5.1: Equilibrium Point Analysis

From our work in Chapter 3, we are able to understand the solutions of linear systems both qualitatively and analytically. Unfortunately, nonlinear systems are in general much less amenable to the analytic and algebraic techniques that we have developed, but we can use the mathematics of linear systems to understand the behavior of solutions of nonlinear systems near their equilibrium points.

Consider the Van der Pol equation:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + (1 - x^2)y \end{cases}$$

This is nonlinear, so we don't have a lot of tools to analyze it right now. But we can use *linear tools* to analyze what's happening near equilibrium solutions.

To find equilibrium solutions, we're interested in where $x' = 0$ and $y' = 0$ at the same time. These curves are called **nullclines.** Nullclines divide the phase space and give us information about where each component of the direction field is zero. Equilibrium solutions are found where a nullcline from one component a nullcline from another component.

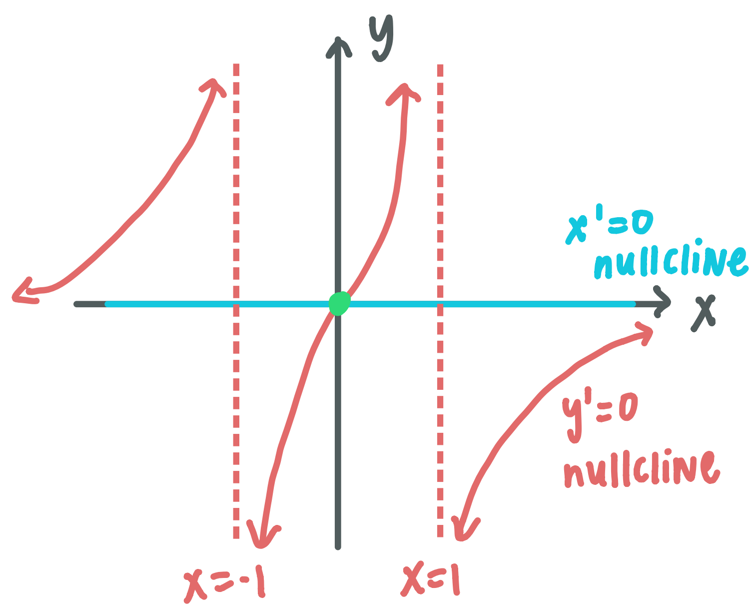
Example 0.1. *Determine the equilibrium solution(s) to the Van der Pol equation above using nullclines.*

$x' = 0$ if $y = 0$

$y' = 0$ if $-x + (1 - x^2)y = 0$

$\Updownarrow$

$y = \dfrac{x}{1 - x^2}$



intersection @ (0,0)

$\Rightarrow$ equil. sol'n @ (0,0).

We can understand why solutions spiral away from the origin by approximating the Van der Pol system with another system that is much easier for us to analyze– a linear system.

So, we can **linearize** the system around the equilibrium point $(0, 0)$. This gives us the linear approximation of the system close to the origin:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + y - x^2 \cdot y \end{cases} \Longrightarrow \text{linear approx. to system around } (0,0): \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + y \end{cases}$$

What does this tell us about the equilibrium point at the origin? How can we classify it?

how can we classify equil. soln @ origin

w/ $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

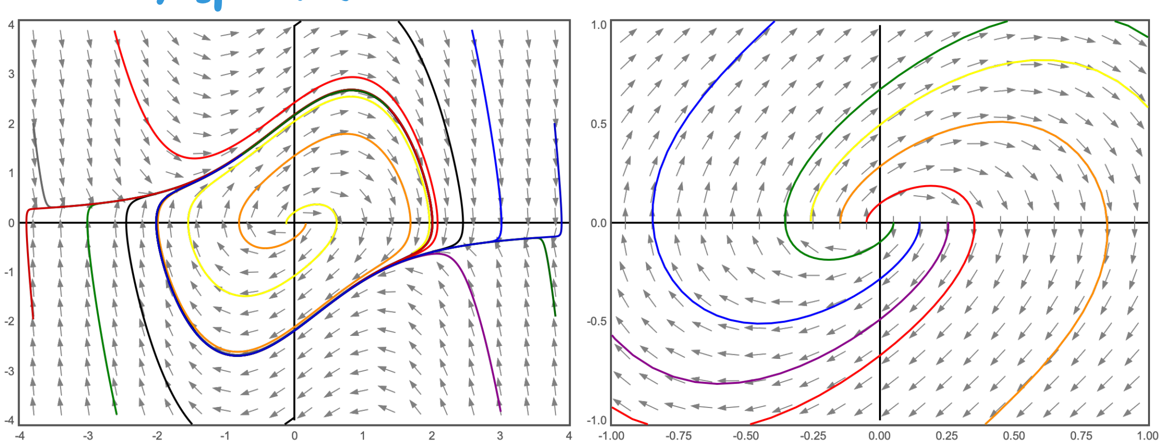
$T = 1$, $D = 1$

$\Rightarrow$ spiral source

$\lambda = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$

complex λ w/ $\mathrm{Re}(\lambda) > 0$

$\Rightarrow$ spiral source



Example 0.2. *Consider x and y as two populations competing for some resource, with the system below governing their behavior:*

$$\begin{cases} \dfrac{dx}{dt} = 2x\left(1 - \dfrac{x}{2}\right) - xy\,, \\ \dfrac{dy}{dt} = 3y\left(1 - \dfrac{y}{3}\right) - 2xy\,. \end{cases}$$

For a fixed x value, if y increases then $\frac{dx}{dt} < 0$ due to the $-xy$ term. Similarly, for a fixed y value, if x increases then $\frac{dy}{dt} < 0$ due to the $-2xy$ term. So, an increase in either population has an adverse effect on the growth rate of the other species.

What are the nullclines and the equilibrium solutions?

$x' = 0$ if $2x\left(1 - \frac{x}{2}\right) - xy = 0$

$x \cdot \left[\, 2 - x - y \,\right] = 0$

$y' = 0$ if $3y\left(1 - \frac{y}{3}\right) - 2xy = 0$

$y \cdot \left[\, 3 - y - 2x \,\right] = 0$

$x' = 0$ @

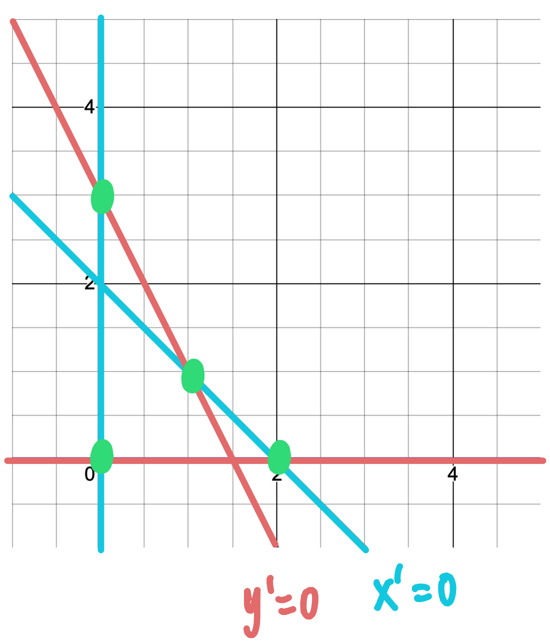
$x = 0$ &

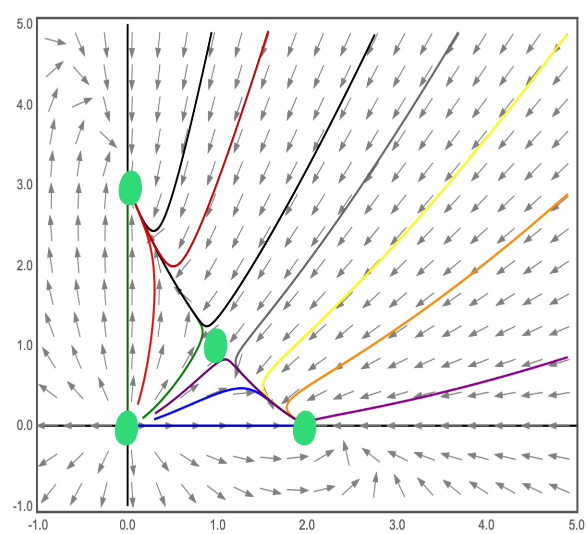
$y = 2 - x$

$y' = 0$ @

$y = 0$ &

$y = 3 - 2x$





The equilibrium point $(1,1)$ is of particular interest to us; its existence indicates that it is possible for these two species to coexist in equilibrium! In the phase portrait, we saw most solutions tended towards $(2,0)$, $(0,3)$, and $(1,1)$, since they all had trajectories pointing toward them.

Two important questions remain though...

First, what solutions tend to the equilibrium point $(1,1)$? In particular, is the set of these solutions large enough that we could hope to find an example of such a solution in nature?

Second, what separates the initial conditions that yield solutions for which x tends to zero from those solutions for which y tends to zero? To answer these questions, we study the system near the equilibrium point $(1,1)$ using linearization.

We've only worked with linear systems and classifying equilibrium points at the origin, so we need to define a *new* system that shifts the equilibrium point we're interested in to the origin! This change of variables would be:

$$\begin{cases} u = x-1 \\ v = y-1 \end{cases} \longleftrightarrow \begin{cases} x = u+1 \\ y = v+1 \end{cases} \qquad (x,y) = (1,1) \downarrow (u,v) = (0,0)$$

With that change, what would $\frac{dx}{dt}$ and $\frac{dy}{dt}$ become in terms of u and v?

$$\frac{dx}{dt} = \frac{d}{dt}(u+1) = \frac{du}{dt} \qquad \frac{dy}{dt} = \frac{d}{dt}(v+1) = \frac{dv}{dt}$$

$$\frac{dx}{dt} = 2x\left(1 - \frac{x}{2}\right) - xy = 2\cdot(u+1)\cdot\left[1 - \frac{u+1}{2}\right] - (u+1)(v+1)$$

$$\Rightarrow \frac{du}{dt} = -u - v - u^2 - uv$$

$$\frac{dy}{dt} = 3y\left(1 - \frac{y}{3}\right) - 2xy = 3\cdot(v+1)\cdot\left[1 - \frac{v+1}{3}\right] - 2\cdot(u+1)(v+1)$$

$$\Rightarrow \frac{dv}{dt} = -2u - v - 2uv - v^2$$

Now, the point $(x,y) = (1,1)$ has become the origin with the variable $(u,v) = (0,0)$.

With our new nonlinear system, we still don't have tools to describe the behavior of our system. However, with $(u, v) \approx (0, 0)$, we can reduce the system to a *linear* system by using **linearization**:

$$\begin{cases} \frac{du}{dt} = -u - v - u^2 - uv \\ \frac{dv}{dt} = -2u - v - 2uv - v^2 \end{cases} \Rightarrow \text{linear approx. about } (u,v)=(0,0): \begin{cases} \frac{du}{dt} = -u - v \\ \frac{dv}{dt} = -2u - v \end{cases}$$

This is a valid *local approximation* about the point $(0, 0)$ since near the origin $(u, v) \approx (0, 0)$, the nonlinear terms are much smaller than the linear terms. This allows us to classify the equilibrium solution $(x, y) = (1, 1)$ by looking at the classification of $(u, v) = (0, 0)$.

What would the equilibrium solution at $(u, v) = (0, 0)$ be classified as? Then look at how this relates to the phase space from earlier.

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow \begin{matrix} T = -2 \\ D = -1 \end{matrix} \Rightarrow \text{saddle}$$

$$\downarrow$$

$$(-1-\lambda)(-1-\lambda) - 2 = 0$$

$$\lambda^2 + 2\lambda + 1 = 1 + 1$$

$$(\lambda+1)^2 = 2$$

$$\lambda = -1 \pm \sqrt{2}$$

$$\lambda_1 = -1 + \sqrt{2} > 0$$

$$\lambda_2 = -1 - \sqrt{2} < 0$$

$$\Rightarrow \text{saddle}$$

$(u,v) = (0,0)$ is a saddle

$\Downarrow$

$(x,y) = (1,1)$ is a saddle

Sometimes our systems won't have polynomials... How can we deal with systems like this?

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -y - \sin(x)\,. \end{cases}$$

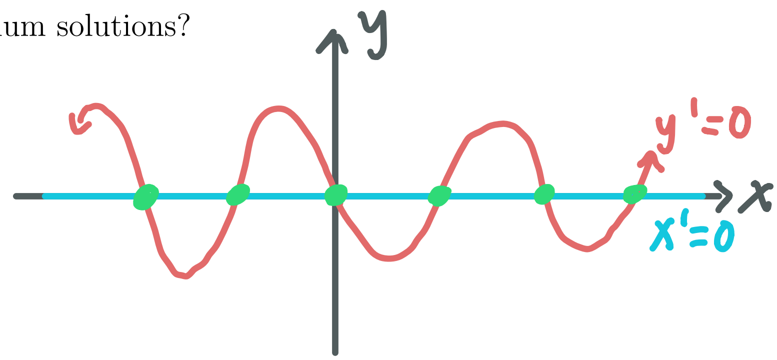
What are the nullclines and equilibrium solutions?

$x' = 0$ if $y = 0$

$y' = 0$ if $y = -\sin(x)$

equil. solns @ $(\pm n\pi, 0)$

$n = 0, 1, 2, \ldots$



To classify the equilibrium point at the origin of the nonlinear system, we'd need to linearize the system so that we can use the tools we've developed in Chapter 3. How could we do that with $\sin(x)$?

Taylor's thm:

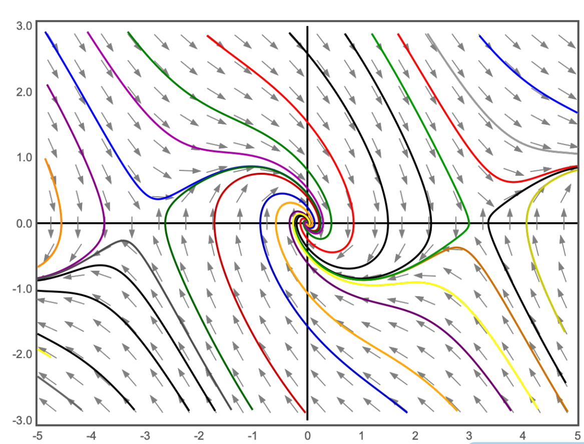
$$f(x) \approx f(a) + f'(a)\cdot(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)\cdot(x-a)^n}{n!} + \cdots$$

- linearize system about $(0,0)$:

$$\sin(x) \approx 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots$$

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -y - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots\right) \end{cases}$$



linear approx about $(0,0)$:

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -y - x \end{cases}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad T = -1,\; D = 1$$

$\Rightarrow$ origin is spiral sink

Linearization in the general case

Consider the nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y)\,, \\ \dfrac{dy}{dt} = g(x, y)\,, \end{cases}$$

$$\frac{dx}{dt} = 0 = f(x_0, y_0)$$

$$\frac{dy}{dt} = 0 = g(x_0, y_0)$$

with the equilibrium point (x_0, y_0). What would a change of variables with u, v look like for the transformation (x_0, y_0) to the origin $(u, v) = (0, 0)$?

$$u = x - x_0 \qquad x = u + x_0$$

$$\leftrightarrow$$

$$v = y - y_0 \qquad y = v + y_0$$

Taylor's thm:

$$f(x) \approx f(a) + f'(a)\cdot(x-a) + \ldots$$

What's the new nonlinear system?

$$\begin{cases} du/dt = dx/dt = f(u + x_0, v + y_0) \\ dv/dt = dy/dt = g(u + x_0, v + y_0) \end{cases}$$

With this, we can use a (first-order) linear approximation/Taylor series for a function of two-variables:

$$f(u+x_0, v+y_0) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0) + \ldots$$

$$f(u+x_0, v+y_0) \approx 0 + \frac{\partial f}{\partial x}(x_0,y_0)\cdot u + \frac{\partial f}{\partial y}(x_0,y_0)\cdot v$$

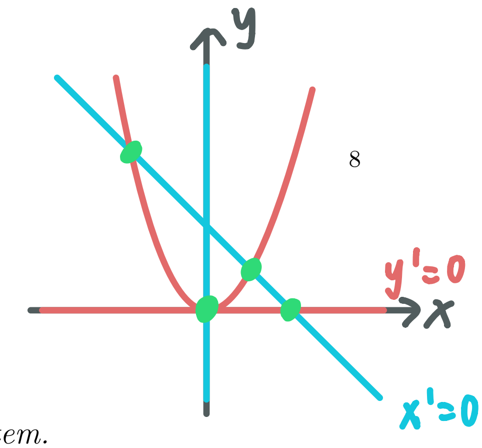
(This is similar to tangent planes from Calc. III!) And what's linearized system from using this approximation?

$$\frac{du}{dt} \approx f(u+x_0, v+y_0) \approx \frac{\partial f}{\partial x}(x_0,y_0)\cdot u + \frac{\partial f}{\partial y}(x_0,y_0)\cdot v$$

$$\frac{dv}{dt} \approx g(u+x_0, v+y_0) \approx \frac{\partial g}{\partial x}(x_0,y_0)\cdot u + \frac{\partial g}{\partial y}(x_0,y_0)\cdot v$$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}} \begin{bmatrix} u \\ v \end{bmatrix}$$

We can use the **Jacobian matrix** to classify the system! This process takes the translation and linearization into one-step.



Example 0.3. *Consider the nonlinear system*

$$\begin{cases} \dfrac{dx}{dt} = x(2-x-y)\,, \\ \dfrac{dy}{dt} = y(y-x^2)\,. \end{cases}$$

Find and classify all of the equilibrium points for this system.

- Nullclines: $x'=0$ if $x=0$ or $y=2-x$; $y'=0$ if $y=0$ or $y=x^2$
- equil. pts: $(0,0)$, $(2,0)$, $(1,1)$, $(-2,4)$

- Jacobian Matrix: $J(x,y)$ ($DF(x,y)$) $= \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix} = \begin{bmatrix} 2-2x-y & -x \\ -2xy & 2y-x^2 \end{bmatrix}$

$f(x,y) = 2x - x^2 - xy$

$\dfrac{\partial f}{\partial x} = 2-2x-y \qquad \dfrac{\partial f}{\partial y} = -x$

$g(x,y) = y^2 - x^2 \cdot y$

$\dfrac{\partial g}{\partial x} = -2x\cdot y \qquad \dfrac{\partial g}{\partial y} = 2y - x^2$

- classifying equilibria:

- $J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$T = 2$, $D = 0$ $\Rightarrow$ $(0,0)$ is a source

- $J(1,1) = \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$

$T = 0$, $D = -3$ $\Rightarrow$ $(1,1)$ is a saddle

- $J(2,0) = \begin{bmatrix} -2 & -2 \\ 0 & -4 \end{bmatrix}$

$T = -6$, $D = 8$ $\Rightarrow$ $(2,0)$ is a sink

- $J(-2,4) = \begin{bmatrix} 2 & 2 \\ 16 & 4 \end{bmatrix}$

$T = 6$, $D = 8-32 < 0$ $\Rightarrow$ $(-2,4)$ is a saddle

Two cases where the behavior of a nonlinear system and its linearization differ:

- When the linearized system is a center (that is, when the real component of a complex eigenvalue is 0),
- When the linear system has a zero eigenvalue.

Section Recap

What are some take-away concepts from this section?