights about the graph can we gain from the form of the rectangular equation you derived?

10.7 Parametric Equations: Graphs

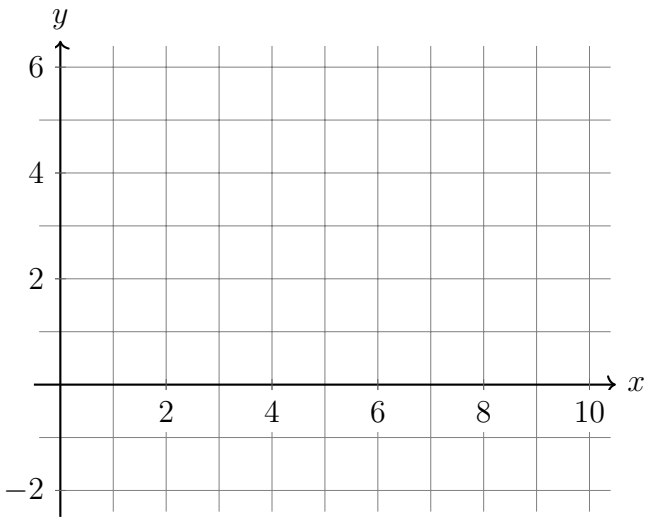
Learning Goals 10.7

After this section, I should be able to:

1. Graph plane curves described by parametric equations by plotting points.
2. Graph parametric equations.

Example 10.40 *Sketch the parametric graph* $(x(t), y(t)) = (t^2 + 1, 2 + t)$.

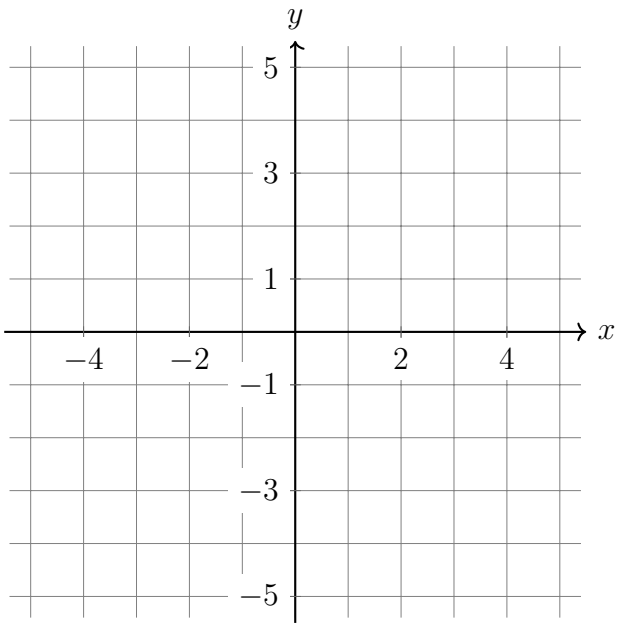
t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5



Example 10.41 *Eliminate the t parameter from the parametric graph above to find y as a function of x.*

Example 10.42 *Sketch the parametric graph* $(x(t), y(t)) = (5\cos(t), 2\sin(t))$.

t	$x(t)$	$y(t)$
-5	1.4	1.9
-4	-3.3	1.5
-3	-4.95	-0.28
-2	-2.1	-1.8
-1	2.7	-1.7
0	5	0
1	2.7	1.7
2	-2.1	1.8
3	-4.95	0.28
4	-3.3	-1.5
5	1.4	-1.5



Example 10.43 *Eliminate the* t *parameter from the parametric graph above to find an expression for the graph in terms of* x *and* y.

It's the bottom of the ninth inning with two outs and two runners on base. The home team is down by two runs. The batter swings and hits the ball at 140 feet per second, launching it at a 45° angle from 4 feet off of the ground. How far will the ball travel? Will it clear the 25 foot outfield fence for a game-winning home run?

Projectile motion (like we saw earlier in the course!) occurs when an object is launched at an angle θ to the horizontal with an initial velocity v_0 and a starting height h.

The motion of the object can be described by the following parametric equations:

$$x(t) = (v_0 \cos(\theta)\, t\,,$$
$$y(t) = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t + h$$

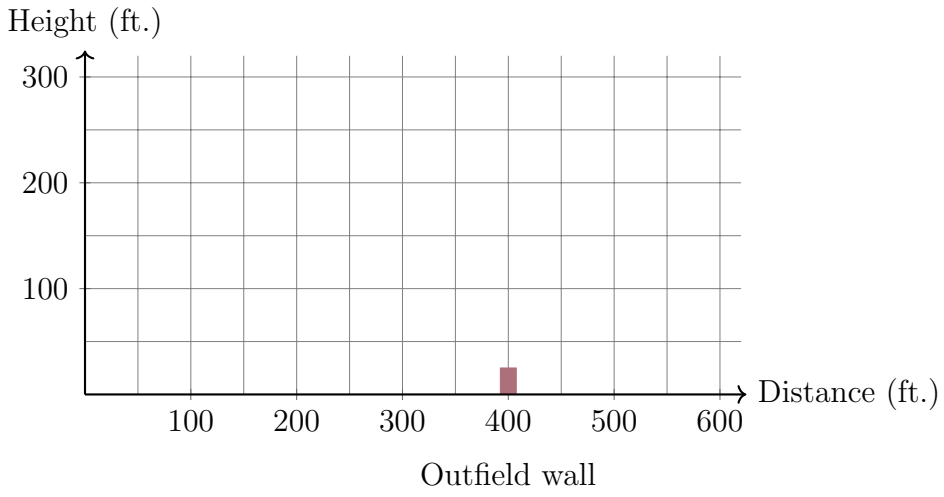
Here, g represents the acceleration due to gravity, and its value depends on the units used: $g = 32\,\text{ft/s}^2$ or $g = 9.8\,\text{m/s}^2$. The equation for x gives the horizontal distance, while the equation for y gives the vertical distance.

Example 10.44 *Find the parametric equations to model the path of the baseball.*

Example 10.45 *Where is the ball after 2 seconds?*

Example 10.46 *How long is the ball in the air?*

Example 10.47 *Is the hit a home run?*



Section Recap

What are some take-away concepts from this section?

10.7 Exercises:

1. Given the parametric equations $x = t^2 - 1$ and $y = 2t + 3$, sketch the graph of the parametric curve. Show how you eliminate the parameter and describe the graph's key features.

2. For the parametric equations $x = 2t + 1$ and $y = 3t^2 - 2$, sketch the graph of the parametric curve. Eliminate the parameter and describe the resulting graph in detail. Show all necessary steps in the process.

3. Sketch the graph of the parametric equations $x = \cos(t)$ and $y = \sin(t)$ for t in the range $0 \leq t \leq 2\pi$. Eliminate the parameter and describe the graph. Show all steps involved in finding the rectangular equation and graphing it.

10.7 Scaffolding Questions:

1. Given the parametric equations $x = t^2 - 1$ and $y = 2t + 3$, sketch the graph of the parametric curve. Show how you eliminate the parameter and describe the graph's key features.

   - Question Comprehension:
     - What is the goal of eliminating the parameter t, and why is it necessary to convert to a rectangular equation?
     - How can we express the relationship between x and y without t?
   - Content Knowledge:
     - How can we isolate t from the equation $y = 2t + 3$ and substitute it into $x = t^2 - 1$?
     - Once t is eliminated, what form should the equation for y in terms of x take?
   - Solution:
     - Did you solve for t in $y = 2t + 3$ and substitute into $x = t^2 - 1$?
     - After substituting, did you simplify the equation to express y in terms of x?
     - Did you sketch the graph based on the derived rectangular equation, identifying key features like intercepts or shape?
   - Reflect:
     - How does the equation relate to the graph's shape (such as parabolic or linear)?
     - What would the graph look like if we modified the parametric equations, for instance, changing the coefficients or constants?

2. For the parametric equations $x = 2t+1$ and $y = 3t^2-2$, sketch the graph of the parametric curve. Eliminate the parameter and describe the resulting graph in detail. Show all necessary steps in the process.

   - Question Comprehension:
     - What does eliminating the parameter t achieve in this context, and how does it help in describing the graph?
     - What are the key steps involved in finding the rectangular equation?
   - Content Knowledge:
     - How can we solve for t in $x = 2t+1$ and substitute it into $y = 3t^2-2$?
     - Once the parameter is eliminated, what form should the resulting rectangular equation take?
   - Solution:
     - Did you solve for t from $x = 2t+1$ and substitute it into $y = 3t^2-2$?
     - Did you simplify the resulting equation and express y in terms of x?
     - Did you sketch the graph based on the simplified equation, identifying key features such as shape and intercepts?
   - Reflect:
     - What is the overall shape of the graph based on the derived equation (e.g., parabola, line)?
     - How would the graph change if the parametric equations were modified, such as changing the powers of t or adding constants?

3. Sketch the graph of the parametric equations $x = \cos(t)$ and $y = \sin(t)$ for t in the range $0 \leq t \leq 2\pi$. Eliminate the parameter and describe the graph. Show all steps involved in finding the rectangular equation and graphing it.

   - Question Comprehension:
     - What is the goal of eliminating the parameter t, and how does it help in finding the graph of the curve?
     - What should we expect the graph to look like based on the parametric equations?
   - Content Knowledge:
     - How can we use the Pythagorean identity $\cos^2(t) + \sin^2(t) = 1$ to eliminate t and find a relationship between x and y?
   - Solution:
     - Did you use the identity $\cos^2(t) + \sin^2(t) = 1$ to eliminate t and obtain the equation $x^2 + y^2 = 1$?
     - Did you recognize that this equation represents a circle with radius 1 centered at the origin?
     - Did you sketch the graph based on this equation, showing that it is a circle with key features like the center and radius?
   - Reflect:
     - How would the graph change if the parametric equations were altered, such as with different functions of t or modified constants?
     - What insights about the graph can be drawn from the form of the equation you derived?