Chapter 3: Probability

Understanding probability is the gateway to understanding inferential statistics because if we know information about a population, then an inference can be made about the likelihood of a sample.

1. Events, Sample Spaces, and Probability

- **Statistical experiment:** a process leading to an outcome with some probability, such as flipping a coin, rolling a die, drawing a card, taking a sample, etc.

- **Outcome:** a result of an experiment, such as flipping heads, rolling a 4, drawing a Queen, etc.

- **Sample space:** the set of all possible outcomes of an experiment.

- **Event:** an outcome or set of outcomes, such as rolling an even number, flipping 2 heads in a row, etc.

- **Probability of an event:** the number of outcomes in the event / total outcomes in the sample space.
  Notation: $P(A)$ represents the probability that event A occurs.

- **Basic Rules of Probability**
  - The probability of any outcome must be no less than 0 (0%) and no more than 1 (100%).
  - The sum of the probabilities of all outcomes must be exactly 1 (100%).

Suppose you are flipping a coin and then rolling a die, and you want to find the probability of flipping heads and rolling an even number.

Determine the following:

Experiment:

flipping coin & then rolling die

Outcome:

flipping coin: Heads or tails
rolling die: 1, 2, 3, 4, 5, 6

Event:

flipping Heads & rolling 2, 4, or 6

Sample space:

$\{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6),$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6).\}$

total possible outcomes = 12
outcomes satisfying event

Probability:

$\frac{3}{12} = \frac{1}{4} = 25\%$

One method of determining the number of sample points for a complex experiment is to develop a counting system. See if you can develop a system for counting the number of ways to select 2 students from a total of 4. If the students are represented by the symbols S_1, S_2, S_3, and S_4, the sample points could be listed in the following pattern:

S_1, S_2, S_3, S_4 : (S_1, S_2) (S_2, S_3) (S_3, S_4)

(4 students) (S_1, S_3) (S_2, S_4)

(S_1, S_4)

A second method of determining the number of sample points for an experiment is to use **combinatorial mathematics.** This branch of mathematics is concerned with developing counting rules for given situations. For example, there is a simple rule for finding the number of different samples of 5 students selected from 1,000. This rule, called the **combinations rule**, is given below.

Theorem 1.1: Combinations Rule

If k samples are drawn from $n \geq k$ options *without replacement*, then the number of possible samples we can observe is:

"n choose k" $$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where } n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \,.$$

! : "factorial"

(Note: $0! = 1$.)

In what situations should this be used: when order does matter, or when order does not matter?

when order does not matter

How many ways are there to select two students out of a group of 10?

2 samples drawn from 10

$10! = 10 \cdot 9 \cdot 8!$

$$\binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10!}{2!\,8!} = \frac{10 \cdot 9 \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{(2 \cdot 1) \cdot (\cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1})}$$

$$= \frac{10 \cdot 9}{2 \cdot 1} = \frac{90}{2} = \boxed{45}$$

2. Unions and Intersections

- **Union:** the union of two events is the event that either (this is an inclusive either, meaning that both events can happen at the same time) event occurs.
  Notation: $A \cup B$ is the union of events A and B.

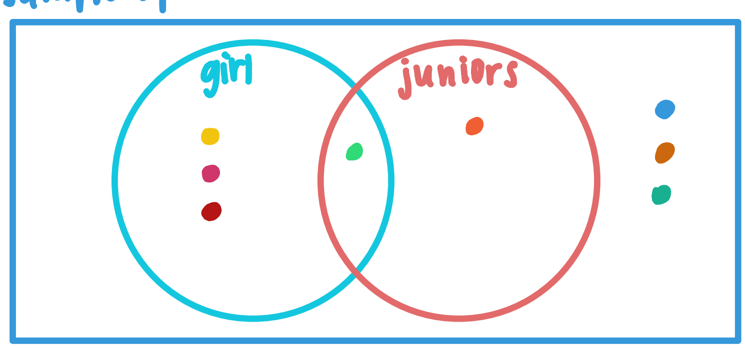
- **Intersection:** the intersection of two events is the event that both events occur.
  Notation: $A \cap B$ is the intersection of events A and B.

- **Complement:** the complement of an event is the event that it does not occur.
  Notation: A^C is the complement of event A:
  $P(A^C) = 1 - P(A)$.

- **Mutually exclusive:** two events are mutually exclusive if they have no shared outcomes (another way of saying this is that their intersection is empty).

Consider drawing a name at random in this class. Consider the events A that the name drawn is a girl, and the event B that the name belongs to a junior. What do $P(A \cup B)$ and $P(A \cap B)$ represent? What tools could we use to help us?

sample space



- fresh, BOY
- fresh, girl
- soph., BOY
- soph., girl
- jun., BOY
- jun., girl
- sen., BOY
- sen., girl

$P(A \cup B) = \frac{5}{8}$

"prob. that a student is a girl <u>or</u> is a junior"

$P(A \cap B) = \frac{1}{8}$

"prob. that a student is a girl <u>and</u> is a junior"

3. Complementary Events

- The **complement** of an event A is the event that A does not occur, denoted A^C. A^C consists of all sample points that do not belong to A.

- The sum of complementary events must equal one; that is, $P(A) + P(A^C) = 1$.

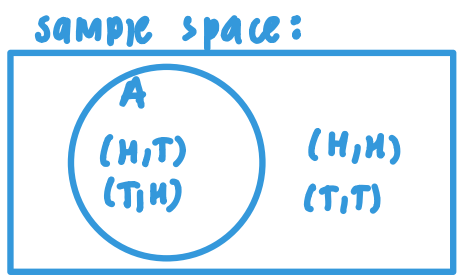
This says: if A denotes an event of an experiment, then any given sample point is either in A or A^C, and not both!

Consider flipping two coins, and define A as the event that exactly one of the coins shows heads. The complement, A^C, then consists of all the possible outcomes where the number of heads showing is not exactly one. In this example that corresponds to either having no heads or two heads showing. What is the data in set A? And what are the points in the set A^C?

sample space: $\{(H,H), (H,T), (T,H), (T,T)\}$

$A: \{(H,T), (T,H)\}$

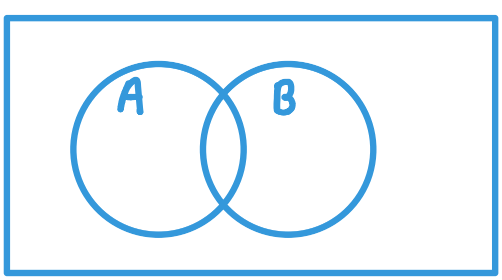
$A^c: \{(H,H), (T,T)\}$



The equation above is a powerful relationship. Often times, it is easier to calculate the complement of an event than the probability of an event itself. Then we can use:

$$P(A) + P(A^C) = 1 \qquad \Longleftrightarrow \qquad P(A) = 1 - P(A^C),$$ $P(A^c) = 1 - P(A)$

What about $(A \cup B)^C$ and $(A \cap B)^C$? What do these look like on Venn diagrams?



$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

How could we find $P(A \cup B)$ with complements?

$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c)$

$P(E) = 1 - P(E^c)$

4. The Additive Rule and Mutually Exclusive Events

Theorem 4.1: Additive Rule of Probability

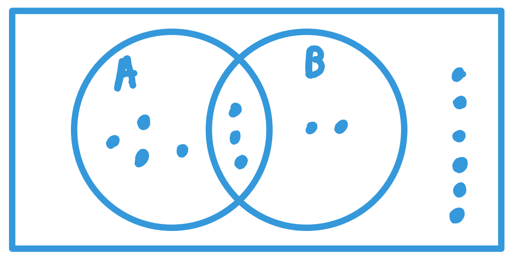
Consider two events A and B. The probability of the union of the two events, $A \cup B$ is equal to the sum of the individual probabilities minus the probability of the intersection:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

What does this look like on a Venn diagram?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{15} + \frac{5}{15} - \frac{3}{15} = \boxed{\frac{9}{15}}$$



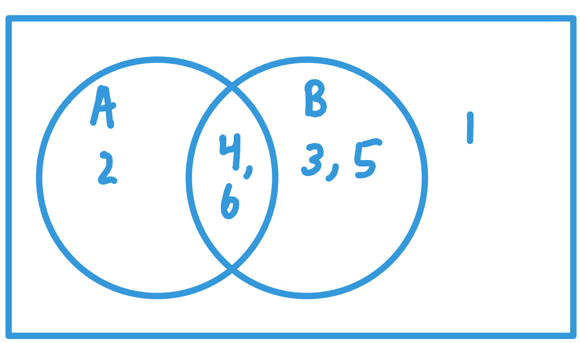
sample space

Take a six-sided die. What is the probability a number that is either even or greater than 2? Use a Venn diagram to map out the sample space and determine the probability, and then use the formula above to determine the probability.

sample space: $\{1, 2, 3, 4, 5, 6\}$

$A: \{2, 4, 6\}$

$B: \{3, 4, 5, 6\}$



sample space

$P(A) = \frac{3}{6} = \frac{1}{2}$ or 50%

$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$ or 33.3%

$P(B) = \frac{4}{6} = \frac{2}{3}$ or 66.7%

$P(A \cup B) = \frac{5}{6}$ $\left(P(A) + P(B) - P(A \cap B)\right)$

$P\left(\text{neither } A \text{ nor } B\right) = P\left((A \cup B)^c\right) = \frac{1}{6}$ or 16.7%

- Two events A and B are **mutually exclusive** if $A \cap B$ contains no sample points; that is, $P(A \cap B) = 0$.

If A and B are mutually exclusive events, what is $P(A \cup B)$?

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{=0} \Rightarrow P(A \cup B) = P(A) + P(B)$$

Consider rolling two dice. What is the probability of either rolling a 10 or greater, or seeing one or more 1's?

Let A be the event of seeing one or more 1's. Write out the sample points of A to determine $P(A)$.

$$A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}$$

$$P(A) = \frac{11}{36} \text{ or } 30.6\%$$

Write out the sample points of B to determine $P(B)$.

$$B = \{(4,6), (5,6), (6,6), (6,4), (6,5), (5,5)\}$$

$$P(B) = \frac{6}{36} = \frac{1}{6} \text{ or } 16.7\%$$

Are these events mutually exclusive? Justify your answer.

yes, A & B are mutually exclusive since there is no overlap b/w them. That is, $P(A \cap B) = 0$ or $A \cap B$ is empty.

To answer the original question asked, we're looking for $P(A \cup B)$. How can we find this?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{11}{36} + \frac{6}{36} - 0$$

$$P(A \cup B) = \frac{17}{36} \text{ or } 47.2\%$$

5. Conditional Probability

Often, we have additional knowledge that might affect the outcome of an experiment, so we may need to alter the probability of an event of interest. A probability that reflects such additional knowledge is called the **conditional probability** of the event.

To develop programs for business travelers staying at convention hotels, Hyatt Hotels Corp. commissioned a study of executives who play golf. The study revealed that 55% of the executives admitted that they had cheated at golf. Also, 20% of the executives admitted that they had cheated at golf and had lied in business. Given that an executive had cheated at golf, what is the probability that the executive had also lied in business?

What is this asking us to find?

if we have an executive who has cheated @ golf, whats probability they've lied in business?

Theorem 5.1: Conditional Probability Formula

To find the conditional probability that event A occurs given that event B occurs, divide the probability that both A and B occur by the probability that B occurs; that is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

This does assume that $P(B) \neq 0$.

So, given that an executive had cheated at golf, how can we find the probability that the executive had also lied in business?

Let A represent the event that the executive has cheated at golf, and B represent the event that the executive has lied in business.

$P(A) = .55$ or 55%

$P(A \cap B) = .2$ or 20%

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{.2}{.55} = .364 \text{ or } 36.4\%$$

The Federal Trade Commission's investigation of consumer product complaints has generated much interest on the part of manufacturers in the quality of their products. A manufacturer of an electromechanical kitchen utensil conducted an analysis of a large number of consumer complaints and found that they fell into the six categories shown in the table below. If a consumer complaint is received, what is the probability that the cause of the complaint was the appearance of the product given that the complaint originated during the guarantee period?

Distribution of Product Complaints				
	Reason for complaint			
	Electrical	**Mechanical**	**Appearance**	**Totals**
During Guarantee Period	*18%*	*13%*	*32%*	*63%*
After Guarantee Period	*12%*	*22%*	*3%*	*37%*
Totals	*30%*	*35%*	*35%*	*100%*

Let A represent the event that the cause of a particular complaint is the appearance of the product, and let B represent the event that the complaint occurred during the guarantee period.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.32}{.63} = \boxed{.51 \text{ or } 51\%}$$

$P(A \cap B) = .32$ or 32%

$P(B) = .63$ or 63%

---

given a Bag of 12 coins & 18 marbles, what is the probability of drawing 2 or more marbles on 3 draws?

2 marbles / 1 coin _ _ _ : $\binom{18}{2} \cdot \binom{12}{1} = 1826$

3 marbles _ _ _ : $\binom{18}{3} = 816$

3 objects _ _ _ : $\binom{30}{3} = 4060$

$P\left(\begin{array}{c}\text{2 or more}\\ \text{marbles}\end{array}\right)$

$$= \frac{1826 + 816}{4060} = .65320\ldots$$

$$\boxed{= 65.32\%}$$

6. Multiplicative Rule and Independent Events

First, let's revisit the conditional probability. Recall that, given events A and B, we have that

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad \text{and} \quad P(B|A) = \frac{P(A\cap B)}{P(A)}.$$

Alternatively, we can write

$$P(A\cap B) = P(A|B)P(B), \quad \text{and} \quad P(A\cap B) = P(B|A)P(A).$$

Why can we write these?

$$P(B)\cdot P(A|B) = \frac{P(A\cap B)}{\cancel{P(B)}}\cdot \cancel{P(B)} \rightarrow P(B)\cdot P(A|B) = P(A\cap B)$$

By using conditional probabilities we have two ways of calculating the probability of an intersection in two different ways:

$$\begin{aligned} P(A\cap B) &= P(A|B)P(B) \\ &= P(B|A)P(A). \end{aligned}$$

The point here is that we can rearrange what we already know so that we can compute certain probabilities with different combinations. In the previous section we wanted to compute conditional probabilities. Here, we can use conditional probabilities to compute other things.

In class of 30, 12 students come to office hours. Of those 12, 10 pass Test 1. Moreover, 21 in total pass Test 1. What fraction of students come to office hours and passed the test?

If T and O are the events of passing the test and coming to office hours, what are we trying to find? $P(T\cap O)$

How can we determine the fraction of students come to office hours and passed the test?

$$P(O) = \frac{12}{30}$$

$$P(T|O) = \frac{10}{12}$$

$$P(T) = \frac{21}{30}$$

$$P(T\cap O) = P(T|O)\cdot P(O) = \underbrace{P(O|T)}_{\uparrow\text{ no idea}}\cdot P(T)$$

$$\downarrow \quad = \frac{10}{12}\cdot\frac{12}{30}$$

$$\boxed{P(T\cap O) = \frac{10}{30} = \frac{1}{3} \text{ or } 33.33\%}$$

When we have two events, there are two possibilities. On the one hand, the two events may be correlated with one another - the probability of one occurring informs the probability of the other occurring. However, we may find that two events have nothing to do with one another - they are totally uncorrelated from one another. For instance consider a person and two events: A - the person goes to Trinity and B - the person is wearing blue. In fact, many people walking around campus are wearing blue, so these are probably correlated.

However, let's consider another event: C - the person likes cheeseburgers. These two events are likely unrelated, and therefore their probabilities are likely unrelated.

Definition 6.1: Independent events

Two events A and B are **independent** if the occurrence of B does not affect the occurrence of A and vice versa. This is equivalent to:

$$P(A|B) = P(A), \quad \text{and} \quad P(B|A) = P(B).$$

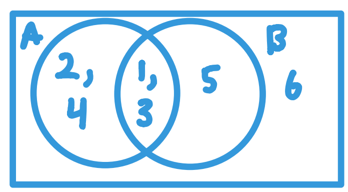
We can use the definition of $P(A|B)$ to find:

$$P(A|B) = P(A) \to \frac{P(A \cap B)}{P(B)} = P(A) \to P(A \cap B) = P(A)P(B).$$

The interpretation here is– the likelihood of two independent events occurring is equal to the product of the two individual probabilities.

Consider the experiment of rolling a dice and the two events: A as rolling a 4 or less, and B as rolling an odd number. What are $P(A)$ and $P(B)$?

$P(A) = \frac{4}{6}$ $\qquad$ $P(B) = \frac{3}{6}$



Are the two events independent?

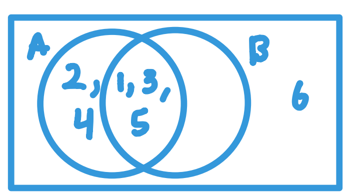
$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$

$$\frac{1}{3} = \frac{2}{6} \stackrel{?\checkmark}{=} \frac{4}{6} \cdot \frac{3}{6} = \frac{1}{3}$$

$\Rightarrow$ A, B are independent events.

If we change event A to rolling a 5 or less, what changes and how?

$$P(A \cap B) \stackrel{?}{=} P(A) \cdot P(B)$$



$$\frac{1}{2} = \frac{3}{6} \stackrel{?}{\neq} \frac{5}{6} \cdot \frac{3}{6} = \frac{5}{12}$$

$\Rightarrow$ A, B are not independent events.

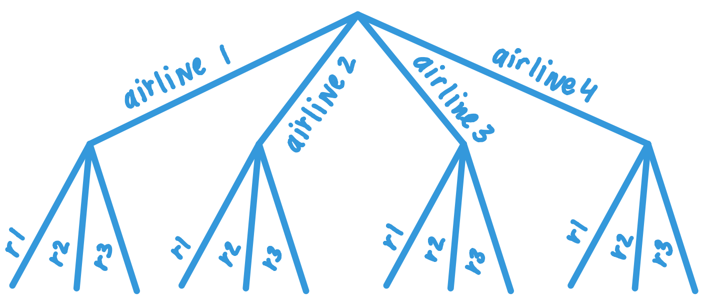
7. Counting Rules

Earlier in this chapter, we pointed out that experiments sometimes have so many sample points that it is impractical to list them all. However, many of these experiments possess sample points with identical characteristics. If we can develop a counting rule to count the number of sample points for such an experiment, it can be used to aid in the solution of the problems. The **combinations rule** for selecting k elements from n elements without regard to order was presented in Theorem 1.1.

Here, we give three additional counting rules: the **multiplicative rule**, **the permutations rule**, and the **partitions rule**.

A product can be shipped by four airlines, and each airline can ship via three different routes (Route 1, Route 2, Route 3).

How many distinct ways exist to ship the product?



$\underline{4} \cdot \underline{3} = \boxed{\text{12 distinct ways}}$

Assuming the airline and route are selected at random, what is the probability that Route 2 is used?

$$\frac{\text{4 ways using route 2}}{\text{12 distinct ways}} = \frac{4}{12} = \boxed{\frac{1}{3} \text{ or } 33.33\%}$$

Definition 7.1: Multiplicative Rule

You have k sets of elements, n_1 in the first set, n_2 in the second set, ..., and n_k in the kth set. Suppose you wish to form a sample of k elements by taking one element from each of the k sets. Then the number of different samples that can be formed is the product

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$$

Suppose there are five companies with internships, and 100 students applying to all five positions. In how many ways can the five students be assigned to these positions?

students 1 2 3 4 5 / 2 1 3 4 5 — different ordering! ⇒ order does matter

company 1 2 3 4 5

$\underline{100} \cdot \underline{99} \cdot \underline{98} \cdot \underline{97} \cdot \underline{96}$

= 9,034,502,400 distinct orderings.

The arrangement of elements in a distinct order is called a **permutation**. There are more than 9 billion different permutations of 5 elements (soldiers) drawn from a set of 100 elements!

Definition 7.2: Permutations Rule

Given a single set of n different elements, you wish to select k elements from the n and *arrange* them within k positions. The number of different permutations of the k elements taken k at a time is denoted by P_k^n and is equal to

$$P_k^n = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ combination

Suppose there are ten athletes, and we're responsible for awarding first and second place to two of them. How many ways can first and second place be awarded to 10 people?

$\underset{1st}{\underline{10}} \cdot \underset{2nd}{\underline{9}}$ = 90 distinct ways to organize 10 athletes into 1st, 2nd place.

. using 'permutation notation':

$$P_2^{10} = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{10 \cdot 9 \cdot \cancel{8!}}{\cancel{8!}} = 10 \cdot 9 = 90 \text{ distinct orderings}$$

Suppose we are supervising four construction workers. We must assign three to job 1 and one to job 2. What are some ways we can make this assignment? How does order matter?

if order did matter, $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to order.

1 2 3 4
1 3 2 4

same ordering. $\Rightarrow$ order doesn't matter within each job site.

$\underbrace{\_\ \_\ \_}_{\text{job 1}}\ \underbrace{\_}_{\text{job 2}}$

In how many different ways can we make this assignment?

$N = 4$ $\quad n_1 = 3$ (job 1)
$k = 2$ $\quad n_2 = 1$ (job 2)

w/ partitions rule

$$\frac{N!}{n_1!\, n_2!} = \frac{4!}{(3!)(1!)} = \frac{4 \cdot \cancel{3!}}{\cancel{3!}} = \boxed{4 \text{ ways to make assignment}}$$

The numerator is the number of different ways (permutations) the workers could be assigned four distinct jobs. In the denominator, the division by (3)(2)(1) is to remove the duplicated permutations resulting from the fact that three workers are assigned the same job, and the division by 1 is associated with the worker assigned to job 2.

Definition 7.3: Partitions Rule

Suppose you wish to partition a single set of N different elements into k sets, with the first set containing n_1 elements, the second containing n_2 elements, . . ., and the kth set containing n_k elements. Then the number of different partitions is

$$\frac{N!}{n_1!\, n_2! \ldots n_k!} \quad \text{where } n_1 + n_2 + n_3 + \ldots + n_k = N\,.$$

Suppose you have 12 construction workers and you wish to assign 3 to job site 1, 4 to job site 2, and 5 to job site 3. In how many different ways can you make this assignment?

(12! = 479,001,600 if order mattered)

$\underbrace{\_\ \_\ \_}_{\text{job 1}}\quad \underbrace{\_\ \_\ \_\ \_}_{\text{job 2}}\quad \underbrace{\_\ \_\ \_\ \_\ \_}_{\text{job 3}}$

$4! = 4\cdot 3\cdot 2$
$= 12\cdot 2$

$3! = 3\cdot 2\cdot 1 = 6$

$N = 12$ $\quad n_1 = 3$
$k = 3$ $\quad n_2 = 4$
$n_3 = 5$

$$\frac{N!}{n_1!\,n_2!\,n_3!} = \frac{12!}{3!\,4!\,5!} = \frac{\cancel{12}\cdot 11\cdot 10\cdot 9\cdot \overset{=8}{4\cdot\cancel{2}}\cdot 7\cdot \cancel{6}\cdot \cancel{5!}}{\cancel{3!}\,\cancel{4!}\,\cancel{5!}}$$

$= \boxed{11\cdot 10\cdot 9\cdot 4\cdot 7}$

$\boxed{= 27{,}720 \text{ ways to make assignment}}$

A consumer testing service is commissioned to rank the top 3 brands of laundry detergent. Ten brands are to be included in the study. In how many different ways can the consumer testing service arrive at the final ranking?

- if order matters:

$\underset{1^{st}}{\underline{10}}\cdot \underset{2^{nd}}{\underline{9}}\cdot \underset{3^{rd}}{\underline{8}} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10\cdot 9\cdot 8 = \boxed{720 = P_3^{10}}$

- if order doesn't matter:

$\underbrace{\_\ \_\ \_}_{\text{top 3 group}} : \binom{10}{3} = \frac{10!}{3!\,7!} = \boxed{120}$

Chapter 3 Understanding

Chapter 3 goals:

After this chapter, you should be able to understand:

(1) The difference between experiment, outcomes, events, and probabilities;

(2) How to create and interpret tree diagrams and Venn diagrams;

(3) The Additive Rule and taking a union of two events;

(4) The Multiplicative Rule and taking an intersection of two events;

(5) The difference between mutually exclusive and independent events;

(6) Conditional probability;

(7) Counting rules including the difference between combinations, permutations, and partitions;

(8) That counting can be difficult sometimes!

Section Recap

What are some take-away concepts from this section?