Examples: Lotka-Volterra, and the Pendulum

1. Lotka-Volterra: model of competition

The Lotka-Volterra model describes how the population of two species evolve in time as the compete for some resource (like the same food supply) when a limited amount of the resource available. This model would work with two species like rabbits and sheep as they compete for grass to eat. There are two main effects we consider:

(1) Each species would grow to its carrying capacity in the absence of each other. For example, the rabbits would 'reproduce like rabbits' if there were no sheep present. Additionally, rabbits reproduce at a faster rate that sheep. To take this into account with our model, we could assign rabbits a higher birth-rate than sheep.
(2) When sheep and rabbits encounter each other, trouble starts. Sometimes the rabbit gets to eat, but more usually, the sheep nudges the rabbit aside and starts nibbling n the grass. We'll assume these conflicts occur at a rate proportional to the size of each population. For example, if there are twice as many sheep, the odds of a rabbit encountering a sheep are twice as great. Further, we assume that the conflicts reduce the growth rate for each species, but the effect is more severe for rabbits.

(The classic Lotka-Volterra model does neglect a number of realistic assumptions, like predators, seasonal effects, and other sources of food. However, this model provides us with a system simple enough to study!)

One such model could be:

$$\begin{cases} \dfrac{dr}{dt} = \overbrace{r\,(3-r)}^{\text{logistic growth}} - 2rs\,, = r\cdot[3-r-2s] \\ \dfrac{ds}{dt} = s\,(2-s) \underbrace{-\,rs}_{\text{conflict}}\,, = s\cdot[2-s-r] \end{cases}$$

with $r(t), s(t) \geq 0$.

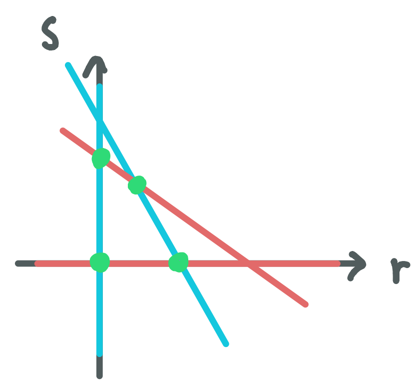
Example 1.1. *Find the equilibrium solution(s) of the model above.*

$r' = 0$ when $r = 0$ or $r = 3 - 2s$

$s' = 0$ when $s = 0$ or $r = 2 - s$

equil. solns:

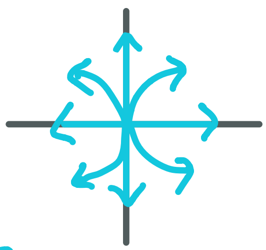
$(0,0), (3,0), (0,2), (1,1)$



Example 1.2. *Classify the equilibrium solutions from the model by determining the eigenvalues and eigenvectors. Then, sketch the phase portrait using that information.*

$$J(r,s) = \begin{bmatrix} 3-2r-2s & -2r \\ -s & 2-2s-r \end{bmatrix}$$

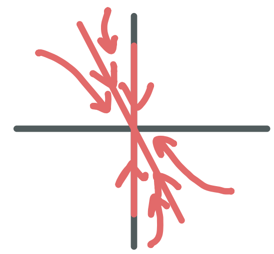
$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$



$\lambda_1 = 3 \qquad \lambda_2 = 2$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

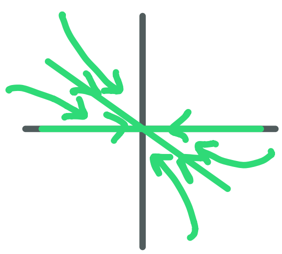
$$J(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$$



$\lambda_1 = -1 \qquad \lambda_2 = -2$

$\bar{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

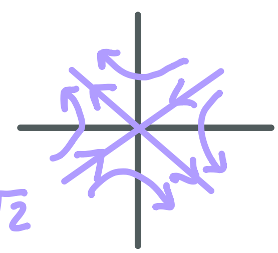
$$J(3,0) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$$



$\lambda_1 = -1 \qquad \lambda_2 = -3$

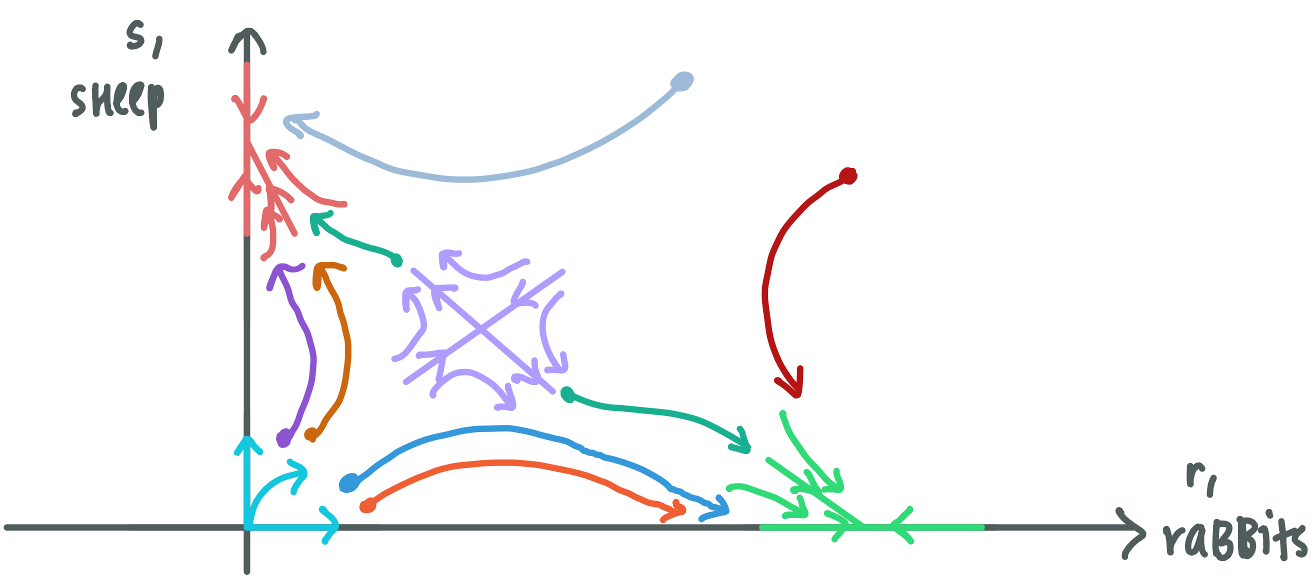
$\bar{v}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$J(1,1) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$$



$\lambda_1 = -1+\sqrt{2} \qquad \lambda_2 = -1-\sqrt{2}$

$\bar{v}_1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$



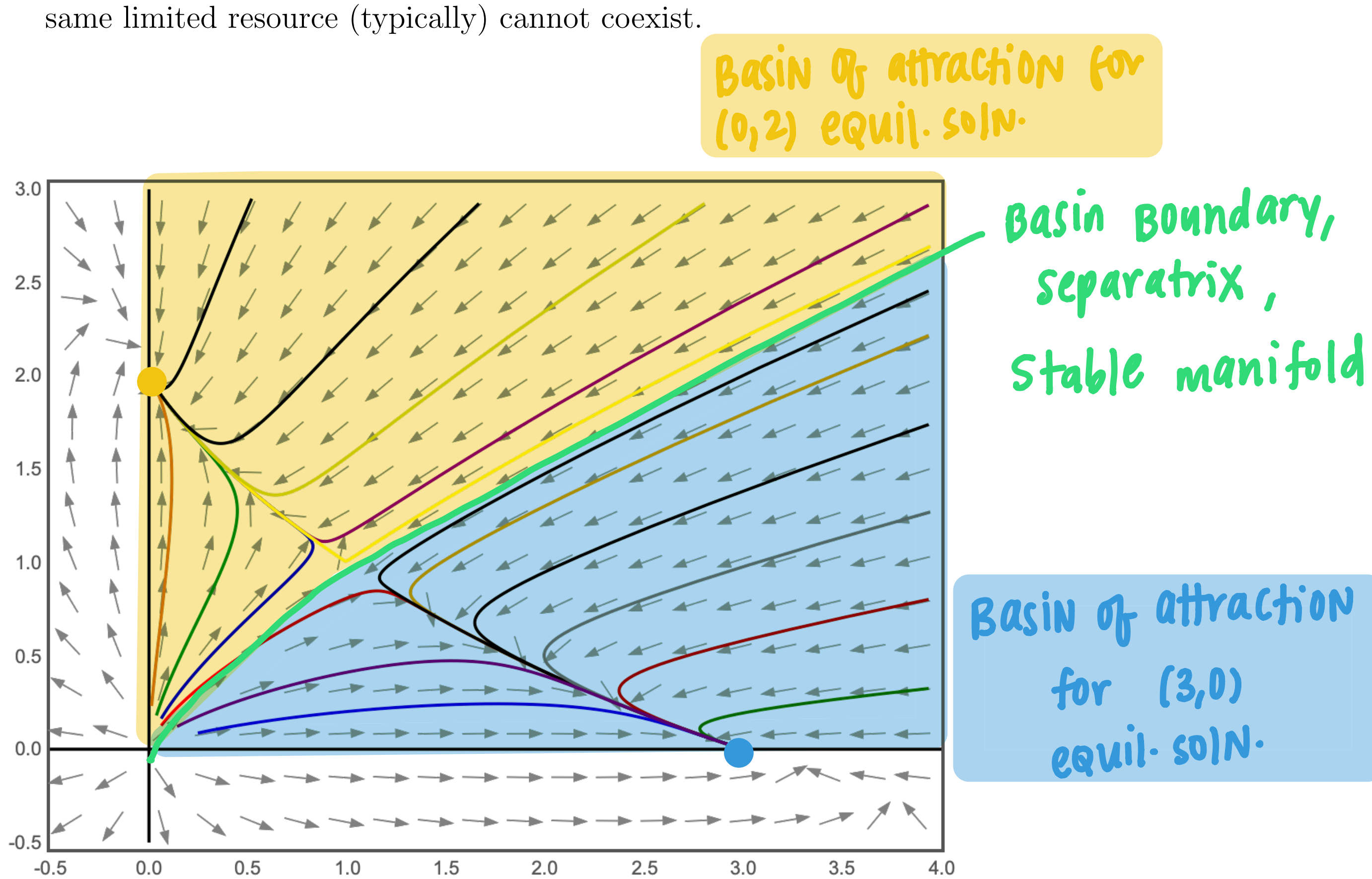
Some interesting notes:

- The r- and s-axes both contain straight-line trajectories. Why would this make intuitive sense?

- Trajectories with initial conditions/initial populations near the origin go to the sink at $(0, 2)$ and some go to the sink at $(3, 0)$. In between them, there must be a special trajectory that cannot decide which way to turn, and so it dives into the saddle point. This trajectory is part of the **stable manifold** of the saddle point.

- This stable manifold separates out trajectories from going to either of the sinks at $(0, 2)$ and $(3, 0)$. Each side of the manifold is a **basin of attraction** for the respective equilibrium solutions. This leads to the additional names of the stable manifold: **basin boundary** and **separatrix**. Trajectories like this split up the phase portrait into regions with different long-term behavior.

- A biological interpretation of our phase portrait suggests that one species will drive the other to extinction, and is dependent on the initial populations of the two species. This, and other models of competition, led biologists to formulate the **principle of competitive exclusion**, stating that two species competing for the same limited resource (typically) cannot coexist.



2. Hamiltonian systems and Conserved quantities

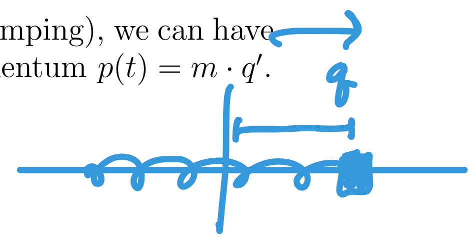
Definition 2.1: Hamiltonian system

A system is a **Hamiltonian system** if there exists some Hamiltonian function $H(x, y)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \text{and} \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

This is most often seen in examples with object displacement– mass-spring systems, pendulums, etc.– where the variables of interest are object's position q and momentum p (mass times velocity). In these examples, the Hamiltonian function $H(q, p)$ is the total energy of the system, the kinetic and potential energies added together.

In the mass-spring systems with harmonic oscillators (without damping), we can have $q(t)$ as the displacement from the resting position, and the momentum $p(t) = m \cdot q'$.



We can define kinetic energy T and the potential energy U as

$$T = \frac{1}{2} m\, v^2 = \frac{1}{2m} p^2, \qquad U = \frac{1}{2} k\, q^2,$$

where k is the spring constant from Hooke's law. The total energy of a system is the sum of the kinetic and potential energy, giving us the Hamiltonian function describing the total energy, as

$$H(q, p) = \frac{1}{2m} p^2 + \frac{1}{2} k\, q^2. \leftarrow \text{energy of system, conserved.}$$

Example 2.1. *Take the appropriate partial derivatives of the Hamiltonian function above to determine the system with* $\frac{dq}{dt}$ *and* $\frac{dp}{dt}$ *in terms of* q, p.

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{1}{m} \cdot p \qquad \begin{bmatrix} q \\ p \end{bmatrix}' = \begin{bmatrix} 0 & 1/m \\ -k & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -kq$$

Example 2.2. *How does this relate to the equation we used in Chapter 4 for the undamped mass-spring system* $mq'' + kq = 0$*, where* q *is the displacement of the mass?*

w/ $p = m \cdot q'$

$$\Rightarrow p' = m \cdot q'' = -kq \qquad \begin{bmatrix} q \\ p \end{bmatrix}' = \begin{bmatrix} 0 & 1/m \\ -k & 0 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

$$\Rightarrow \begin{cases} q' = \frac{1}{m} p \\ p' = -kq \end{cases}$$

3. The Pendulum

If you've studied the pendulum before, it's likely been with a linear approximation of the nonlinear system for specific parameter regimes. With the tools we've developed in this class so far though, we can study the full nonlinear system!

In the absence of damping/friction and external forcing, the pendulum's motion is governed by

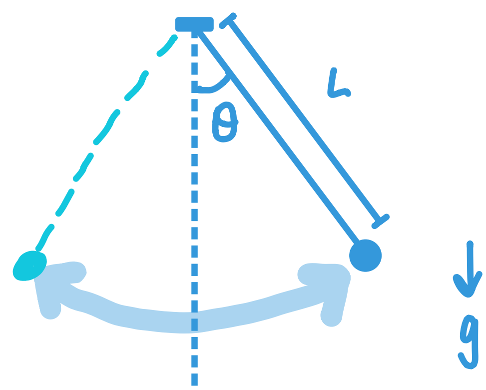
$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0,$$

where $\theta(t)$ is the angle from the downward vertical line, g is the gravitational constant, and L is the length of the pendulum. How can we can write this as the nonlinear first-order system?

$v = \dot{\theta}$ ← angular velocity

$$\dot{v} = \ddot{\theta} = -\frac{g}{L}\sin(\theta)$$

$$\Rightarrow \begin{cases} \dot{\theta} = v \\ \dot{v} = -\frac{g}{L}\sin(\theta) \end{cases}$$



What are equilibrium solutions to this system? What would they represent physically?

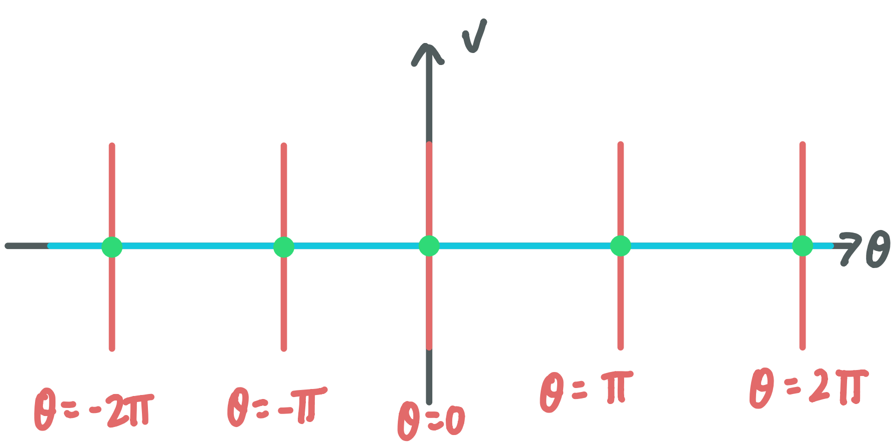
$\dot{\theta} = 0$ if $v = 0$

$\dot{v} = 0$ if $-\frac{g}{L}\sin(\theta) = 0 \leftrightarrow \sin(\theta) = 0$

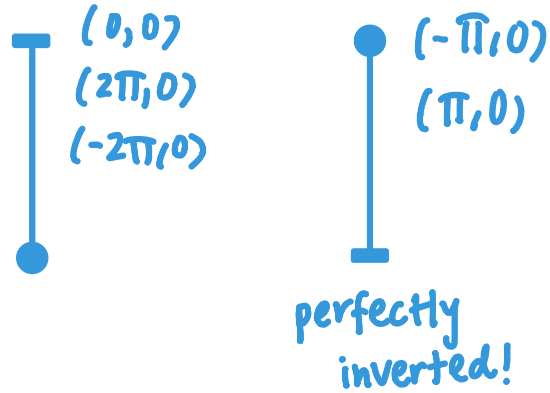
$\theta = n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$

equil. solns.: $(\theta, v) = (n\pi, 0)$



equil. solns:



Since there's no physical difference between angles that differ by 2π, we'll focus on two of the equilibrium solutions $(0, 0)$ and $(\pi, 0)$. What's the Jacobian look like for both cases? How could information from the Jacobian help us sketch the phase plane?

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -\frac{g}{L}\sin(\theta) \end{cases} \qquad J(\theta, v) = \begin{bmatrix} 0 & 1 \\ -w^2\cos(\theta) & 0 \end{bmatrix} \quad \text{w/} \quad \frac{g}{L} = w^2$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix} \qquad J(\pi, 0) = \begin{bmatrix} 0 & 1 \\ w^2 & 0 \end{bmatrix}$$

$T = 0$, $D = w^2$ $\Rightarrow$ center

actually is a center in this case

$T = 0$, $D = -w^2$ $\Rightarrow$ saddle

If this system is conservative (we intuitively know that it is!), what quantity is being conserved?

$$\begin{cases} \dot{\theta} = v = \frac{\partial H}{\partial v} \\ \dot{v} = -\frac{g}{L}\sin(\theta) = -\frac{\partial H}{\partial \theta} \end{cases}$$

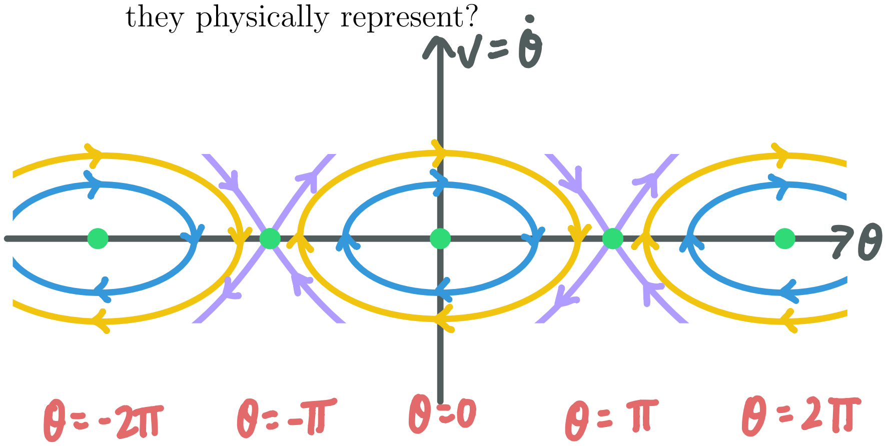
$$\frac{\partial H}{\partial v} = v \Rightarrow H(\theta, v) = \frac{1}{2}v^2 + f(\theta)$$

$$\frac{\partial H}{\partial \theta} = \frac{g}{L}\sin(\theta) \Rightarrow H(\theta, v) = -\frac{g}{L}\cos(\theta) + f(v)$$

$$\Downarrow$$

$$\boxed{H(\theta, \dot{\theta}) = \underbrace{\frac{1}{2}\dot{\theta}^2}_{\text{kinetic energy}} - \underbrace{\frac{g}{L}\cos(\theta)}_{\text{potential energy}}}$$

With the information that this system is *conservative*, the linearized center is, in fact, a *nonlinear center*. This tells us the Hamiltonian function is constant along the contours/trajectories in the phase portrait. What is the lowest energy/Hamiltonian function value the system attains? Where are those trajectories located and what do they physically represent?

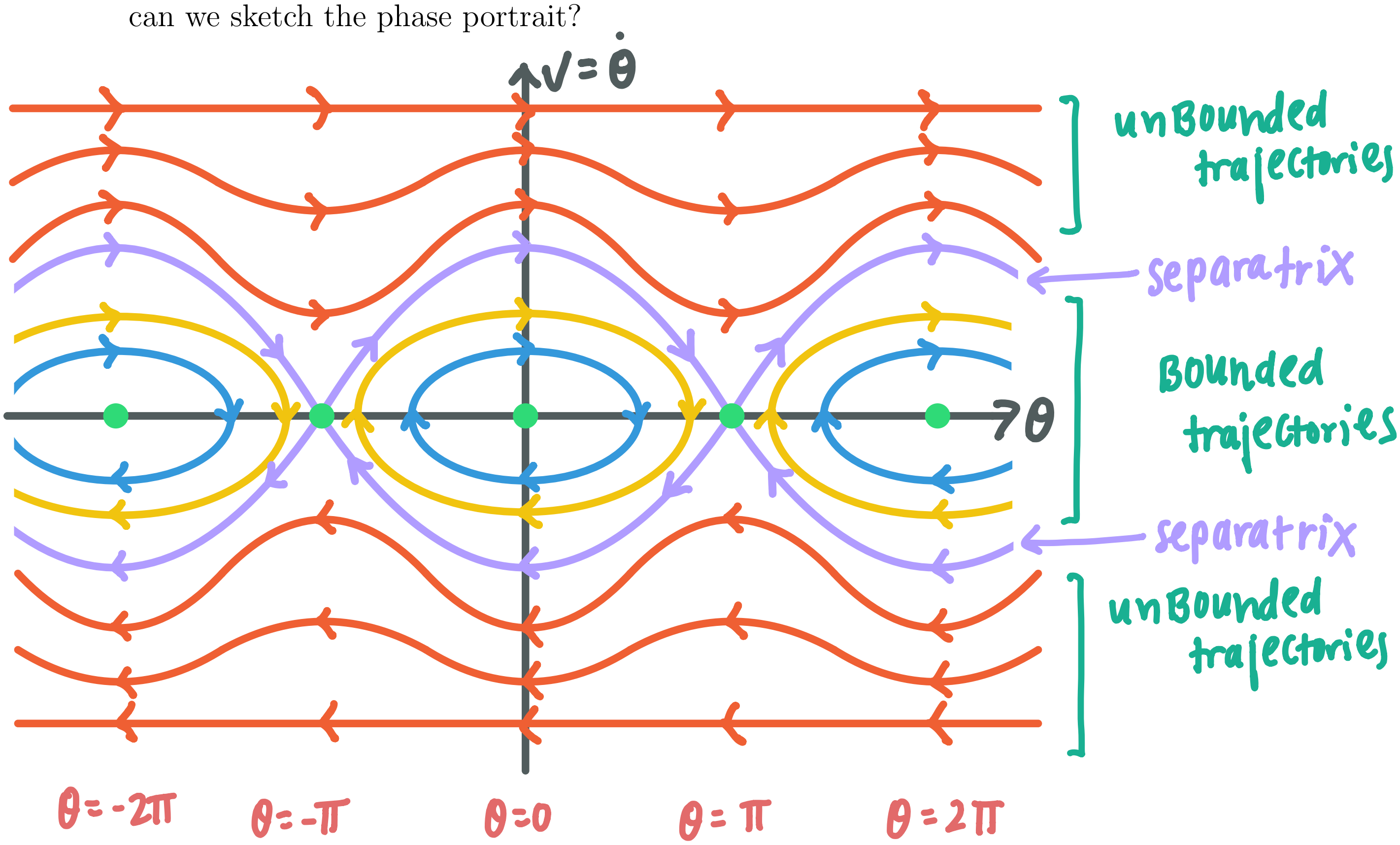


What does the system look like for $|\dot{\theta}| >> 1$? $\Rightarrow$ angular velocity is large in magnitude

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -\frac{g}{L}\sin(\theta) \end{cases}$$

$|\dot{\theta}| >> 1$, $|\dot{v}| \leq \omega^2$ $\Rightarrow$ ~horizontal trajectories in phase portrait

With information about the equilibrium solutions and then 'far-field' behavior, how can we sketch the phase portrait?



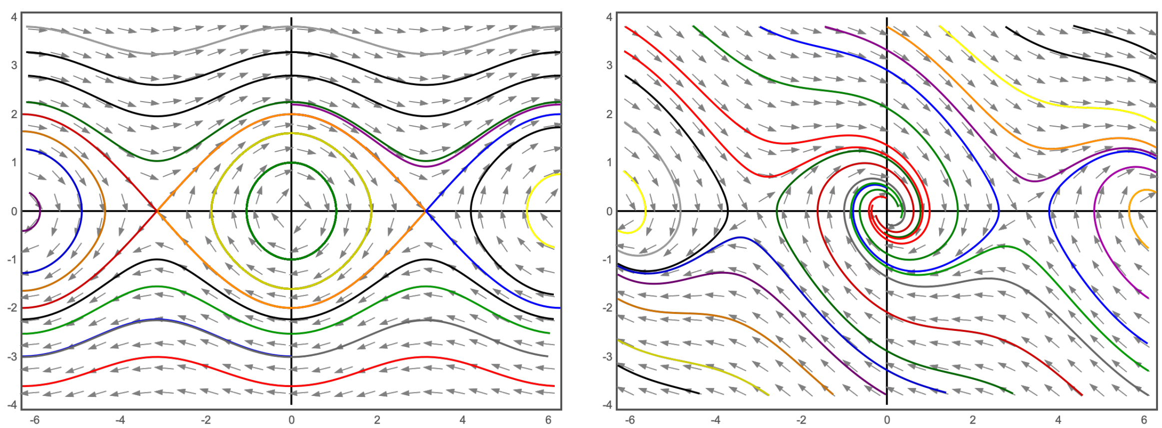
- The physically-relevant parameter regime is inside the separatrices; these trajectories correspond to a pendulum acting like 'normal.'
- At the saddle points, the pendulum is completely inverted and completely at rest. The trajectories along the separatrices represent the 'delicate motions in which the pendulum slows to a halt precisely as it approaches the inverted position.'
- Lastly, the unbounded trajectories represent an inverted pendulum where the pendulum whirls repeatedly over the top.

4. Pendulum with damping

Now, consider a pendulum with damping/friction governed by the system

$$\ddot{\theta} + \frac{b}{2}\dot{\theta} + \frac{g}{L}\sin(\theta) = 0,$$

where $b > 0$ is a coefficient related to the damping strength. In comparing the phase portraits of the undamped and damped cases below, we see the centers have become spiral sinks and the saddles have remained saddles.



In the damped situation, the pendulums are continuously losing energy due to friction. If we think back to the *undamped* case, the energy of the system was constant for all time since there were no outside forces acting upon the system. With the energy of the system as

$$E(t) = \frac{1}{2}\dot{\theta}^2 - \frac{g}{L}\cos(\theta),$$

how can we show the system is always losing energy?

$$\frac{dE}{dt} = \frac{1}{2}\cdot 2\dot{\theta}\,\ddot{\theta} + \frac{g}{L}\sin(\theta)\dot{\theta}$$

$$= \dot{\theta}\left(\ddot{\theta} + \frac{g}{L}\sin(\theta)\right)$$

$$= \dot{\theta}\left(-\frac{b}{2}\dot{\theta}\right)$$

$$= -\frac{b}{2}\dot{\theta}^2$$

$$\frac{dE}{dt} \leq 0$$

energy of system is always decreasing.

w/ damping:

$(0,0)$
$(2\pi,0)$
$(-2\pi,0)$

stable

$(-\pi,0)$
$(\pi,0)$

perfectly inverted!

unstable

Section Recap

What are some take-away concepts from this section?