3.9: Related Rates

If two quantities that change over time are related to each other, then their rates of change over time are related as well. For example, consider an expanding circle. Then, the radius $r = r(t)$ and the area $A = A(t)$ both change with time and are related with

$$A = \pi r^2 .$$

As the circle expands over time, the rate $\frac{dr}{dt}$ at which the radius increases is related to the rate $\frac{dA}{dt}$ at which its area increases. We can find the equation that relates these rates by using implicit differentiation to the formula that relates r and A:

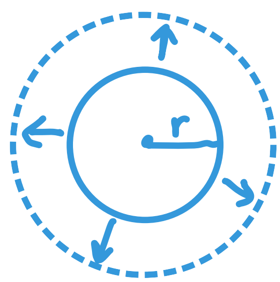
$$A(t) = \pi \cdot [r(t)]^2$$

$$\frac{d}{dt}\left[A(t)\right] = \frac{d}{dt}\left[\pi \cdot [r(t)]^2\right] \longrightarrow \frac{dA}{dt} = \pi \cdot 2\, r(t) \cdot \frac{dr}{dt}$$

Suppose the circle has an initial radius of 2inches and the radius is increasing at a constant rate of 3 inches per second. How fast is the area of the circle initially increasing?

Can we draw the situation?

(could be like a puddle on the ground in rain)



What information are we looking for? Based on our drawing, what do we expect the answer to look like (i.e., positive, negative, zero)?

goal: $\frac{dA}{dt}$ when $t=0$. we expect $\frac{dA}{dt} > 0$.

What information have we been given? How can we use that to solve for our goal?

given: $r(0) = 2$ in.

$\frac{dr}{dt} = 3 \frac{\text{in.}}{\text{sec.}}$

$$\left.\frac{dA}{dt}\right|_{t=0} = \pi \cdot 2\, r(0) \cdot \left.\frac{dr}{dt}\right|_{t=0}$$

$$= \pi \cdot 2 \cdot (2 \text{ in.}) \cdot \left(3 \frac{\text{in.}}{\text{sec.}}\right)$$

$$\boxed{\left.\frac{dA}{dt}\right|_{t=0} = 12\pi \frac{\text{in}^2}{\text{sec.}}}$$

Many related-rates problems involve geometric quantities such as volume, area, and surface area. Many of the formulas may (or should?) already be familiar to you.

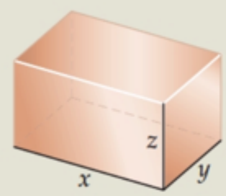
Some common formulas are:

Volume and Surface Area Formulas

The formulas that follow describe the volume V and surface area S of a rectangular box, sphere, right circular cylinder, and right circular cone. The lateral (side) surface area L is also given for the cylinder and cone.

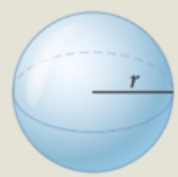
(a) The volume and surface area of a ***rectangular box*** of length x, width y, and height z are

$$V = xyz$$
$$S = 2xy + 2yz + 2xz$$



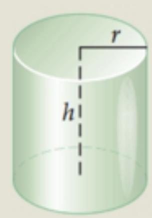
(b) The volume and surface area of a ***sphere*** of radius r are

$$V = \frac{4}{3}\pi r^3$$
$$S = 4\pi r^2$$



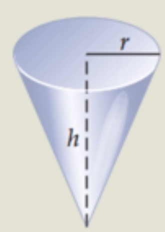
(c) The volume, surface area, and lateral surface area of a ***right circular cylinder*** of radius r and height h are

$$V = \pi r^2 h$$
$$S = 2\pi rh + 2\pi r^2$$
$$L = 2\pi rh$$

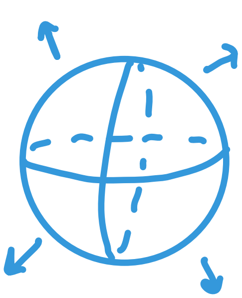


(d) The volume, surface area, and lateral surface area of a ***right circular cone*** of radius r and height h are

$$V = \frac{1}{3}\pi r^2 h$$
$$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$
$$L = \pi r\sqrt{r^2 + h^2}$$



Example 0.1. *With the volume of a sphere as* $V(t) = \frac{4}{3}\pi\left[r(t)\right]^3$, *determine* $\dfrac{dV}{dt}$.



$$V(t) = \frac{4}{3}\pi \cdot \left[r(t)\right]^3$$

$$\frac{d}{dt}\left[V(t)\right] = \frac{d}{dt}\left[\frac{4}{3}\pi \cdot \left[r(t)\right]^3\right]$$

$$\boxed{\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3 \cdot \left[r(t)\right]^2 \cdot \frac{dr}{dt} = 4\pi \cdot \left[r(t)\right]^2 \cdot \frac{dr}{dt}}$$

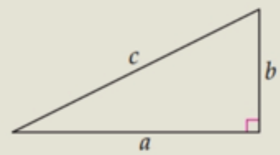
Its also common for related-rates problems to involve right triangles, either with the Pythagorean theorem or with similar triangles.

Two Theorems About Right Triangles

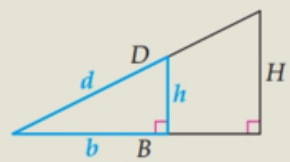
The following two theorems describe well-known relationships between the side lengths of right triangles:

(a) The ***Pythagorean theorem*** states that if a right triangle has legs of lengths a and b and hypotenuse of length c, then:

$$a^2 + b^2 = c^2$$



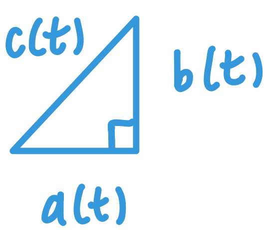
(b) The ***law of similar triangles*** states that if two right triangles have the same three angle measures, so that one is just a scaled-up version of the other, then the ratios of side lengths on one triangle are equal to the ratios of corresponding side lengths on the other. Specifically, with the side lengths shown in the diagram at the right, we have



$$\frac{h}{b} = \frac{H}{B}, \qquad \frac{d}{b} = \frac{D}{B}, \qquad \frac{d}{h} = \frac{D}{H}$$

The reason these theorems about triangles arise in related-rates problems is that both theorems give us ways to relate quantities that might change together over time. Finding an equation that relates two quantities is often the first step in finding an equation that relates the rates of change of those quantities.

Example 0.2. *If* $a(t), b(t), c(t)$ *all change in time and satisfy* $a^2 + b^2 = c^2$, *find* $\dfrac{dc}{dt}$.



$$[a(t)]^2 + [b(t)]^2 = [c(t)]^2$$

$$\frac{d}{dt}\left[[a(t)]^2 + [b(t)]^2\right] = \frac{d}{dt}\left[[c(t)]^2\right]$$

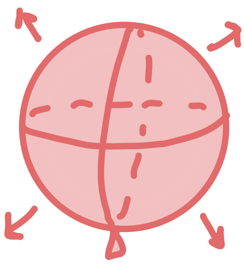
$$2 \cdot a(t) \cdot \frac{da}{dt} + 2 \cdot b(t) \cdot \frac{db}{dt} = 2 \cdot c(t) \cdot \frac{dc}{dt}$$

$$\boxed{\frac{a(t) \cdot \frac{da}{dt} + b(t) \cdot \frac{db}{dt}}{c(t)} = \frac{dc}{dt}}$$

working with sphere

Example 0.3. *Suppose a pink spherical party balloon is being inflated at a constant rate of 44 cubic inches per second.*

(1) *How fast is the radius of the balloon increasing at the instant that the balloon has a radius of 4 inches?*



given: $\frac{dV}{dt} = 44 \frac{in^3}{sec.}$

goal: Find $\frac{dr}{dt}$ when $r(t) = 4 in$

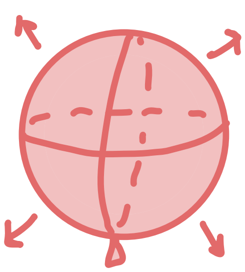
$$\frac{dV}{dt} = 4\pi (r(t))^2 \cdot \frac{dr}{dt}$$

$$44 \frac{in^3}{sec.} = 4\pi \cdot (4 \text{ in.})^2 \frac{dr}{dt}$$

$$44 \frac{in^3}{sec.} = 64\pi \text{ in}^2 \cdot \frac{dr}{dt}$$

$$44 \frac{in^3}{sec} \cdot \frac{1}{64\pi \text{ in}^2} = \boxed{\frac{11}{16\pi} \frac{in.}{sec.} = \frac{dr}{dt}}$$

(2) *How fast is the radius of the balloon increasing at the instant that the balloon contains 100 cubic inches of air?*



given: $\frac{dV}{dt} = 44 \frac{in^3}{sec.}$

goal: Find $\frac{dr}{dt}$ when $V = 100 \text{ in}^3$

use $V = \frac{4}{3}\pi r^3$ to get radius!

$$100 = \frac{4}{3}\pi r^3$$

$$\frac{300}{4\pi} = r^3 \rightarrow \left(\frac{300}{4\pi}\right)^{1/3} = r$$

$$\frac{dV}{dt} = 4\pi (r(t))^2 \cdot \frac{dr}{dt}$$

$$44 \frac{in^3}{sec.} = 4\pi \cdot (r(t))^2 \cdot \frac{dr}{dt}$$

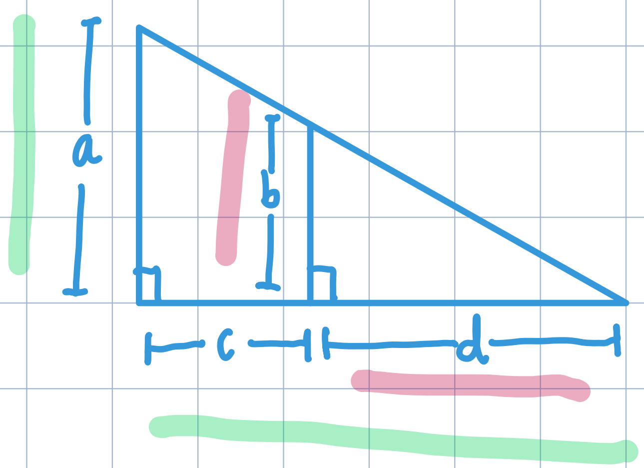
$$44 \frac{in^3}{sec.} = 4\pi \cdot \left(\frac{300}{4\pi}\right)^{2/3} \text{in}^2 \cdot \frac{dr}{dt}$$

$$\boxed{\frac{44}{4\pi \cdot \left(\frac{300}{4\pi}\right)^{2/3}} \frac{in.}{sec.} = \frac{dr}{dt}}$$

warm-up:

- find $\frac{d}{dx}\left[(f(x))^{-1}\right] = -1\cdot(f(x))^{-2}\cdot f'(x)$

- with similar triangles, find a relationship between the variables: a, b, c, d.



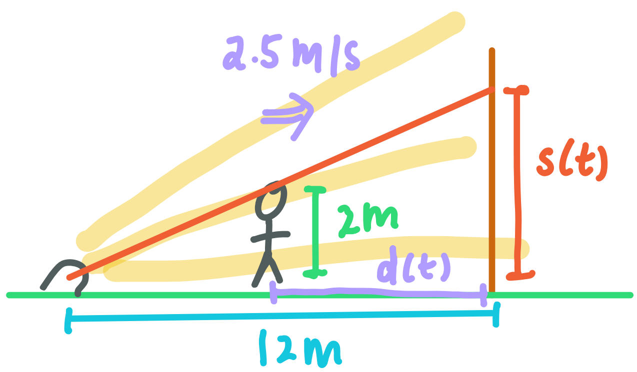
$$\frac{b}{d} = \frac{a}{c+d}$$

small Δ   large Δ

$$\frac{b}{a} = \frac{d}{c+d}$$

heights   lengths

Example 0.4. *A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 2.5 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building? (Round your answer to one decimal place.)*



1. Draw picture.
2. Find given info.
3. Determine goal.
4. identify formula to relate information.

constants:

distance b/w light & wall = 12m

height of man = 2m

speed of man: $\Rightarrow d'(t) = -2.5$ m/s

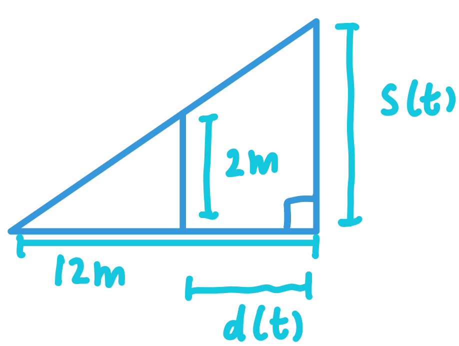
variables:

distance b/w man & wall : $d(t)$

length of shadow: $s(t)$

goal: want $s'(t)$ when $d(t) = 4$m.

need to relate $s(t)$, $d(t)$:



$$\frac{s(t)}{12} = \frac{2}{12 - d(t)}$$

$$\Rightarrow s(t) = 24 \cdot (12 - d(t))^{-1}$$

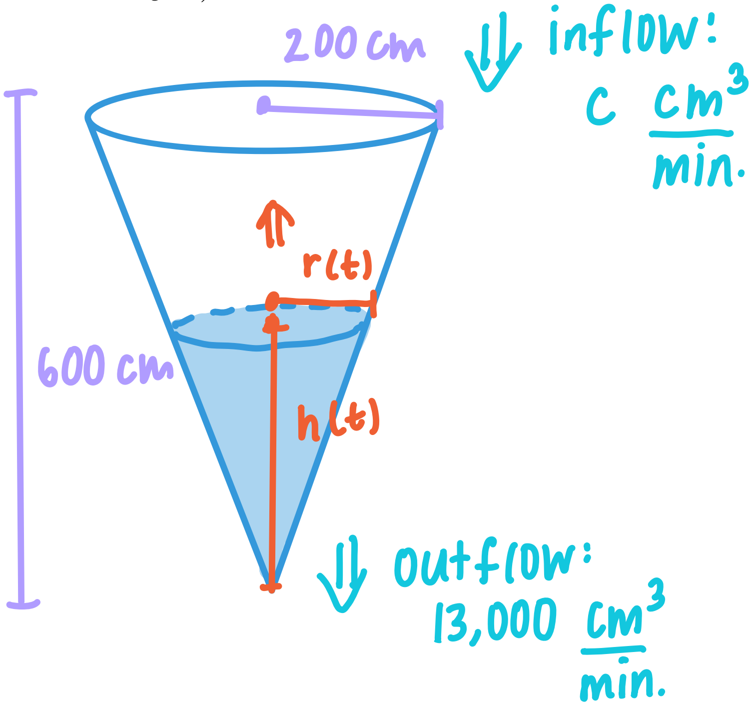
$$s'(t) = -24\cdot(12-d(t))^{-2}\cdot[0-d'(t)]$$

$$\Rightarrow s'(t) = \frac{24\cdot d'(t)}{(12-d(t))^2}$$

$$s'(t) = \frac{24\cdot(-2.5)}{(12-4)^2}$$

$$\boxed{s'(t) = -\frac{15}{16}\ \frac{\text{meters}}{\text{sec.}}}$$

Example 0.5. *Water is leaking out of an inverted conical tank at a rate of 13,000 cm^3/min at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank. (Round your answer to the nearest integer.)*



given:

$$\frac{dh}{dt} = 20 \ \frac{\text{cm}}{\text{min.}}$$

when $h(t) = 200$ cm

goal:

Find c (inflow rate)

$$\frac{dV}{dt} = c - 13{,}000$$

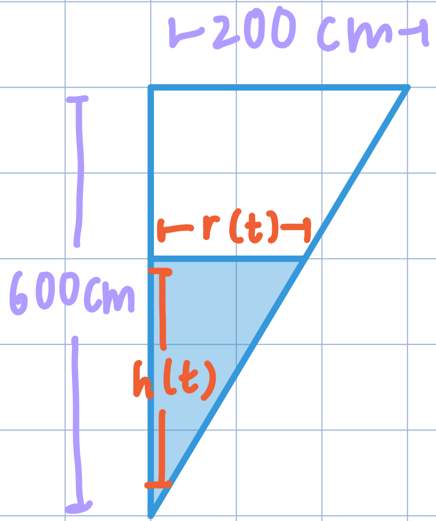
goal: Find

$$c = \frac{dV}{dt} + 13000$$

volume of water in tank:

$$V(t) = \frac{\pi}{3} \cdot [r(t)]^2 \cdot h(t)$$

* only have info re: $h(t)$, let's get $V(t)$ just in terms of $h(t)$



$$\frac{200}{r(t)} = \frac{600}{h(t)}$$

$$r(t) = \frac{h(t)}{3}$$

$$\Rightarrow \quad V(t) = \frac{\pi}{3} \cdot \left[\frac{h(t)}{3}\right]^2 \cdot h(t) = \frac{\pi}{27} \cdot [h(t)]^3$$

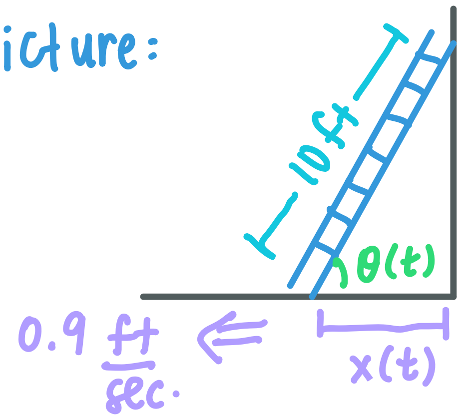
$$C = \frac{dV}{dt} + 13000$$

$$C = \frac{\pi}{27} \cdot 3 \cdot [h(t)]^2 \cdot \frac{dh}{dt} + 13000$$

$$C = \frac{\pi}{27} \cdot 3 \cdot [200\ cm]^2 \cdot \left[\frac{20\ cm}{min.}\right] + 13000 \ \frac{cm^3}{min.}$$

Example 0.6. *A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 0.9 ft/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6 ft from the wall? (That is, find the angle's rate of change when the bottom of the ladder is 6 ft from the wall.)*

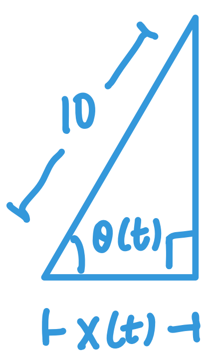
picture:



given:

- constants:
  10ft: length of ladder
  $x'(t) = 0.9$ ft./sec.
- variables:
  $x(t)$: distance B/W bottom of ladder & wall.
  $\theta(t)$: angle B/W bottom of ladder & ground

goal: $\theta'(t)$ when $x(t) = 6$



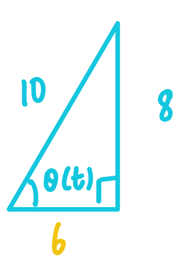
• relate $\theta(t), x(t)$: $\cos(\theta(t)) = \frac{x(t)}{10}$

$$\frac{d}{dt}\left[\cos(\theta(t))\right] = \frac{d}{dt}\left[\frac{x(t)}{10}\right]$$

$$-\sin(\theta(t)) \cdot \theta'(t) = \frac{1}{10} \cdot x'(t)$$

$$\Rightarrow \theta'(t) = -\frac{1}{10} \cdot \frac{1}{\sin(\theta(t))} \cdot x'(t)$$

Find $\sin(\theta(t))$ when $x(t) = 6$:



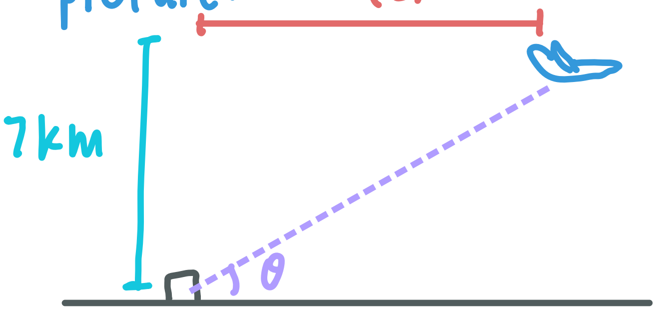
$\Rightarrow \sin(\theta(t)) = \frac{8}{10}$

$$\theta'(t) = -\frac{1}{10} \cdot \frac{1}{8/10} \cdot (0.9)$$

$$\boxed{\theta'(t) = -\frac{.9}{8} \frac{\text{rad.}}{\text{sec.}}}$$

Example 0.7. *A plane flies horizontally at an altitude of 7 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of $\pi/4$ rad/min. How fast is the plane traveling at that time?*

picture: $x(t)$

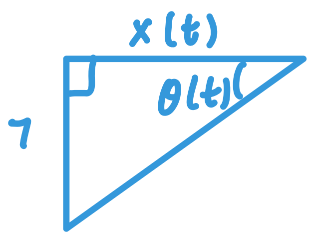


given:
- constants:
  7km: altitude of plane
- variables:
  $\theta(t)$: angle of elevation
  $x(t)$: horizontal distance B/W plane & telescope

goal: Find $x'(t)$ WHEN

$\theta(t) = \frac{\pi}{3}$ rad. & $\theta'(t) = -\frac{\pi}{4} \frac{\text{rad.}}{\text{min.}}$

- relate $x(t)$, $\theta(t)$:

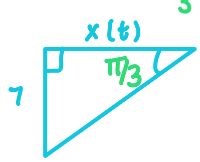


$$\tan(\theta(t)) = \frac{7}{x(t)} = 7 \cdot [x(t)]^{-1}$$

$$\frac{d}{dt}\left[\tan(\theta(t))\right] = \frac{d}{dt}\left[7 \cdot [x(t)]^{-1}\right]$$

$$\sec^2(\theta(t)) \cdot \theta'(t) = -7 \cdot [x(t)]^{-2} \cdot x'(t)$$

Finding $x(t)$ WHEN $\theta(t) = \frac{\pi}{3}$:



$$\tan\left(\frac{\pi}{3}\right) = \frac{7}{x(t)}$$

$$x(t) = \frac{7}{\tan(\pi/3)}$$

$$x(t) = \frac{7}{\sqrt{3}}$$

$$\Rightarrow x'(t) = -\frac{1}{7} \cdot \sec^2(\theta(t)) \cdot \theta'(t) \cdot [x(t)]^2$$

$$x'(t) = -\frac{1}{7} \cdot \sec^2\left(\frac{\pi}{3}\right) \cdot \left[-\frac{\pi}{4}\right] \cdot \left[\frac{7}{\sqrt{3}}\right]^2$$

$$= +\frac{1}{\cancel{7}} \cdot \cancel{(2)^2} \cdot \left[+\frac{\pi}{\cancel{4}}\right] \cdot \left[\frac{\cancel{7} \cdot 7}{3}\right]$$

$$\boxed{x'(t) = \frac{7}{3}\pi \ \frac{\text{km.}}{\text{min.}}}$$