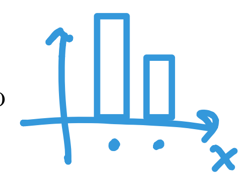
Chapter 5: Continuous Random Variables

In Chapter 4, we focused on discrete random variables. What were some of the properties of outcomes? How can we compare those to the outcomes of continuous random variables?

1. Continuous Probability Distributions

- **Continuous random variable**: a random variable with an infinite number of outcomes in any interval, such as time, length, weight, etc.

discrete:



While discrete random variables can be graphically represented by a histogram, continuous random variables are graphically represented as a function.

For example, the graph below shows a function of a continuous random variable, also called a **probability density function.** PDF

Definition 1.1: Properties of probability density functions

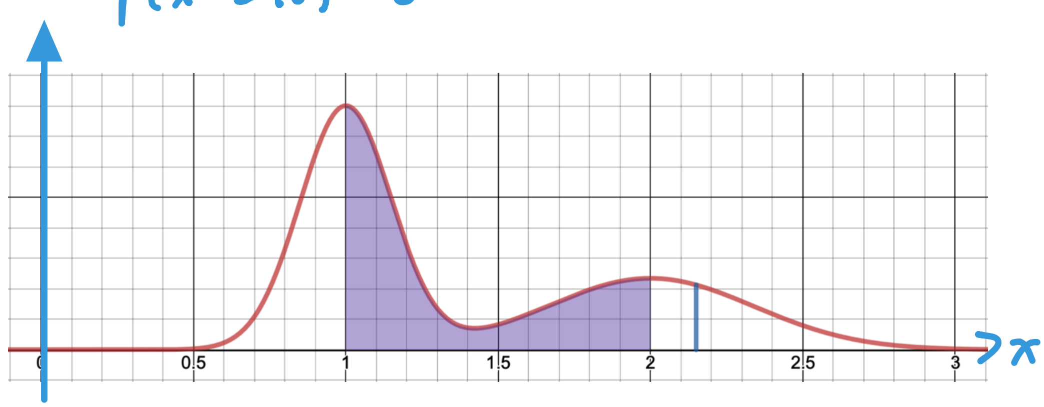
- $P(a) = 0$ for any outcome a.
- $P(a < x < b) = P(a \leq x \leq b)$ is probability that outcomes are between a and b
- Area under the curve equals 1.

How can we interpret the shaded area in purple of the graph below show?

$P(1 < x < 2)$: probability of an outcome between $x=1$ & $x=2$.

What about the blue line at $x = 2.15$?

$P(x = 2.15) = 0$



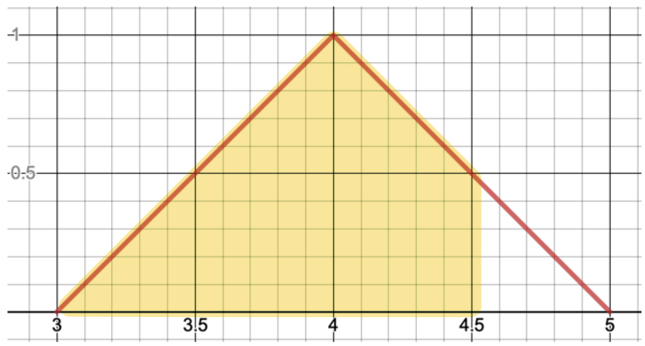
outcomes

Example 1.1. *Let* $x = \{$*the length of toy cars in a factory in inches*$\}$.

The graph below represents a probability density function of x. *How can we tell this is for a continuous random variable and not a discrete random variable?*

discrete → histogram

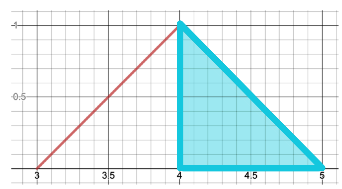
continuous → curve



Find the following probabilities:

- $P(x = 3)$ $= 0$

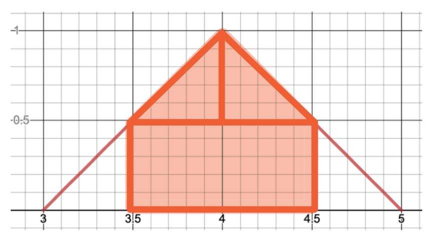
- $P(4 < x < 5)$ $= \frac{1}{2} \cdot 1 \cdot 1 = .5$ or 50%



- $P(3.5 < x < 4.5)$ $= 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

  $= \frac{1}{2} + \frac{1}{8} + \frac{1}{8}$

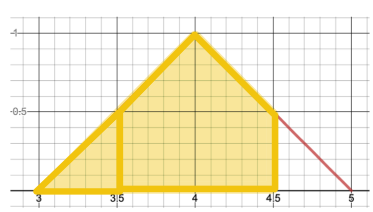
  $= \frac{3}{4}$ or 75%



- $P(x \leq 4.5)$ $= \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

  $= \frac{3}{4} + \frac{1}{8}$

  $= \frac{7}{8}$ or 87.5%

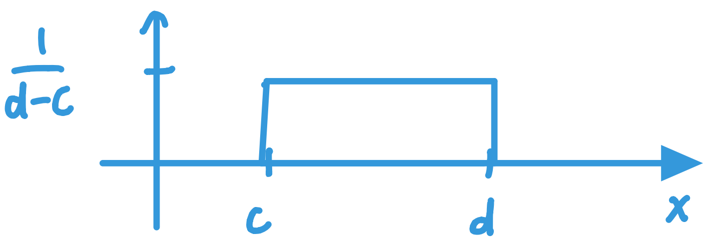


2. The Uniform Distribution

- **Uniform probability distribution random variable**: a continuous probability distribution in which all values in an interval are equally likely to occur.

Since all values in the interval have the same likelihood of occurring, the probability density function is constant over that set of intervals. So, the uniform probability distribution has a rectangular shape.

We still have the property that the area under the curve must equal 1. If a uniform distribution has ranges from $x = c$ to $x = d$, how can we determine the height of the uniform probability distribution?

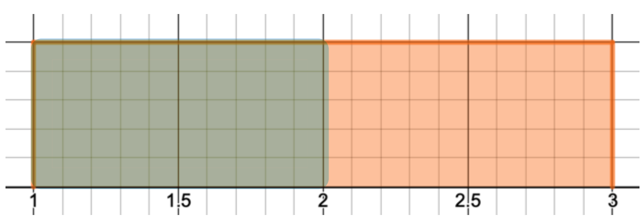


base · height = 1

$$(d-c)\cdot\frac{1}{d-c} = 1$$

Example 2.1. *Determine the height of the two uniform probability distributions below.*

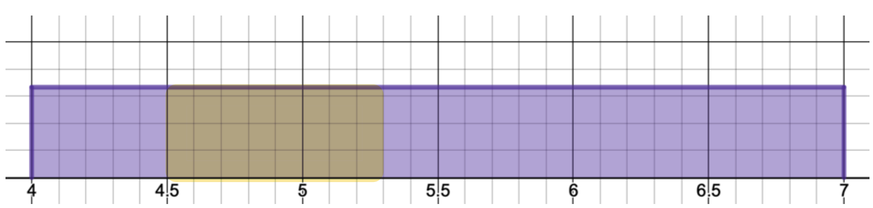
$$\frac{1}{3-1} = \frac{1}{2}$$



$$P(1 < x < 2) = 1\cdot\frac{1}{2} = \frac{1}{2}$$

outcomes

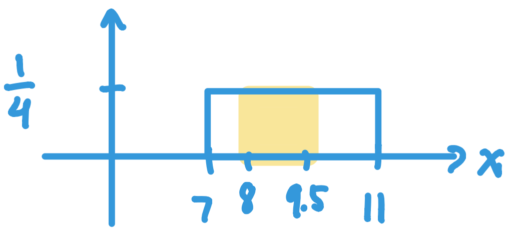
$$\frac{1}{7-4} = \frac{1}{3}$$



$$P(4.5 < x < 5.25) = \frac{3}{4}\cdot\frac{1}{3} = \frac{3}{12} = \frac{1}{4} \text{ or } 25\%$$

Example 2.2. *Suppose* $x = \{minutes\ waiting\ for\ a\ bus\}$ *is uniformly distributed between* $7 \leq x \leq 11$.

Draw out the probability distribution. What is the height of the function?



height $= \frac{1}{11-7} = \frac{1}{4}$

What is the probability of waiting between 8 and 9.5 minutes? How could we write this mathematically?

$P(8 < x < 9.5) = (1.5)\cdot(.25) = \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{8}$ or 37.5%.

Determine $P(7.5 < x \leq 8.07)$*. What does this represent?*

$P(7.5 < x < 8.07) = (8.07 - 7.5)\cdot(.25) = (.57)\cdot(.25) = .1425$ or 14.25%

prob. of waiting B/W 7.5 & 8.07 min

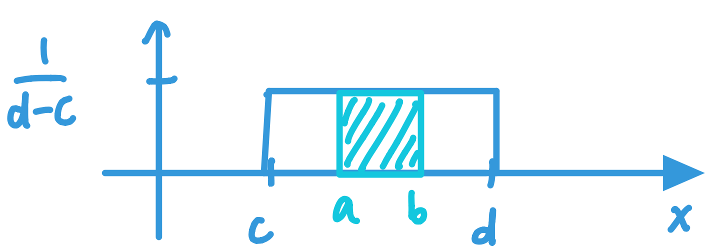
Definition 2.1: Prob. distribution for a Uniform Random Variable x

Probability density function: $f(x) = \dfrac{1}{d-c} \quad (c \leq x \leq d)$

Mean: $\mu = \dfrac{c+d}{2}$ **Standard deviation:** $\sigma = \dfrac{d-c}{\sqrt{12}}$

Finding probability between $x = a$ and $x = b$:

$$P(a < x < b) = \frac{b-a}{d-c} \qquad \text{where } c \leq a < b \leq d\,.$$



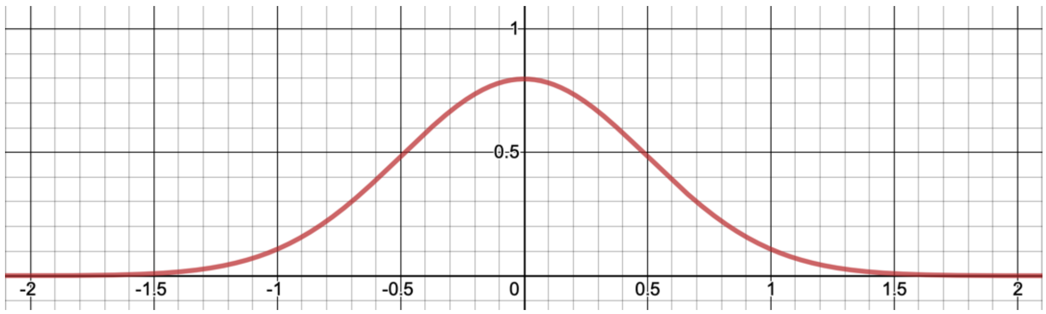
$P(a < x < b) =$ base $\cdot$ height

$\downarrow \quad = (b-a)\cdot\frac{1}{d-c}$

$P(a < x < b) = \frac{b-a}{d-c}$

3. The Normal Distribution

One of the most commonly observed continuous random variables has a **bell-shaped probability distribution** (or **bell curve**). It is known as a normal random variable and its probability distribution is called a **normal distribution**.



What are some properties of distribution curves like this?

symmetric, evenly distributed, area under curve = 1

Definition 3.1: Prob. Distribution for a Normal Random Variable x

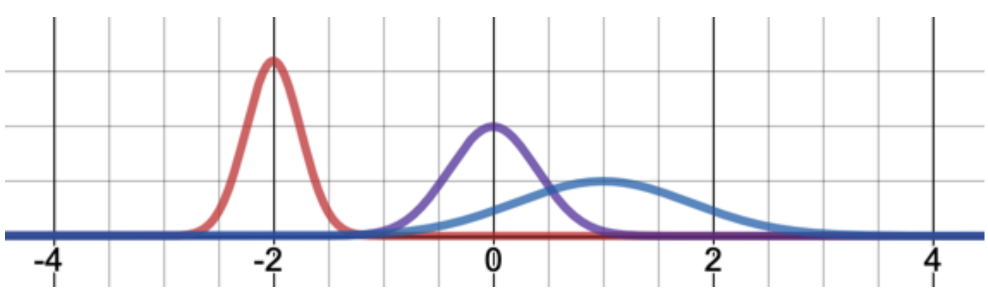
Probability density function: $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where

- μ: mean of the normal random variable,
- σ : standard deviation of the variable,
- π , e : mathematical numbers.

$P(x < a)$ is obtained from a table of normal probabilities.

Example 3.1. *Find the μ value of each curves. Which has the largest σ value?*

Blue curve has largest σ value - it's the most spread out data



$\mu = -2$ $\mu = 0$ $\mu = 1$

Definition 3.2: Standard Normal Distribution

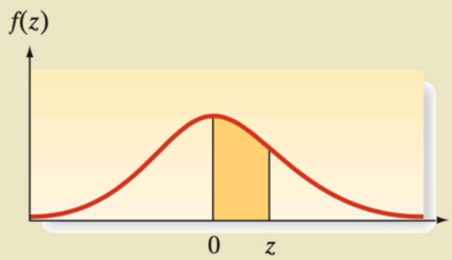
Standard normal distribution function: $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$,

where z is called the **normal random variable.**

This is a normal distribution with mean 0 and a standard deviation of 1.

We will (later) need to convert all normal random variables to *standard* normal variables in order to use the table below to find probabilities. The entries in the body of the table give the area (probability) between 0 and z, where we can find z by looking at the left-hand column (for the first number after the decimal pints) with the top row (for the second number after the decimal point).

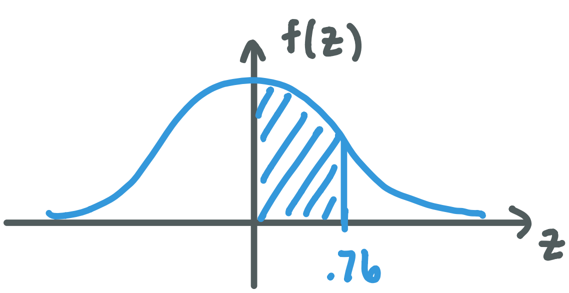
only for standard normal distributions



$\mu = 0$
$\sigma = 1$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441

Example 3.2. *Find the probability that the standard normal random variable z falls between 0 and .76.*



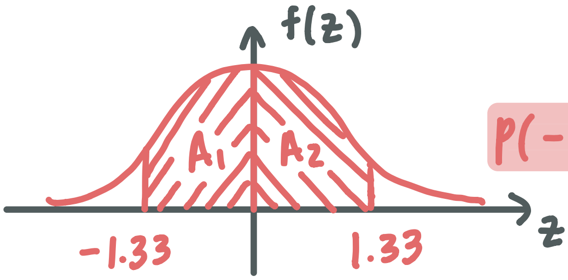
$P(0 < z < .76) = .2764$

Example 3.3. *Find the probability that the standard normal random variable z falls between -1.33 and $+1.33$.*

$$P(-1.33 < z < 1.33) = \overbrace{P(-1.33 < z < 0)}^{A_1} + \overbrace{P(0 < z < 1.33)}^{A_2}$$

$$= P(0 < z < 1.33) + P(0 < z < 1.33)$$

$$= .4082 + .4082$$

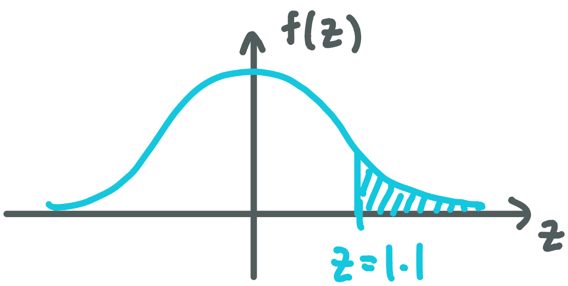


$$\boxed{P(-1.33 < z < 1.33) = .8164}$$

Example 3.4. *Find the probability that a standard normal random variable exceeds* 1.10. *What would our goal be written as in mathematical term?*

$$P(z > 1.1) = \frac{1}{2} - P(0 < z < 1.1)$$

$$= \frac{1}{2} - .3643$$

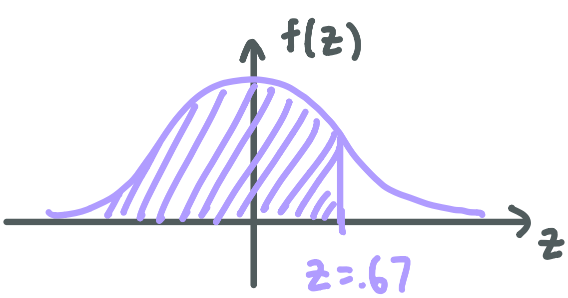


$$P(z > 1.1) = .1357$$

Example 3.5. *Find the probability that a standard normal random variable lies to the left of* .67.

$$P(z < .67) = \frac{1}{2} + P(0 < z < .67)$$

$$= \frac{1}{2} + .2486$$



$$P(z < .67) = .7486$$

Definition 3.3: Property of Normal Distributions

If x is a normal random variable with mean μ and standard deviation σ, then the random variable z defined by the formula

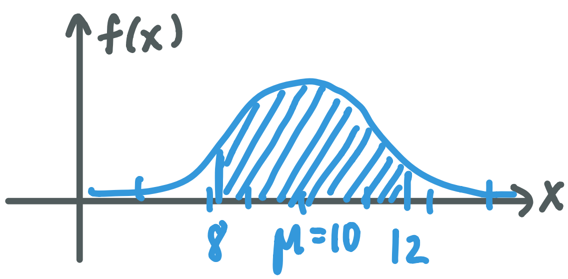
$$z = \frac{x - \mu}{\sigma}$$

has a standard normal distribution. The value z describes the number of standard deviations between x and μ.

Example 3.6. *Assume that the length of time, x, between charges of a cellular phone is normally distributed with a mean of 10 hours and a standard deviation of 1.5 hours. Find the probability that the cell phone will last between 8 and 12 hours between charges.*

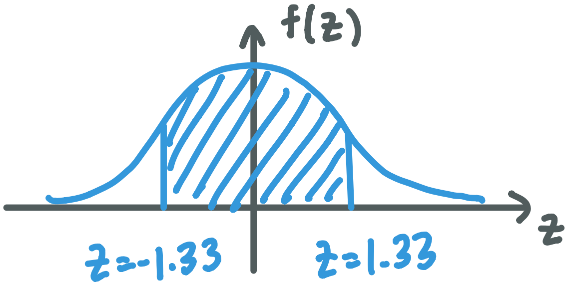
given: $\mu = 10$, $\sigma = 1.5$

goal: find $P(8 < X < 12)$



z-score of $x=8$: $z = \frac{8-10}{1.5} = -1.33$

z-score of $x=12$: $z = \frac{12-10}{1.5} = 1.33$

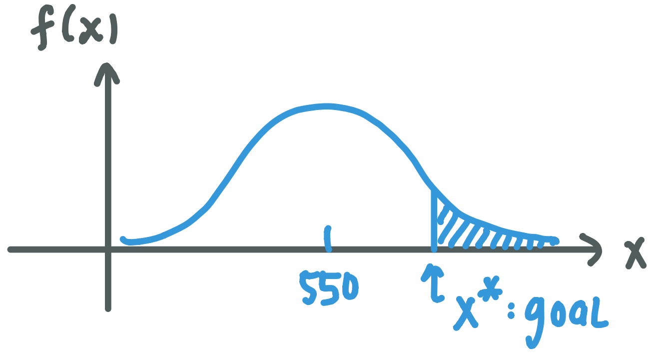


$P(8 < X < 12) = P(-1.33 < z < 1.33) = .8164$

(from ex. 3.3)

there is an 81.64% probability that a cell phone's charge will last b/w 8 & 12 hours.

Example 3.7. *Suppose the scores x on a college entrance examination are normally distributed with a mean of 550 and a standard deviation of 100. A certain prestigious university will consider for admission only those applicants whose scores exceed the 90th percentile of the distribution. Find the minimum score an applicant must achieve in order to receive consideration for admission to the university.*

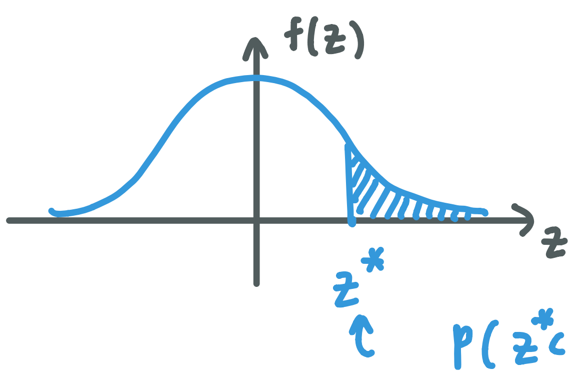


given: $\mu = 550$, $\sigma = 100$

goal: find x^* such that

$P(x^* < x) = .1$

$\Downarrow$



$P(z^* < z) = .1$

$P(z > 1.28) = .1$

$\Rightarrow$ looking for x^* with $z = 1.28$:

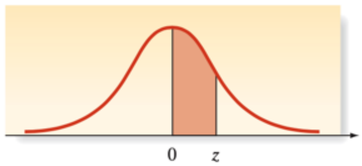
$$1.28 = \frac{x^* - 550}{100}$$

$$128 = x^* - 550$$

$$\boxed{678 = x^*}$$

$P(\ z < 1.28\ ) = .3997 \rightarrow$ 10.03% above 678

$P(\ z < 1.29) = .4015 \rightarrow$ 9.85% above 679



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Key Questions

(1) The crash test rating of a new car can be determined by investigating how severe a head injury is (to a crash test dummy) when the car crashes head-on into a barrier at 35mph. The scores of these tests are approximately normally distributed with a mean of 605 points and a standard deviation of 185 points.
 (a) Find $P(500 < x < 700)$.
 (b) Find $P(x > 1000)$.
 (c) Only 10% of cars will have above what score?

#1 a) .4073
b) .0166
c) 841.8

#2
a) .0047
b) .0136
c) .8186

(2) A nationwide exam has an average score of 77 points with a standard deviation of 7.7 points.
 (a) What is the probability that a random student scores a 97 or above?
 (b) What is the probability that a random student scores a 60 or below?
 (c) If scoring at least a 70 on the exam is considered passing, what is the probability that a random student passes the exam?

(3) In the United States, the average adult male is 69 inches tall (5'9") with a standard deviation of 4.5 inches and the average adult female is 63.5 inches tall (5'3.5") with a standard deviation of 4.5 inches. Assume both data sets are normally distributed. Let m be the height of a random male and f be the height of a random female.
 (a) Find $P(m > 6'6")$.
 (b) Find $P(m < 63.5")$.
 (c) Find $P(5' < m < 6')$.
 (d) Find $P(f < 5')$.
 (e) Find $P(f > 69")$.
 (f) Find $P(5' < f < 6')$.

#3 a) .0228 d) .2177
b) .1112 e) .1112
c) .7258 f) .7529

#4
a) .7019

(4) California's electoral college votes. During a presidential election, each state is allotted a different number of votes to the electoral college depending on population. For example, California is allotted 55 votes (the most) while several states (including the District of Columbia) are allotted 3 votes each (the least). When a presidential candidate wins the popular vote in a state, the candidate wins all the electoral college votes in that state (except in Nebraska and Maine). To become president, a candidate must win 270 of the total of 538 votes in the electoral college. Assuming a candidate wins California's 55 votes, the number of additional electoral college votes the candidate will win can be approximated by a normal distribution with $\mu = 241.5$ votes and $\sigma = 49.8$ votes. If a presidential candidate wins the popular vote in California, what are the chances that he or she becomes the next U.S. president?

Chapter 5 Understanding

Chapter 5 goals:

After this chapter, you should be able to understand:

(1) The definition and properties of a continuous random variable;

(2) The Uniform Distribution;

(3) The Normal Distribution and interpreting the normal probability table.

Section Recap

What are some take-away concepts from this section?